复分析作业

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Complex Analysis, Stein & Shakarchi

https://xiaoshuo-lin.github.io

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习题 1.1.6 设 $|a|<1,|z|<1$. 证明:

(3) $\dfrac{||z|-|a||}{1-|a||z|}\leqslant\left|\dfrac{z-a}{1-\bar{a}z}\right|\leqslant\dfrac{|z|+|a|}{1+|a||z|}$.

证明 $\left|\dfrac{z-a}{1-\bar{a}z}\right|^2=\dfrac{(z-a)(\bar{z}-\bar{a})}{(1-\bar{a}z)(1-a\bar{z})}=\dfrac{|z|^2+|a|^2-\bar{a}z-a\bar{z}}{1+|a|^2|z|^2-\bar{a}z-a\bar{z}}$. 令 $f(t)=\dfrac{\alpha+t}{\beta+t}=1+\dfrac{\alpha-\beta}{\beta+t}$, 其中 $\alpha<\beta$, 则当 $t>-\beta$ 时, $f(t)$ 单调递增. 由于 $|a|<1,|z|<1$, 我们有 $\big(|z|^2-1\big)\big(|a|^2-1\big)>0$, 即 $|z|^2+|a|^2<1+|a|^2|z|^2$, 因此取 $\alpha=|z|^2+a^2,\beta=1+|a|^2|z|^2$, 则有

$$|a|^2|z|^2+1-2\operatorname{Re}(a\bar{z})=[\operatorname{Re}(a\bar{z})-1]^2+[\operatorname{Im}(a\bar{z})]^2>0\implies t=-\bar{z}-a\bar{z}>-\beta,$$

从而

$$\left|\frac{z-a}{1-\bar{a}z}\right|^2=f(-\bar{a}z-a\bar{z})=f(2\operatorname{Re}(-\bar{a}z))<f(2|a||z|)=\frac{|z|^2+|a|^2+2|a||z|}{1+|a|^2|z|^2+2|a||z|}=\left(\frac{|z|+|a|}{1+|a||z|}\right)^2.$$

又

$$(|a||z|-1)^2>0\implies 2|a||z|<1+|a|^2|z|^2\implies t=-2|a||z|>-\beta,$$

因此

$$\left(\frac{|z|-|a|}{1-|a||z|}\right)^2=\frac{|z|^2+|a|^2-2|a||z|}{1+|a|^2|z|^2-2|a||z|}=f(-2|a||z|)\leqslant f(-2\operatorname{Re}(\bar{a}z))=\left|\frac{z-a}{1-\bar{a}z}\right|^2.$$

故

$$\frac{||z|-|a||}{1-|a||z|}\leqslant\left|\frac{z-a}{1-\bar{a}z}\right|\leqslant\frac{|z|+|a|}{1+|a||z|}.$$

□

习题 1.2.6 证明: 三点 z_1,z_2,z_3 共线的充要条件为

$$\begin{vmatrix} z_1 & \overline{z_1} & 1\\ z_2 & \overline{z_2} & 1\\ z_3 & \overline{z_3} & 1\end{vmatrix}=0.$$

证明 记 $z_j=x_j+\mathrm{i}y_j$, 则 z_1,z_2,z_3 共线 $\iff\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=0\iff\begin{vmatrix} x_1 & x_1-\mathrm{i}y_1 & 1\\ x_2 & x_2-\mathrm{i}y_2 & 1\\ x_3 & x_3-\mathrm{i}y_3 & 1\end{vmatrix}=0\iff$ $\begin{vmatrix} 2x_1 & x_1-\mathrm{i}y_1 & 1\\ 2x_2 & x_2-\mathrm{i}y_2 & 1\\ 2x_3 & x_3-\mathrm{i}y_3 & 1\end{vmatrix}=0\iff\begin{vmatrix} z_1 & \overline{z_1} & 1\\ z_2 & \overline{z_2} & 1\\ z_3 & \overline{z_3} & 1\end{vmatrix}=0.$ □

习题 1.2.12 设 z_1,z_2,z_3 是单位圆周上的三个点, 证明: 这三个点是一个正三角形三个顶点的充要条件为

$$z_1+z_2+z_3=0.$$

证明 不妨设沿逆时针方向次序为 z_1,z_2,z_3.

(⇒) 由于 z_1,z_2,z_3 恰三等分单位圆周, $z_2=\omega z_1, z_2=\omega^2 z_1$, 其中 $\omega=\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$. 因此 $z_1+z_2+z_3=z_1(1+\omega+\omega^2)=0$.

(⇐) 由于三点绕原点同方向旋转相同角度不影响正三角形的判定, 通过除以 z_1, 可不妨设 $z_1=1$, 则 $z_2+z_3=-1$. 因此 $|\operatorname{Im} z_2|=|\operatorname{Im} z_3|$, 再由 $|z_2|=|z_3|=1$ 得 $\operatorname{Re} z_2=\operatorname{Re} z_3=-\dfrac{1}{2}$, 于是 $z_2=\omega, z_3=\omega^2$, z_1,z_2,z_3 构成正三角形的三个顶点. □

习题 1.3.1 证明: 在复数的球面表示下, z 和 $\dfrac{1}{\bar z}$ 的球面像关于复平面对称.

证明 在球极投影下, 对 $z\in\mathbb{C}$, 有

$$z\mapsto\left(\frac{z+\bar z}{|z|^2+1},\frac{z-\bar z}{\mathrm{i}(|z|^2+1)},\frac{|z|^2-1}{|z|^2+1}\right),$$

$$\frac{1}{\bar z}\mapsto\left(\frac{\frac{1}{\bar z}+\frac{1}{z}}{\left|\frac{1}{\bar z}\right|^2+1},\frac{\frac{1}{\bar z}-\frac{1}{z}}{\mathrm{i}\left(\left|\frac{1}{\bar z}\right|^2+1\right)},\frac{\left|\frac{1}{\bar z}\right|^2-1}{\left|\frac{1}{\bar z}\right|^2+1}\right)=\left(\frac{z+\bar z}{|z|^2+1},\frac{z-\bar z}{\mathrm{i}(|z|^2+1)},\frac{1-|z|^2}{|z|^2+1}\right).$$

故 z 和 $\dfrac{1}{\bar z}$ 的球面像关于复平面对称. □

习题 1.3.2 证明: 在复数的球面表示下, z 和 w 的球面像是直径对点当且仅当 $z\overline{w}=-1$.

证明 (⇐) 由习题 1.3.1, z 与 $\dfrac{1}{\bar z}=-w$ 的球面像关于复平面对称. 而 w 与 $-w$ 的球面像关于单位球过原点的直径对称, 因此 z 和 w 的球面像是直径对点.

(⇒) 在球极投影下, 对 $(x_1,x_2,x_3)\in\mathbb{S}^2$, 有

$$(x_1,x_2,x_3)\mapsto\frac{x_1+\mathrm{i}x_2}{1-x_3},\quad(-x_1,-x_2,-x_3)\mapsto\frac{-x_1-\mathrm{i}x_2}{1+x_3}=\frac{-1}{\frac{x_1-\mathrm{i}x_2}{1-x_3}}.$$

因此 z 和 w 的球面像是直径对点当且仅当 $z\overline{w}=-1$. □

习题 1.4.2 设 $z=x+\mathrm{i}y\in\mathbb{C}$, 证明:

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=\mathrm{e}^x(\cos y+\mathrm{i}\sin y).$$

证明 注意到

$$\begin{aligned}\lim_{n\to\infty}\left|\left(1+\frac{x+\mathrm{i}y}{n}\right)^n\right|&=\lim_{n\to\infty}\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)^{\frac{n}{2}}=\exp\left[\lim_{n\to\infty}\frac{n}{2}\log\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)\right]\\&=\exp\left[\lim_{n\to\infty}\frac{n}{2}\left(\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)\right]=\mathrm{e}^x\end{aligned}$$

以及

$$\lim_{n\to\infty}\arg\left(1+\frac{x+\mathrm{i}y}{n}\right)^n=\lim_{n\to\infty}n\arctan\frac{\frac{y}{n}}{1+\frac{x}{n}}=y,$$

便有

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=\mathrm{e}^x(\cos y+\mathrm{i}\sin y).$$ □

习题 1.5.3 指出下列点集的内部、边界、闭包和导集:

(1) $\mathbb{N}$.

(2) $E = \left\{\frac{1}{k} : k \in \mathbb{N}\right\}$.

(3) $D = \mathbb{B}(1,1) \cup \mathbb{B}(-1,1)$.

(4) $G = \{z \in \mathbb{C} : 1 < |z| \leqslant 2\}$.

(5) $\mathbb{C}$.

解答 (1) 内部 $= \varnothing$, 边界 $= \mathbb{N}$, 闭包 $= \mathbb{N}$, 导集 $= \varnothing$.

(2) 内部 $= \varnothing$, 边界 $=$ 闭包 $= E \cup \{0\}$, 导集 $= \{0\}$.

(3) 内部 $= D$, 边界 $= \{z \in \mathbb{C} : |z-1| = 1$ 或 $|z+1| = 1\}$, 闭包 $=$ 导集 $= \{z \in \mathbb{C} : |z-1| \leqslant 1$ 或 $|z+1| \leqslant 1\}$.

(4) 内部 $= \{z \in \mathbb{C} : 1 < |z| < 2\}$, 边界 $= \{z \in \mathbb{C} : |z| = 1$ 或 $2\}$, 闭包 $=$ 导集 $= \{z \in \mathbb{C} : 1 \leqslant |z| \leqslant 2\}$.

(5) 内部 $= \mathbb{C}$, 边界 $= \varnothing$, 闭包 $\mathbb{C}$, 导集 $= \mathbb{C}$. □

习题 1.5.5 证明：若 D 为开集, 则 $D' = \overline{D} = \partial D \cup D$.

证明 (1) 由于 $\overline{D} = D \cup D'$, 为证 $D' = \overline{D}$, 只需证 $D \subset D'$. 对任意 $x \in D$, 由 D 是开集, 存在 $r > 0$ 使得 $\mathbb{B}(x,r) \subset D$. 于是对任意 $\varepsilon \in (0,r)$ 都有 $\mathbb{B}^\circ(x,\varepsilon) \subset D$, 故 $x \in D'$, $D \subset D'$, 进而 $D' = \overline{D}$.

(2) 由于 D 是开集, $\partial D \cap D = \varnothing$, 而 $\overline{D} = D \cup D'$, 为证 $\overline{D} = \partial D \cup D$, 只需证 $\partial D \subset D'$. 对任意 $x \in \partial D$ 与 $r > 0$, $\mathbb{B}(x,r) \cap D = \mathbb{B}^\circ(x,r) \cap D \neq \varnothing$, 因此 $x \in D'$, 进而 $\partial D \subset D'$, $\overline{D} = \partial D \cup D$. □

习题 1.6.1 满足下列条件的点 z 所组成的点集是什么？如果是域, 说明它是单连通域还是多连通域？

(1) $\operatorname{Re} z = 1$.

(2) $\operatorname{Im} z < -5$.

(3) $|z - \mathrm{i}| + |z + \mathrm{i}| = 5$.

(4) $|z - \mathrm{i}| \leqslant |2 + \mathrm{i}|$.

(5) $\arg(z-1) = \frac{\pi}{6}$.

(6) $|z| < 1, \operatorname{Im} z > \frac{1}{2}$.

(7) $\left|\frac{z-1}{z+1}\right| \leqslant 2$.

(8) $0 < \arg \frac{z-\mathrm{i}}{z+\mathrm{i}} < \frac{\pi}{4}$.

解答 (1) 直线 $\{z \in \mathbb{C} : \operatorname{Re} z = 1\}$, 非域.

(2) 半平面 $\{z \in \mathbb{C} : \operatorname{Im} z < -5\}$, 单连通域.

(3) 以 $\pm\mathrm{i}$ 为焦点、5 为长轴长的椭圆, 非域.

(4) 以 i 为圆心、$\sqrt{5}$ 为半径的闭圆盘, 非域.

(5) 以 1 为起点 (不含) 且与实轴夹角为 $\dfrac{\pi}{6}$ 的射线, 非域.

(6) 弓形 $\left\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > \frac{1}{2}\right\}$, 单连通域.

(7) $\left\{z \in \mathbb{C} : |z+3| \geqslant 2\sqrt{2}\right\}$, 非域.

(8) $\left\{z \in \mathbb{C} : \operatorname{Re} z < 0 \text{ 且 } |z+1| > \sqrt{2}\right\}$, 单连通域. □

习题 1.6.2 证明: 非空点集 $E \subset \mathbb{R}$ 为连通集, 当且仅当 E 是一个区间.

证明 ($\Rightarrow$) 设 $\varnothing \neq E \subset \mathbb{R}$ 连通. 若 E 不是一个区间, 则存在 $x < z < y$ 满足 $x, y \in E$ 但 $z \notin E$. 于是

$$E = (E \cap (-\infty, z)) \sqcup (E \cap (z, +\infty))$$

是两个非空不交开集的并, 与 E 连通矛盾. 故 E 是区间.

($\Leftarrow$) 设 $E \subset \mathbb{R}$ 为区间. 若 E 不连通, 则存在不交开集 $U, V \subset \mathbb{R}$ 使得

$$U \cap E \neq \varnothing, \quad V \cap E \neq \varnothing, \quad E \subset U \sqcup V.$$

不失一般性, 假设存在 $a < b$ 使得 $a \in U \cap E$ 且 $b \in V \cap E$. 令

$$A = \{x \in U \cap E : x < b\},$$

并记 $c = \sup A$. 则由 A 是开集可知 $c \neq a$, 于是 $a < c \leqslant b$. 特别地, $c \in E$. 但是

- ◇ $c \notin U$: 若 $c \in U$, 则存在 $\varepsilon > 0$ 使得 $b > c + \varepsilon \in U$. 由 E 是区间知 $c + \varepsilon \in U \cap E$, 但这与 $c = \sup A$ 矛盾.
- ◇ $c \notin V$: 若 $c \in V$, 则存在 $\varepsilon > 0$ 使得 $(c - \varepsilon, c] \subset V$. 因为 $c > a$, 所以可取 ε 充分小使得 $(c - \varepsilon, c] \subset E$, 从而 $c - \varepsilon < c$ 也是 A 的上界 (因 $(c - \varepsilon, c] \cap U = \varnothing$), 与 $c = \sup A$ 矛盾.

故 $c \notin U \cup V$, 进而 $c \notin E$, 矛盾. □

习题 1.6.5 证明: 若 D 是有界单连通域, 则 ∂D 连通. 举例说明, 若 D 是无界单连通域, 则 ∂D 可能不连通.

证明 先给出 D 是无界单连通域时的反例: 令 $D = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$, 它是无界单连通域, 但 $\partial D = \{z \in \mathbb{C} : \operatorname{Im} z = \pm 1\}$ 不连通. 下证原命题.

引理 1 若 $D \subset \mathbb{C}$ 是有界单连通域, 则 $\mathbb{C} \setminus D$ 连通.

引理 2 ([Mun] Theorem 63.1(a)) 设 X 是两个开集 U 和 V 之并, 且 $U \cap V$ 可以表示成两个不交开集 A 和 B 之并. 假设有一条 U 中的道路 α 从 A 的一个点 a 到 B 的一个点 b, 并且有一条 V 中的道路 β 从 b 到 a. 记 $f = \alpha * \beta$, 则 f 是一条回路, 且道路同伦类 $[f]$ 生成 $\pi_1(X, a)$ 的一个无限循环子群.

[Mun] J. R. Munkres, *Topology*, 2nd ed., Pearson Education Limited, 2019.

原命题 设 ∂D 不连通, 不妨设 $\partial D = D_1 \cup D_2$, 其中 D_1, D_2 是两个不相交的闭集. 由于 D 有界, D_1, D_2 都是紧致的, 设 $\varepsilon = \frac{1}{3} d(D_1, D_2) > 0$, 构造开集 $A = \bigcup\limits_{z \in D_1} \mathbb{B}(z, \varepsilon)$, $B = \bigcup\limits_{z \in D_2} \mathbb{B}(z, \varepsilon)$, 显然 $D_1 \subset A$, $D_2 \subset B$, 并且仍然有 $A \cap B = \varnothing$. 令 $U = D \cup A \cup B$, $V = (\mathbb{C} \setminus D) \cup A \cup B$, 显然 U 是开集. 对任意 $z \in \partial D$, 都有 $\mathbb{B}(z, \varepsilon) \subset V$, 因此 V 也是开的. 因为 D 连通, 所以 $\overline{D}$ 连通, $U = \overline{D} \cup \bigcup\limits_{z \in \partial D} \mathbb{B}(z, \varepsilon)$,

其中每个开球 $\mathbb{B}(z,\varepsilon)$ 连通, 且和 $\overline{D}$ 至少相交于 z, 故 U 连通. 由引理 1 知 $\mathbb{C}\setminus D$ 连通. 同理, V 连通. 注意到

$$U\cup V=\mathbb{C},\quad U\cap V=A\cup B,\quad U,V \text{ 道路连通 (因它们是连通开集)}.$$

选取 $a\in A$, $b\in B$, 由 U 道路连通, 存在一条 U 中的道路 α 从 a 到 b. 同理存在一条 V 中的道路 β 从 b 到 a. 由引理 2, $f=\alpha*\beta$ 是一条回路, 并且 $[f]$ 生成了 $\pi_1(U\cup V,u)=\pi_1(\mathbb{C},u)$ 的一个无限循环子群. 但因 $\mathbb{C}$ 是单连通的, 其基本群平凡, 没有无限循环子群, 矛盾. 因此 ∂D 是连通的. □

习题 2.2.2 设 $f\in\mathcal{H}(D)$, 并且满足下列条件之一:

(1) $\operatorname{Re} f(z)$ 是常数.

(2) $\operatorname{Im} f(z)$ 是常数.

(3) $|f(z)|$ 是常数.

(4) $\arg f(z)$ 是常数.

(5) $\operatorname{Re} f(z)=[\operatorname{Im} f(z)]^2, z\in D$.

那么 f 是一常数.

证明 (1) 用 u 和 v 记 $f(z)$ 的实部和虚部, 则 $\dfrac{\partial u}{\partial x}\equiv\dfrac{\partial u}{\partial y}\equiv 0$, 由 Cauchy-Riemann 方程, $\dfrac{\partial v}{\partial x}\equiv\dfrac{\partial v}{\partial y}\equiv 0$, 因此 $f'(z)=\dfrac{\partial u}{\partial x}+\mathrm{i}\dfrac{\partial v}{\partial x}\equiv 0$, f 是一常数.

(2) 同 (1) 可得 $\dfrac{\partial u}{\partial x}\equiv\dfrac{\partial u}{\partial y}\equiv\dfrac{\partial v}{\partial x}\equiv\dfrac{\partial v}{\partial y}\equiv 0$, 因此 $f'(z)\equiv 0$, f 是一常数.

(3) 设 $|f(z)|\equiv C$. 若 $C=0$, 则 $f(z)\equiv 0$; 若 $C\neq 0$, 由 $f(z)\overline{f(z)}\equiv C^2$ 得

$$\frac{\partial f}{\partial z}\overline{f(z)}+f(z)\frac{\partial\overline{f}}{\partial z}\equiv\frac{\partial f}{\partial z}\overline{f(z)}\equiv 0.$$

而 $\overline{f(z)}\neq 0$, 因此 $\dfrac{\partial f}{\partial z}=f'(z)=0$, f 是一常数.

(4) 用 u 和 v 记 $f(z)$ 的实部和虚部, 则 $\arg f(z)=\arctan\dfrac{v}{u}$, 且 $u^2+v^2\neq 0$. 由 $\arg f(z)$ 是常数得

$$\begin{cases}\dfrac{\partial}{\partial x}\left(\arctan\dfrac{v}{u}\right)=\dfrac{1}{u^2+v^2}\left(u\dfrac{\partial v}{\partial x}-v\dfrac{\partial u}{\partial x}\right)=0,\\[2ex] \dfrac{\partial}{\partial y}\left(\arctan\dfrac{v}{u}\right)=\dfrac{1}{u^2+v^2}\left(u\dfrac{\partial v}{\partial y}-v\dfrac{\partial u}{\partial y}\right)=0\end{cases}\implies\begin{cases}u\dfrac{\partial v}{\partial x}=v\dfrac{\partial u}{\partial x},\\[2ex] u\dfrac{\partial v}{\partial y}=v\dfrac{\partial u}{\partial y}.\end{cases}$$

而由 Cauchy-Riemann 方程, $\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial y}=\dfrac{\partial u}{\partial x}$, 代入上式即得

$$\begin{cases}-u\dfrac{\partial u}{\partial y}=v\dfrac{\partial u}{\partial x},\\[2ex] u\dfrac{\partial u}{\partial x}=v\dfrac{\partial u}{\partial y}\end{cases}\implies\begin{cases}(u^2+v^2)\dfrac{\partial u}{\partial x}\equiv 0,\\[2ex] (u^2+v^2)\dfrac{\partial u}{\partial y}\equiv 0\end{cases}\xRightarrow{u^2+v^2\neq 0}\frac{\partial u}{\partial x}\equiv\frac{\partial u}{\partial y}\equiv 0.$$

由 (1) 即得 $f(z)$ 是一常数.

(5) 用 u 和 v 记 $f(z)$ 的实部和虚部, 则 $u-v^2\equiv 0$, 因此

$$\begin{cases}\dfrac{\partial u}{\partial x}-2v\dfrac{\partial v}{\partial x}\equiv 0,\\ \dfrac{\partial u}{\partial y}-2v\dfrac{\partial v}{\partial y}\equiv 0.\end{cases}$$

由 Cauchy-Riemann 方程, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$, 代入上式即得

$$\begin{cases}\dfrac{\partial v}{\partial y}=2v\dfrac{\partial v}{\partial x},\\ \dfrac{\partial v}{\partial x}+2v\dfrac{\partial v}{\partial y}\equiv 0\end{cases}\implies\begin{cases}\left(1+4v^2\right)\dfrac{\partial v}{\partial x}\equiv 0,\\ \left(1+4v^2\right)\dfrac{\partial v}{\partial y}\equiv 0\end{cases}\implies\dfrac{\partial v}{\partial x}\equiv\dfrac{\partial v}{\partial y}\equiv 0.$$

由 (1) 即得 $f(z)$ 是一常数. □

习题 2.2.4 设 $z=r(\cos\theta+\mathrm{i}\sin\theta)$, $f(z)=u(r,\theta)+\mathrm{i}v(r,\theta)$, 证明 Cauchy-Riemann 方程为

$$\begin{cases}\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta},\\ \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}.\end{cases}$$

证明 记 $\begin{cases}x=r\cos\theta,\\ y=r\sin\theta,\end{cases}$ 则 $\begin{pmatrix}\dfrac{\partial u}{\partial r}\\ \dfrac{\partial u}{\partial\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial u}{\partial y}\end{pmatrix}$, 因此

$$\begin{pmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial u}{\partial y}\end{pmatrix}=\begin{pmatrix}\cos\theta & -\dfrac{1}{r}\sin\theta\\ \sin\theta & \dfrac{1}{r}\cos\theta\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial r}\\ \dfrac{\partial u}{\partial\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial u}{\partial\theta}\\ \sin\theta\dfrac{\partial u}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial u}{\partial\theta}\end{pmatrix}.$$

同理可得

$$\begin{pmatrix}\dfrac{\partial v}{\partial x}\\ \dfrac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}\cos\theta\dfrac{\partial v}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial v}{\partial\theta}\\ \sin\theta\dfrac{\partial v}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial v}{\partial\theta}\end{pmatrix}.$$

此时 Cauchy-Riemann 方程为

$$\begin{cases}\cos\theta\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial u}{\partial\theta}=\sin\theta\dfrac{\partial v}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial v}{\partial\theta},\\ \sin\theta\dfrac{\partial u}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial u}{\partial\theta}=\dfrac{1}{r}\sin\theta\dfrac{\partial v}{\partial\theta}-\cos\theta\dfrac{\partial v}{\partial r}.\end{cases}$$

整理即得

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\dfrac{\partial v}{\partial\theta}\\ \dfrac{\partial v}{\partial r}+\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\implies\begin{cases}\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta},\\ \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}.\end{cases}$$ □

习题 2.2.7 设 D 是 $\mathbb{C}$ 中的域, $f \in \mathcal{C}^2(D)$. 证明：对每个 $z \in D$, 有

$$\frac{\partial^2 f}{\partial z \partial \bar{z}}(z) = \frac{\partial^2 f}{\partial \bar{z} \partial z}(z).$$

证明 由于 $f \in \mathcal{C}^2(D)$, 其二阶偏导数具有对称性, $\dfrac{\partial^2}{\partial z \partial \bar{z}} = \dfrac{1}{4}\Delta = \dfrac{\partial^2}{\partial \bar{z} \partial z}$. □

习题 2.2.11 设 D 是域, $f : D \to \mathbb{C} \setminus (-\infty, 0]$ 是非常数的全纯函数, 则 $\log|f(z)|$ 和 $\arg f(z)$ 是 D 上的调和函数, 而 $|f(z)|$ 不是 D 上的调和函数.

证明 由

$$\begin{aligned}\Delta \log|f(z)| &= \frac{1}{2}\Delta \log|f(z)|^2 = 2\frac{\partial^2}{\partial z \partial \bar{z}} \log\Big(f(z)\overline{f(z)}\Big) \\ &= 2\frac{\partial}{\partial z}\left(\frac{f(z)\overline{f'(z)}}{f(z)\overline{f(z)}}\right) = 2\frac{\partial}{\partial z}\left(\frac{\overline{f'(z)}}{\overline{f(z)}}\right) \xlongequal{\text{C-R 方程}} 0\end{aligned}$$

知 $\log|f(z)|$ 是 D 上的调和函数. 由

$$\mathrm{e}^{2\mathrm{i} \arg f(z)} = \frac{f(z)}{\overline{f(z)}}$$

可得

$$2\mathrm{i}\mathrm{e}^{2\mathrm{i} \arg f(z)} \frac{\partial}{\partial z} \arg f(z) = \frac{f'(z)}{\overline{f(z)}} \implies \frac{\partial}{\partial z} \arg f(z) = \frac{f'(z)}{2\mathrm{i} f(z)},$$

因此

$$\Delta \arg f(z) = 4\frac{\partial^2}{\partial \bar{z} \partial z} \arg f(z) = \frac{\partial}{\partial \bar{z}}\left(\frac{f'(z)}{2\mathrm{i} f(z)}\right) = 0,$$

即 $\arg f(z)$ 是 D 上的调和函数. 而

$$\frac{\partial}{\partial z}|f(z)| = \frac{f'(z)\overline{f(z)}}{2\sqrt{f(z)\overline{f(z)}}},$$

进而

$$\Delta|f(z)| = 4\frac{\partial^2}{\partial \bar{z} \partial z}|f(z)| = 2f'(z) \cdot \frac{\overline{f'(z)}|f(z)| - \frac{1}{2}|f(z)|\overline{f'(z)}}{|f(z)|^2} = \frac{|f'(z)|^2}{|f(z)|},$$

由 $f(z)$ 非常数, $|f'(z)|$ 不恒为 0, 因此 $|f(z)|$ 不是 D 上的调和函数. □

习题 2.2.13 设 u 是域 D 上的实值调和函数, $|\nabla u| \neq 0$, φ 是 $u(D)$ 上的实函数. 证明：$\varphi \circ u$ 是 D 上的调和函数当且仅当 φ 是线性函数.

证明 记 $\psi = \varphi \circ u$, 则 $\Delta \psi = \varphi''(u)|\nabla u|^2 + \varphi'(u)\Delta u = \varphi''(u)|\nabla u|^2$. 故 $\Delta \psi \equiv 0 \iff \varphi''(u) \equiv 0$. □

习题 2.3.1 求映射 $w = \dfrac{z - \mathrm{i}}{z + \mathrm{i}}$ 在 $z_1 = -1$ 和 $z_2 = \mathrm{i}$ 处的转动角和伸缩率.

解答 由于 $\dfrac{\partial w}{\partial z} = \dfrac{2\mathrm{i}}{(z + \mathrm{i})^2}$, $w'(z_1) = -1$, $w'(z_2) = -\dfrac{\mathrm{i}}{2}$, 映射 w 在 z_1 处的转动角为 π, 伸缩率为 1; 在 z_2 处的转动角为 $-\dfrac{\pi}{2}$, 伸缩率为 $\dfrac{1}{2}$. □

习题 2.3.2 设 f 是域 D 上的全纯函数, 且 $f'(z)$ 在 D 上不取零值. 试证：

(1) 对每一个 $u_0 + \mathrm{i}v_0 \in f(D)$, 曲线 $\operatorname{Re} f(z) = u_0$ 和曲线 $\operatorname{Im} f(z) = v_0$ 正交.

(2) 对每一个 $r_0\mathrm{e}^{\mathrm{i}\theta_0} \in f(D) \setminus \{0\}, -\pi < \theta_0 \leqslant \pi$, 曲线 $|f(z)| = r_0$ 与曲线 $\arg f(z) = \theta_0$ 正交.

证明 (1) 用 u 和 v 记 $f(z)$ 的实部和虚部, 则曲线 $u(x,y)=u_0$ 在 (x,y) 处的法向量为 $\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$, 曲线 $v(x,y)=v_0$ 在 (x,y) 处的法向量为 $\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)\xlongequal{\text{C-R 方程}}\left(-\frac{\partial u}{\partial y},\frac{\partial u}{\partial x}\right)$. 因此在这两条曲线交点处两法向量正交, 即这两条曲线正交.

(2) 设 $f(z)=R(r,\theta)\mathrm{e}^{\mathrm{i}\Theta(r,\theta)}$.

① 对 $\log f(z)=\log R(r,\theta)+\mathrm{i}\Theta(r,\theta)$ 运用习题 2.2.4 即得极坐标系下的 Cauchy-Riemann 方程

$$\begin{cases}\dfrac{\partial R}{\partial r}=\dfrac{R}{r}\dfrac{\partial\Theta}{\partial\theta},\\[2mm]\dfrac{\partial R}{\partial\theta}=-Rr\dfrac{\partial\Theta}{\partial r}.\end{cases}$$

② 曲线 $R(r,\theta)=r_0$ 在 (r,θ) 处的法向量为 $\frac{\partial R}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial R}{\partial\theta}\mathbf{e}_\theta$, 曲线 $\Theta(r,\theta)=\theta_0$ 在 (r,θ) 处的法向量为 $\frac{\partial\Theta}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial\Theta}{\partial\theta}\mathbf{e}_\theta\xlongequal{\text{C-R 方程}}-\frac{1}{Rr}\frac{\partial R}{\partial\theta}\mathbf{e}_r+\frac{1}{R}\frac{\partial R}{\partial r}\mathbf{e}_\theta$. 因此在这两条曲线交点处两法向量正交, 即这两条曲线正交. □

习题 2.3.3 设 $f\in\mathcal{H}(\mathbb{B}(0,1)\cup\{1\})$, 且 $f(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$, $f(1)=1$. 证明: $f'(1)\geqslant 0$.

证明 由于 $f(z)$ 在 $z=1$ 处全纯,

$$f(z)=f(1)+f'(1)(z-1)+o(|z-1|)=1+f'(1)+o(|z-1|),\quad z\to 1.$$

由题设, 当 $|z|<1$ 时 $|f(z)|<1$, 因此

$$|1+f'(1)+o(|z-1|)|<1,\quad \mathbb{B}(0,1)\ni z\to 1.$$

展开即得

$$\operatorname{Re}(f'(1)(z-1))+o(|z-1|)<0,\quad \mathbb{B}(0,1)\ni z\to 1.$$

令 $z-1=r\mathrm{e}^{\mathrm{i}\theta}$, $\theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, 上式化为

$$\operatorname{Re}\big(f'(1)r\mathrm{e}^{\mathrm{i}\theta}\big)+o(r)<0\iff\operatorname{Re}\big(f'(1)\mathrm{e}^{\mathrm{i}\theta}\big)+o(1)<0,\quad r\to 0^+.$$

于是

$$\operatorname{Re}\big(f'(1)\mathrm{e}^{\mathrm{i}\theta}\big)\leqslant 0,\quad \forall\theta\in\left(\tfrac{\pi}{2},\tfrac{3\pi}{2}\right).$$

令 $f'(1)=|f'(1)|\mathrm{e}^{\mathrm{i}\arg f'(1)}$. 若 $|f'(1)|\neq 0$, 则

$$\operatorname{Re}\Big(\mathrm{e}^{\mathrm{i}\left(\arg f'(1)+\theta\right)}\Big)\leqslant 0,\quad \forall\theta\in\left(\tfrac{\pi}{2},\tfrac{3\pi}{2}\right).$$

因此

$$\arg f'(1)+\theta\in\left[\tfrac{\pi}{2},\tfrac{3\pi}{2}\right],\quad \forall\theta\in\left(\tfrac{\pi}{2},\tfrac{3\pi}{2}\right),$$

由此可见 $\arg f'(1)=0$, 从而 $f'(1)=|f'(1)|>0$. 故 $f'(1)\geqslant 0$. □

习题 2.4.2 求 $\left|\mathrm{e}^{z^2}\right|$ 和 $\arg\mathrm{e}^{z^2}$.

解答 $\left|\mathrm{e}^{z^2}\right|=\mathrm{e}^{(\operatorname{Re}z)^2-(\operatorname{Im}z)^2}$, $\arg\mathrm{e}^{z^2}=2\operatorname{Re}z\operatorname{Im}z$. □

习题 2.4.4 设 f 是整函数, $f(0)=1$. 证明:

(1) 若 $f'(z)=f(z)$ 对每个 $z\in\mathbb{C}$ 成立, 则 $f(z)\equiv \mathrm{e}^z$.

(2) 若对每个 $z,w\in\mathbb{C}$, 有 $f(z+w)=f(z)f(w)$, 且 $f'(0)=1$, 则 $f(z)\equiv \mathrm{e}^z$.

证明 (1) 由

$$\frac{\partial}{\partial z}\left(\frac{f(z)}{\mathrm{e}^z}\right)=\frac{f'(z)-f(z)}{\mathrm{e}^z}\equiv 0,\quad \left.\frac{f(z)}{\mathrm{e}^z}\right|_{z=0}=1$$

即知 $f(z)\equiv \mathrm{e}^z$.

(2) 由于

$$\frac{f(z+w)-f(z)}{w}=f(z)\cdot\frac{f(w)-f(0)}{w-0}\xRightarrow[f'(0)=1]{w\to 0}f'(z)\equiv f(z),$$

由 (1) 即知 $f(z)=\mathrm{e}^z$. □

习题 2.4.15 称 $\varphi(z)=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$ 为 Rokovsky 函数. 证明下面四个域都是 φ 的单叶性域:

(1) 上半平面 $\{z\in\mathbb{C}:\operatorname{Im}z>0\}$.

(2) 下半平面 $\{z\in\mathbb{C}:\operatorname{Im}z<0\}$.

(3) 无心单位圆盘 $\{z\in\mathbb{C}:0<|z|<1\}$.

(4) 单位圆盘的外部 $\{z\in\mathbb{C}:|z|>1\}$.

证明 设 $z_1,z_2\in\mathbb{C}$ 使得 $\varphi(z_1)=\varphi(z_2)$, 则 $(z_1z_2-1)(z_1-z_2)=0$, 因此只要域 D 中任意两点不满足 $z_1z_2=1$, D 就是 $\varphi(z)$ 的单叶性域.

(1) 对任意 $z_1,z_2\in\{z\in\mathbb{C}:\operatorname{Im}z>0\}$, 由 $\arg z_1,\arg z_2\in(0,\pi)$ 得 $\arg(z_1z_2)\in(0,2\pi)$, 因此 $z_1z_2\neq 1$.

(2) 通过 $z_1z_2=1 \iff \overline{z_1z_2}=1$ 转化为 (1).

(3) 对任意 $z_1,z_2\in\{z\in\mathbb{C}:0<|z|<1\}$, 由 $|z_1|,|z_2|<1$ 得 $|z_1z_2|<1$, 因此 $z_1z_2\neq 1$.

(4) 通过 $z_1z_2=1 \iff \dfrac{1}{z_1}\dfrac{1}{z_2}=1$ 转化为 (3). □

习题 2.4.16 求习题 2.4.15 中的四个域在映射 $\varphi(z)=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$ 下的像.

解答 设 $z=r\mathrm{e}^{\mathrm{i}\theta}$, $\varphi(z)=u+\mathrm{i}v$, 则

$$u=\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta,\quad v=\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta.$$

因此 φ 将圆周 $|z|=r_0\neq 0$ 映为曲线

$$u=\frac{1}{2}\left(r_0+\frac{1}{r_0}\right)\cos\theta,\quad v=\frac{1}{2}\left(r_0-\frac{1}{r_0}\right)\sin\theta.$$

当 $r_0\neq 1$ 时, 这是半轴长为 $a=\dfrac{1}{2}\left(r_0+\dfrac{1}{r_0}\right)$, $b=\dfrac{1}{2}\left|r_0-\dfrac{1}{r_0}\right|$, 且由 $a^2-b^2\equiv 1$ 知 $z=\pm 1$ 为所有椭圆的公共焦点. 当 $r_0\to 1$ 时, $a\to 1, b\to 0$, 椭圆压缩成实轴上的线段 $[-1,1]$; 当 $r_0\to 0^+$ 或 $r_0\to+\infty$ 时, $a,b\to+\infty$, 椭圆扩张为圆周. 故

- ◇ 无心单位圆盘 $\{z \in \mathbb{C} : 0 < |z| < 1\} \xrightarrow{\varphi} \mathbb{C} \setminus [-1, 1]$.
- ◇ 单位圆盘的外部 $\{z \in \mathbb{C} : |z| > 1\} \xrightarrow{\varphi} \mathbb{C} \setminus [-1, 1]$.

再考虑射线 $\arg z = \theta_0$ $(\theta \in [0, 2\pi))$, 它在 φ 下的像为

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta_0, \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta_0.$$

当 $\theta_0 = 0$ 时, 这是射线 $\{u : u \geqslant 1\}$; 当 $\theta_0 = \pi$ 时, 这是射线 $\{u : u \leqslant -1\}$; 当 $\theta_0 = \dfrac{\pi}{2}$ 或 $\dfrac{3\pi}{2}$ 时, 这是虚轴; 当 θ_0 不取上述值时, 这是双曲线

$$\frac{u^2}{\cos^2\theta_0} - \frac{v^2}{\sin^2\theta_0} = 1,$$

且由 $\cos^2\theta_0 + \sin^2\theta_0 \equiv 1$ 知 $z = \pm 1$ 为所有双曲线的公共焦点. 故

- ◇ 上半平面 $\{z \in \mathbb{C} : \operatorname{Im} z > 0\} \xrightarrow{\varphi} \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$.
- ◇ 下半平面 $\{z \in \mathbb{C} : \operatorname{Im} z > 0\} \xrightarrow{\varphi} \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$. □

习题 2.4.18 证明: $w = \cos z$ 将半条形域 $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 2\pi, \operatorname{Im} z > 0\}$ 一一地映为 $\mathbb{C} \setminus [-1, +\infty)$.

证明 记 $\mu(z) = \mathrm{i}z, \eta(z) = \mathrm{e}^z, \varphi(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, 则 $w = \varphi \circ \eta \circ \mu$, 且有

$$\begin{array}{ccc} \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 2\pi, \operatorname{Im} z > 0\} & \xrightarrow[1:1]{\mu} & \{z \in \mathbb{C} : \operatorname{Re} z < 0, 0 < \operatorname{Im} z < 2\pi\} \\ & & \Big\downarrow {\scriptstyle 1:1}\ \eta \\ \mathbb{C} \setminus [-1, +\infty) & \xleftarrow[1:1]{\varphi} & \mathbb{B}(0,1) \setminus [0,1) \end{array}$$

其中第一个箭头为双射是显然的, 第二个箭头为双射可由 $\{z \in \mathbb{C} : \operatorname{Re} z < 0, 0 < \operatorname{Im} z < 2\pi\}$ 是 η 的单叶域得到, 第三个箭头为双射证明如下: 由习题 2.4.15, 无心单位圆盘是 φ 的单叶域, 进而 $\mathbb{B}(0,1) \setminus [0,1)$ 也是 φ 的单叶域; 再由习题 2.4.16, φ 将无心单位圆盘映为 $\mathbb{C} \setminus [-1, 1]$, 因此 φ 将 $\mathbb{B}(0,1) \setminus [0,1)$ 映为

$$(\mathbb{C} \setminus [-1, 1]) \setminus \varphi([0, 1)) = \mathbb{C} \setminus ([-1, 1] \cup (1, +\infty)) = \mathbb{C} \setminus [-1, +\infty),$$

由此得到第三个箭头, 且其为双射. 由双射的复合即得所欲证. □

习题 2.4.19 证明: $w = \sin z$ 将半条形域 $\left\{z \in \mathbb{C} : -\dfrac{\pi}{2} < \operatorname{Re} z < \dfrac{\pi}{2}, \operatorname{Im} z > 0\right\}$ 一一地映为上半平面.

证明 由于 $\sin z = \cos\left(z - \dfrac{\pi}{2}\right)$, 只需考虑 $\{z \in \mathbb{C} : -\pi < \operatorname{Re} z < 0, \operatorname{Im} z > 0\}$ 在函数 $w = \cos z$ 下的像. 记 $\mu(z) = \mathrm{i}z, \eta(z) = \mathrm{e}^z, \varphi(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, 则 $w = \varphi \circ \eta \circ \mu$, 且有

$$\begin{array}{ccc} \{z \in \mathbb{C} : -\pi < \operatorname{Re} z < 0, \operatorname{Im} z > 0\} & \xrightarrow[1:1]{\mu} & \{z \in \mathbb{C} : \operatorname{Re} z < 0, -\pi < \operatorname{Im} z < 0\} \\ & & \Big\downarrow {\scriptstyle 1:1}\ \eta \\ \{z \in \mathbb{C} : \operatorname{Im} z > 0\} & \xleftarrow[1:1]{\varphi} & \{z \in \mathbb{C} : |z| < 1 \text{ 且 } \operatorname{Im} z < 0\} \end{array}$$

其中第一个箭头为双射是显然的, 第二个箭头为双射可由 $\{z \in \mathbb{C} : \operatorname{Re} z < 0, -\pi < \operatorname{Im} z < 0\}$ 是 η 的单叶域得到, 第三个箭头为双射证明如下: 由习题 2.4.15, 无心单位圆盘是 φ 的单叶域, 进而

$\{z\in\mathbb{C}:\operatorname{Re}z<0,-\pi<\operatorname{Im}z<0\}$ 也是 φ 的单叶域; 再由习题 2.4.16 中的讨论可见, φ 将单位圆盘内部的半径为 r_0 下半圆周映为半长轴长 $\frac{1}{2}\left(r_0+\frac{1}{r_0}\right)$、半短轴长 $\frac{1}{2}\left|r_0-\frac{1}{r_0}\right|$ 的上半椭圆, 因此

$$\{z\in\mathbb{C}:\operatorname{Re}z<0,-\pi<\operatorname{Im}z<0\}\xrightarrow{\varphi}\{z\in\mathbb{C}:\operatorname{Im}z>0\},$$

且这是双射. □

习题 2.4.21 当 z 按逆时针方向沿圆周 $\{z\in\mathbb{C}:|z|=2\}$ 旋转一圈后, 计算下列函数辐角的增量:

(1) $(z-1)^{\frac{1}{2}}$.

(2) $\left(1+z^4\right)^{\frac{1}{3}}$.

(3) $\left(z^2+2z-3\right)^{\frac{1}{4}}$.

(4) $\left(\dfrac{z-1}{z+1}\right)^{\frac{1}{2}}$.

(5) $\left(\dfrac{z^2-1}{z^2+5}\right)^{\frac{1}{7}}$.

解答 记 $C=\{z\in\mathbb{C}:|z|=2\}$. 对有理函数 $R(z)=\prod\limits_{j=1}^{m}(z-a_j)^{n_j}$ 与 $F(z)=R(z)^{\frac{1}{n}}$, 若 $C\cap\{a_j\}_{j=1}^m=\varnothing$, 记 $\Lambda=\{j:a_j$ 在 C 内部$\}$, 则有

$$\Delta_C\operatorname{Arg}R(z)=\sum_{j=1}^{m}n_j\Delta_C\operatorname{Arg}(z-a_j)=2\pi\sum_{j\in\Lambda}n_j\implies\Delta_C\operatorname{Arg}F(z)=\frac{2\pi}{n}\sum_{j\in\Lambda}n_j.$$

(1) 由于 1 在 C 的内部, $\Delta_C\operatorname{Arg}(z-1)^{\frac{1}{2}}=\dfrac{2\pi}{2}\cdot1=\pi$.

(2) 由 $1+z^4$ 的根 z 均满足 $|z|^4=|-1|^4=1$, 其 4 个根均位于 C 的内部, $\Delta_C\left(1+z^4\right)^{\frac{1}{3}}=\dfrac{2\pi}{3}\cdot4=\dfrac{8\pi}{3}$.

(3) 由于 $z^2+2z-3=(z+3)(z-1)$, 1 在 C 的内部, 而 -3 在 C 的外部, $\Delta_C\left(z^2+2z-3\right)^{\frac{1}{4}}=\dfrac{2\pi}{4}\cdot1=\dfrac{\pi}{2}$.

(4) 由于 ±1 均在 C 的内部, $\Delta_C\left(\dfrac{z-1}{z+1}\right)^{\frac{1}{2}}=\dfrac{2\pi}{2}\cdot(1-1)=0$.

(5) 由于 ±1 在 C 的内部, 而 $\pm\sqrt{5}\mathrm{i}$ 在 C 的外部, $\Delta_C\left(\dfrac{z^2-1}{z^2+5}\right)^{\frac{1}{7}}=\dfrac{2\pi}{7}\cdot2=\dfrac{4\pi}{7}$. □

习题 2.4.22 设 $f(z)=\dfrac{z^{p-1}}{(1-z)^p}$, $0<p<1$. 证明: f 能在域 $D=\mathbb{C}\setminus[0,1]$ 上选出单值的全纯分支.

证明 由于 $f(z)=\dfrac{1}{z}\left(\dfrac{z}{1-z}\right)^p=\dfrac{1}{z}\exp\left(p\operatorname{Log}\dfrac{z}{1-z}\right)$, 只需证 $\operatorname{Log}\dfrac{z}{1-z}$ 能在 $D=\mathbb{C}\setminus[0,1]$ 上选出单值的全纯分支. 当 $z\notin[0,1]$ 时, $\dfrac{z}{1-z}\notin[0,+\infty)$, 而 $\operatorname{Log}z$ 在 $\mathbb{C}\setminus[0,+\infty)$ 上可选出单值全纯分支, 得证. □

习题 2.4.26 设 D 是 z 平面上去掉线段 $[-1,\mathrm{i}],[1,\mathrm{i}]$ 和射线 $z=\mathrm{i}t\ (1\leqslant t<+\infty)$ 后所得的域, 证明函数 $\operatorname{Log}\left(1-z^2\right)$ 能在 D 上分出单值全纯分支. 设 f 是满足 $f(0)=0$ 的那个分支, 试计算 $f(2)$ 的值.

证明 对任意不经过 ±1 的简单闭曲线,

$$\Delta_C \operatorname{Log}\left(1-z^2\right)=\Delta_C \operatorname{Log}(1+z)+\Delta_C \operatorname{Log}(1-z).$$

- 若 C 仅包含点 1 且沿逆时针方向, 则 $\Delta_C \operatorname{Log}\left(1-z^2\right)=\Delta_C \operatorname{Log}(1-z)=\mathrm{i}\Delta_C \operatorname{Arg}(1-z)=2\pi\mathrm{i}$.
- 若 C 仅包含点 -1 且沿逆时针方向, 则 $\Delta_C \operatorname{Log}\left(1-z^2\right)=\Delta_C \operatorname{Log}(1+z)=\mathrm{i}\Delta_C \operatorname{Arg}(1+z)=2\pi\mathrm{i}$.
- 若 C 同时包含 ±1 且沿逆时针方向, 则 $\Delta_C \operatorname{Log}\left(1-z^2\right)=\Delta_C \operatorname{Log}(1-z)+\Delta_C \operatorname{Log}(1+z)=4\pi\mathrm{i}$.
- 若 C 不包含 ±1, 则 $\Delta_C \operatorname{Log}\left(1-z^2\right)=0$.

由于 D 中任一简单闭曲线无法仅包含 1 或 -1, 也无法同时包含 ±1, 由上述讨论即知 $\operatorname{Log}\left(1-z^2\right)$ 能在 D 上分出单值全纯分支. 对于满足 $f(0)=0$ 的分支 f, 当 z 沿 D 中简单曲线从 0 变动到 2 时,

$$\begin{aligned}f(2)-f(0)=\Delta_\gamma \operatorname{Log}\left(1-z^2\right)&=\left(\log\left|1-2^2\right|-\log 1\right)+\mathrm{i}[\Delta_\gamma \operatorname{Arg}(1+z)+\Delta_\gamma \operatorname{Arg}(1-z)]\\&=\mathrm{i}(0+\pi)=\log 3+\pi\mathrm{i}.\end{aligned}$$

故 $f(2)=\log 3+\pi\mathrm{i}$. □

习题 2.4.27 证明函数 $\sqrt[4]{(1-z)^3(1+z)}$ 能在 $\mathbb{C}\setminus[-1,1]$ 上选出一个单值全纯分支 f, 满足 $f(\mathrm{i})=\sqrt{2}\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}$. 试计算 $f(-\mathrm{i})$ 的值.

证明 承接习题 2.4.21 解答开头的讨论, 我们还有

$$\Delta_C F(z)=|R(z_0)|^{\frac{1}{n}}\mathrm{e}^{\frac{\mathrm{i}}{n}\operatorname{Arg}R(z_0)}\left[\mathrm{e}^{\frac{\mathrm{i}}{n}\Delta_C \operatorname{Arg}R(z)}-1\right],$$

其中 z_0 为环绕曲线 C 时的起点. 因此

$$\Delta_C F(z)=0 \iff \mathrm{e}^{\frac{\mathrm{i}}{n}\Delta_C \operatorname{Arg}R(z)}=1 \iff \Delta_C \operatorname{Arg}R(z)=2kn\pi \iff \sum_{j\in\Lambda}n_j=kn,\quad k\in\mathbb{Z}.$$

本题中, 对于 $R(z)=(1-z)^3(1+z)$ 与 $F(z)=\left[(1-z)^3(1+z)\right]^{\frac{1}{4}}$,

- 由于 3 不是 4 的整数倍, 因此 1 是 $F(z)$ 的枝点.
- 由于 1 不是 4 的整数倍, 因此 -1 是 $F(z)$ 的枝点.
- 由于 $3+1=4$ 是 4 的整数倍, 因此 ∞ 不是 $F(z)$ 的枝点.

因此对 $\mathbb{C}\setminus[-1,1]$ 上的任一简单闭曲线 γ, 要么 γ 同时包含 ±1 两点, 要么 γ 不包含 ±1 两点, 在这两种情况下均有 $\Delta_\gamma F(z)=0$. 又 $(1-\mathrm{i})^3(1+\mathrm{i})=-4\mathrm{i}=\left(\sqrt{2}\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}\right)^4$, 于是能在 $\mathbb{C}\setminus[-1,1]$ 上选出一个满足 $f(\mathrm{i})=\sqrt{2}\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}$ 的单值全纯分支 f. 现取 E 为以 ±1 为焦点、$\pm\mathrm{i}$ 为上下顶点的椭圆的左半部分, 则

$$\begin{gathered}f(-\mathrm{i})-f(\mathrm{i})=\Delta_E F(z)=\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}}{4}\operatorname{Arg}R(\mathrm{i})}\left(\mathrm{e}^{\frac{\mathrm{i}}{4}\Delta_E \operatorname{Arg}R(z)}-1\right),\\ \Delta_E \operatorname{Arg}R(z)=\frac{\pi}{2}\cdot 3+\frac{3\pi}{2}=3\pi.\end{gathered}$$

因此

$$f(-\mathrm{i})=\sqrt{2}\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}+\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}}{4}\operatorname{Arg}(-4\mathrm{i})}\left(\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}}-1\right)=\sqrt{2}\mathrm{e}^{\frac{5\pi}{8}\mathrm{i}}.$$ □

习题 2.5.2 求出把上半平面映为单位圆盘的分式线性变换, 使得 $-1, 0, 1$ 分别映为 $1, \mathrm{i}, -1$.

解答 设所求的分式线性变换将 z 映为 w, 则 $\dfrac{z-0}{z-1} : \dfrac{-1-0}{-1-1} = \dfrac{w-\mathrm{i}}{w-(-1)} : \dfrac{1-\mathrm{i}}{1-(-1)}$, 解得 $w = \dfrac{z-\mathrm{i}}{\mathrm{i}z-1}$.
检验: 对 $x \in \mathbb{R}$ 与 $y > 0$, 有 $\left|\dfrac{(x+\mathrm{i}y)-\mathrm{i}}{\mathrm{i}(x+\mathrm{i}y)-1}\right|^2 = \dfrac{x^2+(y-1)^2}{x^2+(y+1)^2} < 1$. □

习题 2.5.3 设 $a, b, c, d \in \mathbb{R}$, 则分式线性变换 $w = \dfrac{az+b}{cz+d}$ 把上半平面映为上半平面 $\iff ad - bc > 0$.

证明 由于 $a, b, c, d \in \mathbb{R}$, $w = \dfrac{az+b}{cz+d}$ 必将 $\mathbb{R}$ 映为 $\mathbb{R}$. 又欲证两边均蕴含 $ad - bc \neq 0$, 故不妨假设之.

(⇒) 若 $ad - bc < 0$, 则 $w' = \dfrac{ad-bc}{(cz+d)^2} < 0$. 当 z 在 $\mathbb{R}$ 上由 $-\infty$ 趋向 $+\infty$ 时, w 由 $+\infty$ 趋向 $-\infty$, 根据全纯函数的保角性, w 把上半平面映为下半平面, 矛盾. 故 $ad - bc > 0$.

(⇐) 由 $w' = \dfrac{ad-bc}{(cz+d)^2} > 0$, 当 z 在 $\mathbb{R}$ 上由 $-\infty$ 趋向 $+\infty$ 时, w 也由 $-\infty$ 趋向 $+\infty$, 根据全纯函数的保角性, w 把上半平面映为上半平面. □

习题 2.5.4 试求把单位圆盘的外部 $\{z : |z| > 1\}$ 映为右半平面 $\{w : \operatorname{Re} w > 0\}$ 的分式线性变换, 使得

(1) $1, -\mathrm{i}, -1$ 分别变为 $\mathrm{i}, 0, -\mathrm{i}$.

(2) $-\mathrm{i}, \mathrm{i}, 1$ 分别变为 $\mathrm{i}, 0, -\mathrm{i}$.

证明 设所求的分式线性变换将 z 映为 w.

(1) $\dfrac{z-(-\mathrm{i})}{z-(-1)} : \dfrac{1-(-\mathrm{i})}{1-(-1)} = \dfrac{w-0}{w-(-\mathrm{i})} : \dfrac{\mathrm{i}-0}{\mathrm{i}-(-\mathrm{i})} \implies w = \dfrac{z+\mathrm{i}}{z-\mathrm{i}}$. 检验: 对满足 $x^2+y^2 > 1$ 的 $x, y \in \mathbb{R}$, 有 $\operatorname{Re}\dfrac{(x+\mathrm{i}y)+\mathrm{i}}{(x+\mathrm{i}y)-\mathrm{i}} = \dfrac{x^2+y^2-1}{x^2+(y-1)^2} > 0$.

(2) $\dfrac{z-\mathrm{i}}{z-1} : \dfrac{-\mathrm{i}-\mathrm{i}}{-\mathrm{i}-1} = \dfrac{w-0}{w-(-\mathrm{i})} : \dfrac{\mathrm{i}-0}{\mathrm{i}-(-\mathrm{i})} \implies w = \dfrac{z-\mathrm{i}}{(2-\mathrm{i})z+(2\mathrm{i}-1)}$. 检验: 对满足 $x^2+y^2 > 1$ 的 $x, y \in \mathbb{R}$, 有 $\operatorname{Re}\dfrac{(x+\mathrm{i}y)-\mathrm{i}}{(2-\mathrm{i})(x+\mathrm{i}y)+(2\mathrm{i}-1)} = \dfrac{2\big(x^2+y^2-1\big)}{(2x+y-1)^2+(2y-x+2)^2} > 0$. □

习题 2.5.9 证明: z_1, z_2 关于圆周

$$az\bar{z} + \bar{\beta}z + \beta\bar{z} + d = 0$$

对称的充要条件是

$$az_1\overline{z_2} + \bar{\beta}z_1 + \beta\overline{z_2} + d = 0.$$

证明 **(直线)** 此时 $a = 0$. 若 z_1, z_2 关于所给直线对称, 则 $z_2 - z_1 \perp \mathrm{i}\beta$, 即 $\operatorname{Re}(\mathrm{i}\beta\overline{z_2 - z_1}) = 0$, 展开得

$$\mathrm{i}\beta(\overline{z_2} - \overline{z_1}) - \mathrm{i}\bar{\beta}(z_2 - z_1) = 0 \iff \beta\overline{z_2} + \bar{\beta}z_1 = \beta\overline{z_1} + \bar{\beta}z_2.$$

而 $\dfrac{z_1+z_2}{2}$ 满足所给直线方程:

$$\bar{\beta}\frac{z_1+z_2}{2} + \beta\frac{\overline{z_1}+\overline{z_2}}{2} + d = 0.$$

联立以上两式即得

$$\beta\overline{z_1} + \bar{\beta}z_2 + d = 0.$$

反之, 若 z_1, z_2 满足上式, 对上式取共轭得

$$\bar{\beta} z_1 + \beta \overline{z_2} + d = 0,$$

两式相加得

$$\bar{\beta} \frac{z_1 + z_2}{2} + \beta \frac{\overline{z_1} + \overline{z_2}}{2} + d = 0,$$

两式作差得

$$\beta(\overline{z_2} - \overline{z_1}) - \bar{\beta}(z_2 - z_1) = 0,$$

再乘 $\frac{\mathrm{i}}{2}$ 即得 $\operatorname{Re}(\mathrm{i}\beta\overline{z_2 - z_1}) = 0$. 故 z_1, z_2 关于此直线对称.

(圆周) 记圆周的圆心为 z_0、半径为 R. 若 z_1, z_2 关于此圆周对称, 则

$$z_2 - z_0 = \frac{R^2}{\overline{z_1} - \overline{z_0}}.$$

这是因为对上式取模与辐角可得

$$\begin{cases} |z_2 - z_0||z_1 - z_0| = R^2, \\ \operatorname{Arg}(z_2 - z_0) = -\operatorname{Arg}(\overline{z_1} - \overline{z_0}) = \operatorname{Arg}(z_1 - z_0). \end{cases}$$

代入所给圆周方程的等价形式

$$\left| z + \frac{\beta}{a} \right| = \frac{\sqrt{|\beta|^2 - ad}}{|a|}$$

即得

$$z_2 + \frac{\beta}{a} = \frac{\frac{|\beta|^2 - ad}{a^2}}{\overline{z_1} + \frac{\bar{\beta}}{a}},$$

化简即

$$a z_1 \overline{z_2} + \bar{\beta} z_1 + \beta \overline{z_2} + d = 0.$$

反之, 若 z_1, z_2 满足上式, 将上述过程反向即得 z_1, z_2 关于所给圆周对称. □

习题 2.5.10 设 $T(z) = \frac{az + b}{cz + d}$ 是一个分式线性变换, 如果记

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

那么

$$T^{-1}(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

证明 $T^{-1}(z) = \frac{-dz + b}{cz - a}$, 而由题, $\begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, 因此

$$(c\alpha + d\gamma)z^2 + (c\beta + d\delta - a\alpha - b\gamma)z - (a\beta + b\delta) = 0 \iff \frac{-dz + b}{cz - a} = \frac{\alpha z + \beta}{\gamma z + \delta}.$$ □

习题 2.5.11 设 $T_1(z)=\dfrac{a_1z+b_1}{c_1z+d_1}$, $T_2(z)=\dfrac{a_2z+b_2}{c_2z+d_2}$ 是两个分式线性变换, 如果记

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

那么

$$(T_1\circ T_2)(z)=\frac{az+b}{cz+d}.$$

证明 $(T_1\circ T_2)(z)=\dfrac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(a_2c_1+c_2d_1)z+(b_2c_1+d_1d_2)}$, 而由题, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ a_2c_1+c_2d_1 & b_2c_1+d_1d_2 \end{pmatrix}$, 于是 $(T_1\circ T_2)(z)=\dfrac{az+b}{cz+d}$. □

习题 2.5.16 求一单叶全纯映射, 把半条形域 $\left\{z:-\dfrac{\pi}{2}<\operatorname{Re}z<\dfrac{\pi}{2},\operatorname{Im}z>0\right\}$ 映为上半平面, 且把 $\dfrac{\pi}{2},-\dfrac{\pi}{2},0$ 分别映为 $1,-1,0$.

解答 由习题 2.4.19 知 $w=\sin z$ 满足题意. 亦可如下分解求之, 复合结果仍为 $\sin z$.

$$\left\{z\in\mathbb{C}:-\frac{\pi}{2}<\operatorname{Re}z<\frac{\pi}{2},\operatorname{Im}z>0\right\}\xrightarrow{z\mapsto \mathrm{i}z}\left\{z\in\mathbb{C}:-\frac{\pi}{2}<\operatorname{Im}z<\frac{\pi}{2},\operatorname{Re}z<0\right\}$$

$$\xrightarrow{z\mapsto \mathrm{e}^z}\{z\in\mathbb{C}:|z|<1,\operatorname{Re}z>0\}\xrightarrow{z\mapsto \mathrm{i}z}\{z\in\mathbb{C}:|z|<1,\operatorname{Im}z>0\}$$

$$\xrightarrow[(1)]{z\mapsto\frac{z+1}{z-1}}\{z\in\mathbb{C}:\operatorname{Re}z<0,\operatorname{Im}z<0\}\xrightarrow{z\mapsto z^2}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\xrightarrow[(2)]{z\mapsto-\frac{z+1}{z-1}}\{z\in\mathbb{C}:\operatorname{Im}z>0\}$$

其中用到的两个分式线性变换如下:

(1) $w_1(z)=\dfrac{z+1}{z-1}$ 将上半单位圆盘映为第三象限 (由于二者在 Riemann 球上为全等的新月形, 结合保圆性及保角性可知这样的分式线性变换的确存在), 且使 $-1\mapsto 0, 1\mapsto\infty, \mathrm{i}\mapsto-\mathrm{i}$.

(2) $w_2(z)=-\dfrac{z+1}{z-1}$ 将上半平面映为上半平面, 且使 $0\mapsto 1,\infty\mapsto -1,-1\mapsto 0$. □

习题 2.5.17 求一单叶全纯映射, 把除去线段 $[a,a+h\mathrm{i}]$ 的条形域 $\{z:0<\operatorname{Im}z<1\}$ 映为条形域 $\{w:0<\operatorname{Im}w<1\}$, 其中 $a\in\mathbb{R}$, $0<h<1$.

解答 分解如下:

$$\{z:0<\operatorname{Im}z<1\}\setminus[a,a+h\mathrm{i}]\xrightarrow{z\mapsto\pi(z+a)}\{z\in\mathbb{C}:0<\operatorname{Im}z<\pi\}\setminus[0,h\pi\mathrm{i}]$$

$$\xrightarrow{z\mapsto\mathrm{e}^z}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\setminus\{z\in\mathbb{C}:|z|=1,0\leqslant\arg z\leqslant h\pi\}$$

$$\xrightarrow{z\mapsto\frac{z-1}{z+1}}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\setminus\left[0,\tfrac{1-\cos(h\pi)}{\sin(h\pi)}\mathrm{i}\right]$$

$$\xrightarrow{z\mapsto z^2}\mathbb{C}\setminus\left[-\left(\tfrac{1-\cos(h\pi)}{\sin(h\pi)}\right)^2,+\infty\right)\xrightarrow{z\mapsto z+\left(\frac{1-\cos(h\pi)}{\sin(h\pi)}\right)^2}\mathbb{C}\setminus[0,+\infty)$$

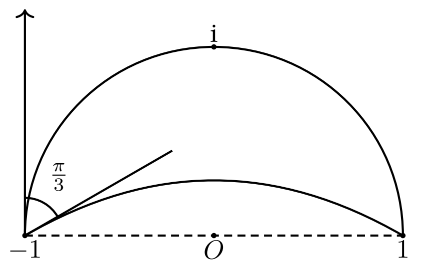
$$\xrightarrow{z\mapsto\sqrt{z}}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\xrightarrow{z\mapsto\frac{1}{\pi}\log z}\{z\in\mathbb{C}:0<\operatorname{Im}z<1\}$$

复合结果为

$$w=\frac{1}{2\pi}\log\left[\left(\frac{\mathrm{e}^{\pi(z+a)}-1}{\mathrm{e}^{\pi(z+a)}+1}\right)^2+\left(\frac{1-\cos(h\pi)}{\sin(h\pi)}\right)^2\right].$$

□

习题 2.5.18 求一单叶全纯映射, 把图示的月牙形域映为 $\mathbb{B}(0,1)$.



题 2.5.18 图

解答 记图示月牙形域为 D, 则有

$$D\xrightarrow{z\mapsto\frac{z+1}{z-1}}\left\{z\in\mathbb{C}:\tfrac{7\pi}{6}<\arg z<\tfrac{3\pi}{2}\right\}\xrightarrow{z\mapsto\log z}\left\{z\in\mathbb{C}:\tfrac{7\pi}{6}<\operatorname{Im}z<\tfrac{3\pi}{2}\right\}$$

$$\xrightarrow{z\mapsto 6z-7\pi}\{z\in\mathbb{C}:0<\operatorname{Im}z<2\pi\}\xrightarrow{z\mapsto\mathrm{e}^z}\mathbb{C}\setminus[0,+\infty)$$

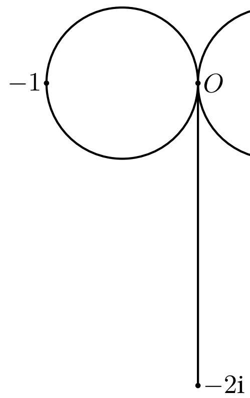
$$\xrightarrow{z\mapsto\sqrt{z}}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\xrightarrow{z\mapsto\frac{z-\mathrm{i}}{z+\mathrm{i}}}\mathbb{B}(0,1)$$

第一个箭头 $z\mapsto\dfrac{z+1}{z-1}$ 将两圆弧映为共起点的两射线, 注意到当 z 在 $\mathbb{R}$ 上由 -1 到 1 时, $w=1+\dfrac{2}{z-1}$ 在 $\mathbb{R}$ 上由 0 到 $-\infty$, 因此由保角性可确定负半实轴到两射线的角度分别为 $\dfrac{\pi}{6}$ 和 $\dfrac{\pi}{2}$. 复合结果为

$$w=\frac{\sqrt{\mathrm{e}^{6\log\frac{z+1}{z-1}-7\pi}}-\mathrm{i}}{\sqrt{\mathrm{e}^{6\log\frac{z+1}{z-1}-7\pi}}+\mathrm{i}}=\frac{(z+1)^3-\mathrm{i}\mathrm{e}^{\frac{7\pi}{2}}(z-1)^3}{(z+1)^3+\mathrm{i}\mathrm{e}^{\frac{7\pi}{2}}(z-1)^3}.$$

□

习题 2.5.20 求一单叶全纯映射, 把图示 $\mathbb{B}\left(-\frac{1}{2}, \frac{1}{2}\right)$ 和 $\mathbb{B}\left(\frac{1}{2}, \frac{1}{2}\right)$ 的外部除去线段 $[-2\mathrm{i}, 0]$ 所成的域映为上半平面.



题 2.5.20 图

解答 记图示区域为 D, 则有

$$D \xrightarrow{z\mapsto \frac{1}{z}} \{z\in\mathbb{C} : -1<\operatorname{Re} z<1\} \setminus \left\{z\in\mathbb{C} : \operatorname{Re} z = 0 \text{ 且 } \operatorname{Im} z \geqslant \tfrac{1}{2}\right\}$$

$$\xrightarrow{z\mapsto \pi\mathrm{i}z+\frac{\pi}{2}+\pi\mathrm{i}} \{z\in\mathbb{C} : 0<\operatorname{Im} z<2\pi\} \setminus \{z\in\mathbb{C} : \operatorname{Im} z = \pi \text{ 且 } \operatorname{Re} z \leqslant 0\}$$

$$\mathbb{C}\setminus[-1,+\infty) \xleftarrow{z\mapsto \mathrm{e}^z} \{z\in\mathbb{C} : 0<\operatorname{Im} z<2\pi\} \setminus \{z\in\mathbb{C} : \operatorname{Im} z = \pi \text{ 且 } \operatorname{Re} z \leqslant 0\}$$

$$\mathbb{C}\setminus[-1,+\infty) \xrightarrow{z\mapsto z+1} \mathbb{C}\setminus[0,+\infty) \xrightarrow{z\mapsto\sqrt{z}} \{z\in\mathbb{C} : \operatorname{Im} z>0\}$$

复合结果为

$$w = \sqrt{\mathrm{e}^{\frac{\pi\mathrm{i}}{z}+\frac{\pi}{2}+\pi\mathrm{i}}+1} = \sqrt{1-\mathrm{e}^{\frac{\pi\mathrm{i}}{z}+\frac{\pi}{2}}}.$$

□

习题 2.5.21 设 $0<r<a$, 求一单叶全纯映射, 把域 $\{z\in\mathbb{C} : \operatorname{Re} z>0, |z-a|>r\}$ 映为同心圆环 $\{w\in\mathbb{C} : \rho<|w|<1\}$.

解答 虚轴与圆周 $|z-a|=r$ 的公共对称点显然在实轴上, 设其为 $\pm x$ $(0<x<a)$, 则 $(a-x)(a+x)=r^2$, 解得 $x=\sqrt{a^2-r^2}$. 因此分式线性变换

$$w = k\cdot\frac{z+\sqrt{a^2-r^2}}{z-\sqrt{a^2-r^2}}, \quad k\in\mathbb{C}$$

将所给域映为同心于原点的圆环. 此时 $0\mapsto -k$, $a-r\mapsto -k\cdot\dfrac{a+\sqrt{a^2-r^2}}{r}$. 取

$$k = \frac{r}{a+\sqrt{a^2-r^2}} = \frac{a-\sqrt{a^2-r^2}}{r},$$

则 w 将所给域映为同心圆环 $\{w\in\mathbb{C} : \rho<|w|<1\}$, 其中 $\rho=k$. □

习题 3.1.2 计算积分 $\displaystyle\int_{|z|=1}\frac{\mathrm{d}z}{z+2}$, 并证明 $\displaystyle\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta = 0$.

解答 由于 $\frac{1}{z+2}$ 在 $\mathbb{B}(0,1)$ 上全纯, 在 $\overline{\mathbb{B}(0,1)}$ 上连续, $\int\limits_{|z|=1} \frac{\mathrm{d}z}{z+2} = 0$. 另一方面, 由

$$0 = \int\limits_{|z|=1} \frac{\mathrm{d}z}{z+2} = \int_0^{2\pi} \frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}+2} = \mathrm{i}\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}+2}\,\mathrm{d}\theta$$

可得

$$\begin{aligned} 0 &= \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}+2}\,\mathrm{d}\theta = \int_0^{2\pi} \frac{(\cos\theta+\mathrm{i}\sin\theta)(2+\cos\theta-\mathrm{i}\sin\theta)}{(2+\cos\theta+\mathrm{i}\sin\theta)(2+\cos\theta-\mathrm{i}\sin\theta)}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{2\cos\theta+1+2\mathrm{i}\sin\theta}{5+4\cos\theta}\,\mathrm{d}\theta = \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta + 2\mathrm{i}\int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta}\,\mathrm{d}\theta, \end{aligned}$$

而

$$\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta = 2\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}, \quad \int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta}\,\mathrm{d}\theta = 0,$$

因此

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta = 0.$$

□

习题 3.1.4 如果多项式 $Q(z)$ 比多项式 $P(z)$ 高两次, 试证:

$$\lim_{R\to\infty} \int\limits_{|z|=R} \frac{P(z)}{Q(z)}\,\mathrm{d}z = 0.$$

证明 设 $\lim\limits_{|z|\to\infty}\left|\frac{z^2P(z)}{Q(z)}\right| = M$, 则存在 $R_0>0$, 使得当 $R>R_0$ 时, $\left|\frac{z^2P(z)}{Q(z)}\right| \leqslant 2M$, 此时

$$\left|\int\limits_{|z|=R} \frac{P(z)}{Q(z)}\,\mathrm{d}z\right| \leqslant \int\limits_{|z|=R} \left|\frac{P(z)}{Q(z)}\right||\mathrm{d}z| \leqslant \int\limits_{|z|=R} \frac{2M}{|z|^2}|\mathrm{d}z| = \frac{4\pi M}{R} \to 0, \quad R\to\infty.$$

□

习题 3.1.5 计算积分 $\int\limits_{|z|=r} z^n\bar{z}^k\,\mathrm{d}z$, 其中 $n,k\in\mathbb{Z}$.

解答 $\int\limits_{|z|=r} z^n\bar{z}^k\,\mathrm{d}z = \int_0^{2\pi} \left(r\mathrm{e}^{\mathrm{i}\theta}\right)^n\left(r\mathrm{e}^{-\mathrm{i}\theta}\right)^k \mathrm{d}r\mathrm{e}^{\mathrm{i}\theta} = \mathrm{i}r^{n+k+1}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-k+1)\theta}\,\mathrm{d}\theta = \begin{cases} 0, & n+1\neq k, \\ 2\pi\mathrm{i}r^{n+k+1}, & n+1=k. \end{cases}$ □

习题 3.2.1 计算积分:

(1) $\int\limits_{|z|=r} \frac{|\mathrm{d}z|}{|z-a|^2}, |a|\neq r.$

(2) $\int\limits_{|z|=2} \frac{2z-1}{z(z-1)}\,\mathrm{d}z.$

(3) $\int\limits_{|z|=5} \frac{z\,\mathrm{d}z}{z^4-1}.$

(4) $\displaystyle\int\limits_{|z|=2a} \frac{\mathrm{e}^z}{z^2+a^2}\,\mathrm{d}z, a>0.$

解答 (1) $\displaystyle\int\limits_{|z|=r} \frac{|\mathrm{d}z|}{|z-a|^2} = \int_0^{2\pi} \frac{r\,\mathrm{d}\theta}{|r\mathrm{e}^{\mathrm{i}\theta}-a|^2} = \int_0^{2\pi} \frac{r\,\mathrm{d}\theta}{(r\mathrm{e}^{\mathrm{i}\theta}-a)(r\mathrm{e}^{-\mathrm{i}\theta}-\bar{a})} = \int_0^{2\pi} \frac{r^2\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{(r\mathrm{e}^{\mathrm{i}\theta}-a)(r^2-\bar{a}r\mathrm{e}^{\mathrm{i}\theta})} =$
$\displaystyle\frac{r}{\mathrm{i}}\int\limits_{|z|=r} \frac{\mathrm{d}z}{(z-a)(r^2-\bar{a}z)} = \frac{r}{\mathrm{i}(r^2-|a|^2)}\int\limits_{|z|=r}\left(\frac{1}{z-a}+\frac{1}{\frac{r^2}{\bar{a}}-z}\right)\mathrm{d}z.$

◇ 若 $|a|<r$, 则 $\displaystyle\int\limits_{|z|=r}\frac{\mathrm{d}z}{z-a}=2\pi\mathrm{i}, \int\limits_{|z|=r}\frac{\mathrm{d}z}{\frac{r^2}{\bar{a}}-z}=0.$

◇ 若 $|a|>r$, 则 $\displaystyle\int\limits_{|z|=r}\frac{\mathrm{d}z}{z-a}=0, \int\limits_{|z|=r}\frac{\mathrm{d}z}{\frac{r^2}{\bar{a}}-z}=-2\pi\mathrm{i}.$

故 $\displaystyle\int\limits_{|z|=r}\frac{|\mathrm{d}z|}{|z-a|^2}=\frac{2\pi r}{|r^2-|a|^2|}.$

(2) $\displaystyle\int\limits_{|z|=2}\frac{2z-1}{z(z-1)}\,\mathrm{d}z = \int\limits_{|z|=2}\left(\frac{1}{z}+\frac{1}{z-1}\right)\mathrm{d}z = 4\pi\mathrm{i}.$

(3) $\displaystyle\int\limits_{|z|=5}\frac{z\,\mathrm{d}z}{z^4-1} = \frac{1}{2}\int\limits_{|z|=5}\frac{\mathrm{d}z^2}{(z^2-1)(z^2+1)} = \frac{1}{4}\int\limits_{|z|=5}\left(\frac{1}{z^2-1}-\frac{1}{z^2+1}\right)\mathrm{d}z^2 = 0.$

(4) 由 Cauchy 积分公式,

$$\int\limits_{|z|=2a}\frac{\mathrm{e}^z}{z^2+a^2}\,\mathrm{d}z = \frac{1}{2a\mathrm{i}}\int\limits_{|z|=2a}\left(\frac{\mathrm{e}^z}{z-a\mathrm{i}}-\frac{\mathrm{e}^z}{z+a\mathrm{i}}\right)\mathrm{d}z = \frac{1}{2a\mathrm{i}}\left(2\pi\mathrm{i}\mathrm{e}^{a\mathrm{i}}-2\pi\mathrm{i}\mathrm{e}^{-a\mathrm{i}}\right) = \frac{2\pi\mathrm{i}\sin a}{a}.$$

□

习题 3.2.2 设 f 在 $\{z: r<|z|<\infty\}$ 中全纯, 且 $\lim\limits_{z\to\infty} zf(z)=A$. 证明:

$$\int\limits_{|z|=R} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}A,$$

其中 $R>r$.

证明 对于 $R'>R$, 有

$$\left|\int\limits_{|z|=R} f(z)\,\mathrm{d}z - 2\pi\mathrm{i}A\right| = \left|\int\limits_{|z|=R'}\left(f(z)\,\mathrm{d}z-\frac{A}{z}\right)\mathrm{d}z\right| \leqslant \int\limits_{|z|=R'}\frac{|zf(z)-A|}{R'}|\mathrm{d}z|$$
$$\leqslant 2\pi\cdot\sup_{|z|=R'}|zf(z)-A|\to 0,\quad R'\to\infty.$$

□

习题 3.4.1 计算下列积分:

(1) $\displaystyle\int\limits_{|z-1|=1}\frac{\sin z}{z^2-1}\,\mathrm{d}z.$

(2) $\displaystyle\int\limits_{|z|=2} \frac{\mathrm{d}z}{1+z^2}$.

(3) $\displaystyle\int\limits_{4x^2+y^2=2y} \frac{\mathrm{e}^{\pi z}}{(1+z^2)^2}\,\mathrm{d}z$.

(4) $\displaystyle\int\limits_{|z|=\frac{3}{2}} \frac{\mathrm{d}z}{(z^2+1)(z^2+4)}$.

(5) $\displaystyle\int\limits_{|z|=2} \frac{\mathrm{d}z}{z^3(z-1)^3(z-3)^5}$.

(6) $\displaystyle\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)}$, 其中 n 为正整数, a,b 不在圆周 $|z|=R$ 上.

解答 (1) $\displaystyle\int\limits_{|z-1|=1} \frac{\sin z}{z^2-1}\,\mathrm{d}z = \int\limits_{|z-1|=1} \frac{\frac{\sin z}{z+1}}{z-1}\,\mathrm{d}z = 2\pi\mathrm{i}\cdot\frac{\sin z}{z+1}\Big|_{z=1} = \pi\mathrm{i}\sin 1.$

(2) 记 $\varepsilon=\dfrac{1}{2}, \gamma_1=\{z:|z-\mathrm{i}|=\varepsilon\}, \gamma_2=\{z:|z+\mathrm{i}|=\varepsilon\}$, 则

$$\int\limits_{|z|=2} \frac{\mathrm{d}z}{1+z^2} = \int\limits_{\gamma_1} \frac{\frac{\mathrm{d}z}{z+\mathrm{i}}}{z-\mathrm{i}} + \int\limits_{\gamma_2} \frac{\frac{\mathrm{d}z}{z-\mathrm{i}}}{z-(-\mathrm{i})} = 2\pi\mathrm{i}\left(\frac{1}{\mathrm{i}+\mathrm{i}}+\frac{1}{-\mathrm{i}-\mathrm{i}}\right) = 0.$$

(3) 记 $E=\{(x,y):4x^2+y^2=2y\}=\{(x,y):4x^2+(y-1)^2=1\}$, 则

$$\int\limits_{4x^2+y^2=2y} \frac{\mathrm{e}^{\pi z}}{(1+z^2)^2}\,\mathrm{d}z = \int\limits_{E} \frac{\frac{\mathrm{e}^{\pi z}}{(z+\mathrm{i})^2}\,\mathrm{d}z}{(z-\mathrm{i})^2} = \frac{2\pi\mathrm{i}}{1!}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{e}^{\pi z}}{(z+\mathrm{i})^2}\right)\Bigg|_{z=\mathrm{i}} = 2\pi\mathrm{i}\cdot\frac{\mathrm{e}^{\pi z}(\pi z+\pi\mathrm{i}-2)}{(z+\mathrm{i})^3}\Bigg|_{z=\mathrm{i}}$$
$$= \frac{\pi(\pi\mathrm{i}-1)}{2}.$$

(4) 记 $\varepsilon=\dfrac{1}{4}, \gamma_1=\{z:|z-\mathrm{i}|=\varepsilon\}, \gamma_2=\{z:|z+\mathrm{i}|=\varepsilon\}$, 则

$$\int\limits_{|z|=\frac{3}{2}} \frac{\mathrm{d}z}{(z^2+1)(z^2+4)} = \int\limits_{\gamma_1} \frac{\frac{\mathrm{d}z}{(z+\mathrm{i})(z^2+4)}}{z-\mathrm{i}} + \int\limits_{\gamma_2} \frac{\frac{\mathrm{d}z}{(z-\mathrm{i})(z^2+4)}}{z-(-\mathrm{i})} = 2\pi\mathrm{i}\left(\frac{1}{6\mathrm{i}}+\frac{1}{-6\mathrm{i}}\right) = 0.$$

(5) 记 $\varepsilon=\dfrac{1}{4}, \gamma_1=\{z:|z|=\varepsilon\}, \gamma_2=\{z:|z-1|=\varepsilon\}$, 则

$$\int\limits_{|z|=2} \frac{\mathrm{d}z}{z^3(z-1)^3(z-3)^5} = \int\limits_{\gamma_1} \frac{\frac{\mathrm{d}z}{(z-1)^3(z-3)^5}}{(z-0)^3} + \int\limits_{\gamma_2} \frac{\frac{\mathrm{d}z}{z^3(z-3)^5}}{(z-1)^3}$$
$$= \frac{2\pi\mathrm{i}}{2!}\cdot\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{(z-1)^3(z-3)^5}\right)\Bigg|_{z=0} + \frac{2\pi\mathrm{i}}{2!}\cdot\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{z^3(z-3)^5}\right)\Bigg|_{z=1}$$
$$= \pi\mathrm{i}\left(\frac{76}{3^6}-\frac{9}{2^6}\right).$$

(6) ① 若 a, b 均在圆周 $|z|=R$ 外, 则由 Cauchy 定理, $\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)} = 0$.

② 若 a 在圆周 $|z|=R$ 外, b 在圆周 $|z|=R$ 内, 记 $\varepsilon = \frac{R-|b|}{2}, \gamma = \{z : |z-b| = \varepsilon\}$, 则

$$\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)} = \int\limits_{\gamma} \frac{\frac{\mathrm{d}z}{(z-a)^n}}{z-b} = \frac{2\pi\mathrm{i}}{(b-a)^n}.$$

③ 若 a 在圆周 $|z|=R$ 内, b 在圆周 $|z|=R$ 外, 记 $\varepsilon = \frac{R-|a|}{2}, \gamma = \{z : |z-a| = \varepsilon\}$, 则

$$\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)} = \int\limits_{\gamma} \frac{\frac{\mathrm{d}z}{z-b}}{(z-a)^n} = \frac{2\pi\mathrm{i}}{(n-1)!} \cdot \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left(\frac{1}{z-b} \right) \Bigg|_{z=a} = -\frac{2\pi\mathrm{i}}{(b-a)^n}.$$

④ 若 a, b 均在圆周 $|z| = R$ 内, 记 $\gamma_1 = \{z : |z-a| = \varepsilon\}, \gamma_2 = \{z : |z-b| = \varepsilon\}$, 其中 $\varepsilon < \min\{R-|a|, R-|b|\}$ 充分小以使 γ_1, γ_2 各自所围区域不交. 于是

$$\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)} = 0 = \int\limits_{\gamma_1} \frac{\frac{\mathrm{d}z}{z-b}}{(z-a)^n} + \int\limits_{\gamma_2} \frac{\frac{\mathrm{d}z}{z-b}}{(z-a)^n} = ③ + ② = 0.$$

□

习题 3.4.4 称

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(z^2-1\right)^n$$

是 Legendre 多项式. 证明:

(1) Legendre 多项式有如下的积分表示:

$$P_n(z) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\gamma} \frac{\left(\zeta^2-1\right)^n}{2^n(\zeta-z)^{n+1}} \mathrm{d}\zeta,$$

其中 γ 是任意内部包含 z 的可求长简单闭曲线.

(2) 如果取

$$\gamma = \left\{ \zeta \in \mathbb{C} : |\zeta - x| = \sqrt{x^2-1} \right\} \quad (1 < x < +\infty),$$

那么有如下的 Laplace 公式:

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left(x + \sqrt{x^2-1} \cos\theta \right)^n \mathrm{d}\theta.$$

证明 (1) 由 Cauchy 积分公式,

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(z^2-1\right)^n = \frac{n!}{2\pi\mathrm{i}} \int\limits_{\gamma} \frac{\left(\zeta^2-1\right)^n}{(\zeta-z)^{n+1}} \mathrm{d}\zeta,$$

整理即得欲证积分表示.

(2) 由 (1) 所得积分表示,

$$P_n(x)=\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}\frac{\left(\zeta^2-1\right)^n}{2^n(\zeta-x)^{n+1}}\,\mathrm{d}\zeta=\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\left(x+\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}\right)^2-1}{2\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}}\right]^n\mathrm{d}\theta.$$

而

$$\int_{\pi}^{2\pi}\left[\frac{\left(x+\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}\right)^2-1}{2\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}}\right]^n\mathrm{d}\theta\xlongequal{\beta=2\pi-\theta}\int_0^{\pi}\left[\frac{\left(x+\sqrt{x^2-1}\mathrm{e}^{-\mathrm{i}\beta}\right)^2-1}{2\sqrt{x^2-1}\mathrm{e}^{-\mathrm{i}\beta}}\right]^n\mathrm{d}\beta,$$

因此

$$\begin{aligned}P_n(x)&=\frac{1}{\pi}\int_0^{\pi}\mathrm{Re}\left\{\left[\frac{\left(x+\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}\right)^2-1}{2\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}}\right]^n\right\}\mathrm{d}\theta\\&=\frac{1}{\pi}\int_0^{\pi}\mathrm{Re}\left\{\left[\frac{\left(x^2-1\right)+\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}\left(\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}+2x\right)}{2\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}}\right]^n\right\}\mathrm{d}\theta\\&=\frac{1}{\pi}\int_0^{\pi}\mathrm{Re}\left\{\left[\frac{\sqrt{x^2-1}\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)+2x}{2}\right]^n\right\}\mathrm{d}\theta\\&=\frac{1}{\pi}\int_0^{\pi}\left(x+\sqrt{x^2-1}\cos\theta\right)^n\mathrm{d}\theta.\end{aligned}$$

□

习题 3.4.5 设 $f\in\mathcal{H}(\mathbb{B}(0,1))\cap\mathcal{C}\left(\overline{\mathbb{B}(0,1)}\right)$. 证明:

(1) $\displaystyle\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\cos^2\left(\tfrac{\theta}{2}\right)\mathrm{d}\theta=2f(0)+f'(0).$

(2) $\displaystyle\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\sin^2\left(\tfrac{\theta}{2}\right)\mathrm{d}\theta=2f(0)-f'(0).$

证明 由 Cauchy 积分公式,

$$f(0)=\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=1}\frac{f(z)}{z}\,\mathrm{d}z=\frac{1}{2\pi}\int_0^{2\pi}\frac{f\left(\mathrm{e}^{\mathrm{i}\theta}\right)}{\mathrm{e}^{\mathrm{i}\theta}}\cdot\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=\frac{1}{2\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta,$$

$$f'(0)=\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=1}\frac{f(z)}{z^2}\,\mathrm{d}z=\frac{1}{2\pi}\int_0^{2\pi}\frac{f\left(\mathrm{e}^{\mathrm{i}\theta}\right)}{\mathrm{e}^{2\mathrm{i}\theta}}\cdot\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=\frac{1}{2\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}\theta.$$

由 Cauchy 定理,

$$\int\limits_{|z|=1}f(z)\,\mathrm{d}z=\mathrm{i}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=0\implies\frac{1}{2\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=0.$$

因此

$$\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\cos^2\left(\tfrac{\theta}{2}\right)\mathrm{d}\theta=\frac{1}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\left(1+\frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}}{2}\right)\mathrm{d}\theta=2f(0)+f'(0),$$

进而

$$\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\sin^2\left(\tfrac{\theta}{2}\right)\mathrm{d}\theta=\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\left[1-\cos^2\left(\tfrac{\theta}{2}\right)\right]\mathrm{d}\theta=4f(0)-\left[2f(0)+f'(0)\right]=2f(0)-f'(0).$$

□

习题 3.4.8 (Schwarz 积分公式) 设 $f \in \mathcal{H}(\mathbb{B}(0,R)) \cap \mathcal{C}\left(\overline{\mathbb{B}(0,R)}\right)$, $f = u + \mathrm{i}v$. 证明：f 可用实部表示为

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R\mathrm{e}^{\mathrm{i}\theta}+z}{R\mathrm{e}^{\mathrm{i}\theta}-z} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta + \mathrm{i}v(0).$$

证明 对于 $z \in \mathbb{B}(0,R)$, 由 Cauchy 积分公式,

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=R} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)R\mathrm{e}^{\mathrm{i}\theta}}{R\mathrm{e}^{\mathrm{i}\theta}-z}\mathrm{d}\theta.$$

记 z 关于圆周 $|z| = R$ 的对称点为 $z^* = \dfrac{R^2}{\bar{z}}$, 则由 Cauchy 定理,

$$\int\limits_{|z|=R} \frac{f(\zeta)}{\zeta - z^*}\mathrm{d}\zeta = 0 \implies \frac{1}{2\pi}\int_0^{2\pi} \frac{f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)R\mathrm{e}^{\mathrm{i}\theta}\bar{z}}{R\mathrm{e}^{\mathrm{i}\theta}\bar{z}-R^2}\mathrm{d}\theta = 0.$$

将以上两式作差即得

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\left[\frac{R\mathrm{e}^{\mathrm{i}\theta}}{R\mathrm{e}^{\mathrm{i}\theta}-z} - \frac{\bar{z}}{\bar{z}-R\mathrm{e}^{-\mathrm{i}\theta}}\right]\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\frac{R^2-|z|^2}{\left|R\mathrm{e}^{\mathrm{i}\theta}-z\right|^2}\mathrm{d}\theta.$$

两端取实部即得

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\frac{R^2-|z|^2}{\left|R\mathrm{e}^{\mathrm{i}\theta}-z\right|^2}\mathrm{d}\theta.$$

注意到

$$\frac{R^2-|z|^2}{\left|R\mathrm{e}^{\mathrm{i}\theta}-z\right|^2} = \mathrm{Re}\left(\frac{R\mathrm{e}^{\mathrm{i}\theta}+z}{R\mathrm{e}^{\mathrm{i}\theta}-z}\right),$$

令

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\frac{R\mathrm{e}^{\mathrm{i}\theta}+z}{R\mathrm{e}^{\mathrm{i}\theta}-z}\mathrm{d}\theta,$$

则 $\mathrm{Re}\, g(z) = u(z)$. 由于 $g(z) \in \mathcal{H}(\mathbb{B}(0,R))$, 令 $h(z) = f(z) - g(z)$, 则 $h(z) \in \mathcal{H}(\mathbb{B}(0,R))$, 且 $\mathrm{Re}\, h(z) \equiv 0$. 由习题 2.2.2 即知 $h(z) \equiv C$ 为常数. 由

$$g(0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta = \mathrm{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta\right\} \xlongequal{\text{平均值公式}} \mathrm{Re}\, f(0) = u(0)$$

即知

$$C = f(0) - g(0) = u(0) + \mathrm{i}v(0) - u(0) = \mathrm{i}v(0).$$

故

$$f(z) = g(z) + \mathrm{i}v(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R\mathrm{e}^{\mathrm{i}\theta}+z}{R\mathrm{e}^{\mathrm{i}\theta}-z} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta + \mathrm{i}v(0).$$

□

习题 3.5.1 设 f 是有界整函数, z_1, z_2 是 $\mathbb{B}(0,r)$ 中任意两点. 证明：

$$\int\limits_{|z|=r} \frac{f(z)}{(z-z_1)(z-z_2)}\mathrm{d}z = 0.$$

并由此得出 Liouville 定理.

证明 由 Cauchy 积分公式,

$$\int\limits_{|z|=r} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z = \frac{1}{z_1-z_2}\int\limits_{|z|=r}\left(\frac{f(z)}{z-z_1}-\frac{f(z)}{z-z_2}\right)\mathrm{d}z = 2\pi\mathrm{i}\cdot\frac{f(z_1)-f(z_2)}{z_1-z_2}.$$

由于 f 有界, 存在 $M>0$ 使得 $|f(z)|\leqslant M$. 又 $f\in\mathcal{H}(\mathbb{C})$, 由 Cauchy 定理与长大不等式, 对 $R>r$, 有

$$\left|\int\limits_{|z|=r}\frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z\right| = \left|\int\limits_{|z|=R}\frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z\right| \leqslant \frac{2\pi RM}{(R-|z_1|)(R-|z_2|)}\to 0,\quad R\to+\infty.$$

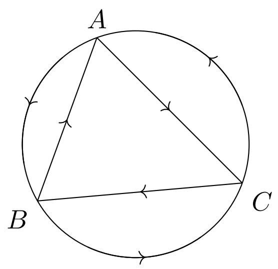
故 $\displaystyle\int\limits_{|z|=r}\frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z=0$, 进而 $f(z_1)=f(z_2)$, 由 z_1,z_2 的任意性即证 Liouville 定理. □

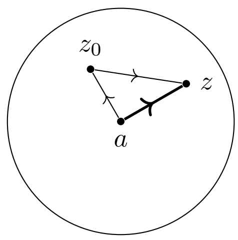
习题 3.5.4 设 f 是整函数, 如果 $f(\mathbb{C})\subset\{z\in\mathbb{C}:\operatorname{Im}z>0\}$, 证明 f 是一个常值函数.

证明 令 $g(z)=\dfrac{f(z)-\mathrm{i}}{f(z)+\mathrm{i}}$, 由题设即得 $g(z)\in\mathcal{H}(\mathbb{C})$ 且 $|g(z)|\leqslant 1$. 根据 Liouville 定理, $g(z)$ 为常值函数, 从而 $f(z)$ 亦为常值函数. □

习题 3.5.8 设 f 是域 D 上的连续函数, 如果对于任意边界和内部都位于 D 中的弓形域 G, 总有 $\displaystyle\int\limits_{\partial G} f(z)\,\mathrm{d}z=0$, 那么 f 是 D 上的全纯函数. 如果把弓形域换成圆盘, 结论是否仍然成立?

证明 (1) 沿弓形域积分为 0 蕴含沿圆盘积分为 0, 进而沿任意外切圆在 D 中的三角形积分为 0. 而 D 中任意三角形均可被剖分为若干个外切圆在 D 中的三角形, 因此沿 D 中任意三角形积分为 0.





为证 f 在 D 上全纯, 只需证 f 在 D 中每个开球上全纯, 因此可不妨设 $D=\mathbb{B}(a,R)$. 任取 $z\in D$, 设 $F(z)=\displaystyle\int\limits_{[a,z]} f(w)\,\mathrm{d}w$. 固定 $z_0\in G$, 由沿三角形积分为 0 可得

$$F(z)=\int\limits_{[a,z_0]} f(w)\,\mathrm{d}w+\int\limits_{[z_0,z]} f(w)\,\mathrm{d}w \implies \frac{F(z)-F(z_0)}{z-z_0}=\frac{1}{z-z_0}\int\limits_{[z_0,z]} f(w)\,\mathrm{d}w.$$

因此

$$\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)=\frac{1}{z-z_0}\int\limits_{[z,z_0]}[f(w)-f(z_0)]\,\mathrm{d}w,$$

进而由长大不等式,

$$\left|\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)\right|\leqslant\max_{w\in[z,z_0]}|f(w)-f(z_0)|.$$

由于 $f \in \mathcal{C}(D)$, 对任意 $\varepsilon > 0$, 存在 $\delta > 0$, 当 $z \in \mathbb{B}(z_0, \delta) \cap D$ 时, 就有 $|f(z) - f(z_0)| < \varepsilon$. 此时

$$\max_{w \in [z, z_0]} |f(w) - f(z_0)| < \varepsilon,$$

故

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$

于是 $F(z)$ 在 D 上全纯, 从而 $f(z) = F'(z)$ 在 D 上全纯.

(2) 若把弓形域换成圆盘, 结论仍成立.

① 先考虑 $f = u + \mathrm{i}v \in \mathcal{C}^1(D)$ 的情形. 对任意 $\mathbb{B}(z_0, r) \subset D$, 有

$$0 = \int_{\partial \mathbb{B}(z_0, r)} f(z) \, \mathrm{d}z = \int_{\partial \mathbb{B}(z_0, r)} (u \, \mathrm{d}x - v \, \mathrm{d}y) + \mathrm{i} \int_{\partial \mathbb{B}(z_0, r)} (u \, \mathrm{d}y + v \, \mathrm{d}x).$$

由 Green 公式可得

$$0 = -\int_{\partial \mathbb{B}(z_0, r)} (u \, \mathrm{d}x - v \, \mathrm{d}y) = \iint_{\mathbb{B}(z_0, r)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \mathrm{d}x \, \mathrm{d}y,$$

$$0 = \int_{\partial \mathbb{B}(z_0, r)} (u \, \mathrm{d}y + v \, \mathrm{d}x) = \iint_{\mathbb{B}(z_0, r)} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y.$$

将以上两式两边同除以 πr^2, 并令 $r \to 0^+$, 由 $u, v \in \mathcal{C}^1(D)$ 即得

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0,$$

这是 Cauchy-Riemann 方程, 故 f 在 D 上全纯.

② 现考虑一般的 $f \in \mathcal{C}(D)$. 设 $\phi(z)$ 为 $\mathbb{C}$ 上的实值函数, 且满足

- ◇ $\phi(z) \geqslant 0$.
- ◇ $\iint_{\mathbb{C}} \phi(z) \, \mathrm{d}x \, \mathrm{d}y = 1$.
- ◇ $\phi \in \mathcal{C}^1(\mathbb{C})$.
- ◇ $\mathrm{supp}(\phi) \subset \overline{\mathbb{B}(0, 1)}$.

对 $\varepsilon > 0$, 定义 $\phi_\varepsilon(z) = \dfrac{\phi\left(\frac{z}{\varepsilon}\right)}{\varepsilon^2}$, 则 $\phi_\varepsilon(z)$ 同样满足上述前三点性质, 且 $\mathrm{supp}(\phi_\varepsilon) \subset \overline{\mathbb{B}(0, \varepsilon)}$. 设

$$f_\varepsilon(z) = \iint_{\mathbb{C}} f(z - \zeta) \phi_\varepsilon(\zeta) \, \mathrm{d}\xi \, \mathrm{d}\eta, \quad \zeta = \xi + \mathrm{i}\eta,$$

则当 $\varepsilon \to 0^+$ 时, $f_\varepsilon(z)$ 局部一致收敛到 $f(z)$, 且对任意 $\mathbb{B}(z_0, r) \subset D$, 有

$$\begin{aligned}
\int_{\partial \mathbb{B}(z_0, r)} f_\varepsilon(z) \, \mathrm{d}z &= \int_{\partial \mathbb{B}(z_0, r)} \iint_{\mathbb{C}} f(z - \zeta) \phi_\varepsilon(\zeta) \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}z \\
&= \iint_{\mathbb{C}} \left\{ \int_{\partial \mathbb{B}(z_0, r)} f(z - \zeta) \, \mathrm{d}z \right\} \phi_\varepsilon(\zeta) \, \mathrm{d}\xi \, \mathrm{d}\eta
\end{aligned}$$

$$
\begin{aligned}
&= \iint_{\mathbb{C}} \left\{ \int\limits_{\partial\mathbb{B}(z_0-\zeta,r)} f(z)\,\mathrm{d}z \right\} \phi_\varepsilon(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= 0.
\end{aligned}
$$

由①即知 $f_\varepsilon(z) \in \mathcal{H}(D)$. 由于 $f(z)$ 是 $f_\varepsilon(z)$ 的局部一致极限, $f(z) \in \mathcal{H}(D)$. □

习题 4.2.2 求下列幂级数的收敛半径：

(3) $\displaystyle\sum_{n=0}^{\infty} [3+(-1)^n]^n z^n$.

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!} z^n$.

解答 (3) $\displaystyle\limsup_{n\to\infty} \sqrt[n]{[3+(-1)^n]^n} = \lim_{n\to\infty} \sqrt[n]{4^n} = 4 \implies$ 收敛半径 $R = \dfrac{1}{4}$.

(4) $\displaystyle\limsup_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{n}{(2\pi n)^{\frac{1}{2n}} \frac{n}{\mathrm{e}}} = \mathrm{e} \implies$ 收敛半径 $R = \dfrac{1}{\mathrm{e}}$. □

习题 4.2.4 设正数列 $\{a_n\}$ 单调收敛于 0. 证明：

(1) $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ 的收敛半径 $R \geqslant 1$.

(2) $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ 在 $\partial\mathbb{B}(0,1)\setminus\{1\}$ 上处处收敛.

证明 (1) 由于 $a_n \downarrow 0$, 存在正整数 N, 当 $n > N$ 时, $a_n < 1$, 从而 $\sqrt[n]{a_n} < 1$, 因此 $\displaystyle\limsup_{n\to\infty} \sqrt[n]{a_n} \leqslant 1$, 收敛半径 $R \geqslant 1$.

(2) 当 $z \in \partial\mathbb{B}(0,1)\setminus\{1\}$ 时, $\displaystyle\left|\sum_{k=0}^{n} z^k\right| = \left|\frac{1-z^{n+1}}{1-z}\right| \leqslant \frac{2}{|1-z|}$, 而 $a_n \downarrow 0$, 由 Dirichlet 判别法得证. □

习题 4.2.7 证明：若 $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ 是 $\mathbb{B}(0,1)$ 上的有界全纯函数, 则 $\displaystyle\sum_{n=0}^{\infty} |a_n|^2 < +\infty$.

证明 设 $|f(z)| \leqslant M, \forall z \in \mathbb{B}(0,1)$. 对 $r \in (0,1)$, 有

$$
\begin{aligned}
2\pi M^2 &\geqslant \int_0^{2\pi} \left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2 \mathrm{d}\theta = \int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n \mathrm{e}^{\mathrm{i}n\theta} \sum_{m=0}^{\infty} \overline{a_m} r^n \mathrm{e}^{-\mathrm{i}m\theta}\,\mathrm{d}\theta \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \overline{a_m} r^{n+m} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n},
\end{aligned}
$$

因此

$$
\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leqslant M^2 \implies \sum_{n=0}^{m} |a_n|^2 r^{2n} \leqslant M^2, \quad \forall m \geqslant n \implies \sum_{n=0}^{m} |a_n|^2 R^{2n} \leqslant M^2, \quad \forall m \geqslant n.
$$

故

$$
\sum_{n=0}^{\infty} |a_n|^2 < +\infty.
$$

□

习题 4.3.1 设 D 是域, $a \in D$, 函数 $f \in \mathcal{H}(D \setminus \{a\})$. 证明: 若 $\lim\limits_{z \to a}(z-a)f(z) = 0$, 则 $f \in \mathcal{H}(D)$.

证明 设 $\varphi(z) = \begin{cases} (z-a)f(z), & z \in D \setminus \{a\}, \\ 0, & z = a. \end{cases}$ 则 $\varphi \in \mathcal{C}(D) \cap \mathcal{H}(D \setminus \{a\})$. 任取 D 中可求长简单闭曲线 γ, 且 γ 所围区域在 D 中, 则不论 a 与 γ 的位置关系, 均有 $\int\limits_{\gamma} \varphi(z)\,\mathrm{d}z = 0$ (当 a 在 γ 所围区域中时, 可添加过 a 的曲线). 由 Morera 定理得 $\varphi \in \mathcal{H}(D)$. 于是, 当补充定义 $f(a) = \varphi'(a)$ 后便有

$$\lim_{z \to a} f(z) = \lim_{z \to a} \frac{\varphi(z) - \varphi(a)}{z - a} = \varphi'(a) = f(a),$$

因此 $f \in \mathcal{C}(D) \cap \mathcal{H}(D \setminus \{a\})$, 同前可得 $f \in \mathcal{H}(D)$. □

习题 4.3.5 是否存在 $f \in \mathcal{H}(\mathbb{B}(0,1))$, 使得下述条件之一成立:

(2) $f\left(\frac{1}{2n}\right) = 0, f\left(\frac{1}{2n-1}\right) = 1, n = 1, 2, 3, \cdots$.

(3) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2}, n = 2, 3, 4 \cdots$.

解答 (2) 不存在. 令 $n \to \infty$, 由 f 在 $z = 0$ 处连续即得矛盾.

(3) 不存在. 因为由唯一性定理, $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ 要求 $f(z) = z^2$, 但这与 $f\left(-\frac{1}{n}\right) = \frac{1}{n^2}$ 矛盾. □

习题 4.3.6 设 $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ 的收敛半径 $R > 0$, $0 < r < R$, $A(r) = \max\limits_{|z|=r} \operatorname{Re} f(z)$. 证明:

(1) $a_n r^n = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} \left[\operatorname{Re} f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right] \mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta, \forall n \in \mathbb{N}$.

(2) $|a_n| r^n \leqslant 2A(r) - 2\operatorname{Re} f(0), \forall n \in \mathbb{N}$.

证明 (1) 由 Cauchy 积分公式,

$$a_n = \frac{1}{2\pi\mathrm{i}} \sum_{m=0}^{\infty} \int\limits_{|z|=r} \frac{a_m}{z^{m+1}}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)}{r^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}} r\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \frac{1}{2\pi r^n} \int_0^{2\pi} f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta.$$

而

$$0 = \int\limits_{|z|=r} f(z) z^{n-1}\,\mathrm{d}z = \mathrm{i}r^n \int_0^{2\pi} f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\theta \implies \frac{1}{2\pi r^n} \int_0^{2\pi} \overline{f(r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta = 0,$$

因此

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} \left[\operatorname{Re} f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right]\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta.$$

(2) 利用 $\displaystyle\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta = 0$ 可得

$$\begin{aligned} |a_n| r^n &= \frac{1}{\pi} \left| \int_0^{2\pi} \left[\operatorname{Re} f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) - A(r)\right]\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta \right| \\ &\leqslant \frac{1}{\pi} \int_0^{2\pi} \left|\operatorname{Re} f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) - A(r)\right|\mathrm{d}\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} \left[A(r) - \operatorname{Re} f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right]\mathrm{d}\theta \end{aligned}$$

$$
\begin{aligned}
&= 2A(r) - \frac{1}{\pi}\operatorname{Re}\left\{\int_0^{2\pi} f\big(re^{i\theta}\big)\,d\theta\right\} \\
&= 2A(r) - 2\operatorname{Re} f(0),
\end{aligned}
$$

其中最后一个等式用到了平均值公式. □

习题 4.3.14 设 D 是域, $a \in D$, $f \in \mathcal{H}(D)$, 并且 $\sum\limits_{n=0}^{\infty} f^{(n)}(a)$ 收敛. 证明:

(1) f 是整函数.

(2) $\sum\limits_{n=0}^{\infty} f^{(n)}(z)$ 在 $\mathbb{C}$ 上内闭一致收敛.

证明 (1) 由于 $f \in \mathcal{H}(D)$, 存在 $\varepsilon > 0$, 使得在 $\mathbb{B}(a, \varepsilon)$ 上有展开式

$$
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.
$$

由 $\sum\limits_{n=0}^{\infty} f^{(n)}(a)$ 收敛可知 $\lim\limits_{n\to\infty}\left|f^{(n)}(a)\right| = 0$, 因此

$$
\limsup_{n\to\infty} \sqrt[n]{\left|\frac{f^{(n)}(a)}{n!}\right|} = 0 \implies \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \text{ 的收敛半径为 } +\infty.
$$

设 $S(z) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(z-a)^n, z \in \mathbb{C}$, 则 $S(z)$ 是 $f(z)$ 在 $\mathbb{C}$ 上的解析延拓 (由零点孤立性知延拓唯一). 故 f 可延拓为整函数.

(2) 由于 $\sum\limits_{n=0}^{\infty} f^{(n)}(a)$ 收敛, 对任意 $\varepsilon > 0$, 存在正整数 N, 使得

$$
\left|f^{(p+1)}(a) + \cdots + f^{(q)}(a)\right| < \varepsilon, \quad \forall q > p > N.
$$

对任意紧集 $K \subset \mathbb{C}$, 记 $M = \max\limits_{z\in K}\left\{e^{|z-a|}\right\}$, 则

$$
\begin{aligned}
\left|\sum_{k=p+1}^{q} S^{(k)}(z)\right| &= \left|\sum_{k=p+1}^{q}\sum_{n=0}^{\infty} \frac{f^{(n+k)}(a)}{n!}(z-a)^n\right| = \left|\sum_{n=0}^{\infty} \frac{f^{(n+p+1)}(a) + \cdots + f^{(n+q)}(a)}{n!}(z-a)^n\right| \\
&\leqslant \varepsilon \sum_{n=0}^{\infty} \frac{|z-a|^n}{n!} = \varepsilon e^{|z-a|} \leqslant M\varepsilon.
\end{aligned}
$$

因此 $\sum\limits_{n=0}^{\infty} S^{(n)}(z)$ 在 K 上一致收敛. 再由 K 的任意性即得 $\sum\limits_{n=0}^{\infty} S^{(n)}(z)$ 在 $\mathbb{C}$ 上内闭一致收敛. □

习题 4.4.6 设 $0 < r < 1$. 证明: 当 n 充分大时, 多项式 $1 + 2z + 3z^2 + \cdots + nz^{n-1}$ 在 $\mathbb{B}(0, r)$ 中没有根.

证明 由于级数 $\sum\limits_{k=0}^{\infty}(k+1)z^k$ 的收敛半径为 1, 当 $|z| < 1$ 时, $\sum\limits_{k=0}^{\infty}(k+1)z^k = \left(\sum\limits_{k=0}^{\infty} z^{k+1}\right)' = \dfrac{1}{(1-z)^2}$. 由

于此级数在 $\mathbb{B}(0,r)$ 中内闭一致收敛, 由 Hurwitz 定理, 当 n 充分大时, 部分和 $\sum_{k=0}^{n}(k+1)z^k$ 在 $\mathbb{B}(0,r)$ 中的零点个数与 $\dfrac{1}{(1-z)^2}$ 相同, 即无零点. □

习题 4.4.7 设 $r>0$. 证明: 当 n 充分大时, 多项式 $1+z+\dfrac{1}{2!}z^2+\cdots+\dfrac{1}{n!}z^n$ 在 $\mathbb{B}(0,r)$ 中没有根.

证明 由于级数 $\sum_{k=0}^{\infty}\dfrac{z^k}{k!}$ 在 $\mathbb{B}(0,r)$ 中内闭一致收敛到 e^z, 由 Hurwitz 定理, 当 n 充分大时, 部分和 $\sum_{k=0}^{n}\dfrac{z^k}{k!}$ 在 $\mathbb{B}(0,r)$ 中的零点个数与 e^z 相同, 即无零点. □

习题 4.4.8 设 $f(z)\in\mathcal{H}\left(\overline{\mathbb{B}(0,1)}\right)$, 且 $f'(z)$ 在 $\partial\mathbb{B}(0,1)$ 上无零点. 证明: 当 n 充分大时, $F_n(z)=n\left[f\left(z+\frac{1}{n}\right)-f(z)\right]$ 与 $f'(z)$ 在 $\mathbb{B}(0,1)$ 中的零点个数相等.

证明 对任意 $0<r<1$, 由于 $f'(z)\in\mathcal{C}\left(\overline{\mathbb{B}(0,r)}\right)$, $F_n(z)$ 在 $\overline{\mathbb{B}(0,r)}$ 上一致收敛到 $f'(z)$, 即 $F_n(z)$ 在 $\mathbb{B}(0,1)$ 中内闭一致收敛到 $f'(z)$. 又 $f'(z)$ 在 $\partial\mathbb{B}(0,1)$ 上无零点, 由 Hurwitz 定理, 当 n 充分大时, $F_n(z)$ 在 $\mathbb{B}(0,1)$ 中的零点个数与 $f'(z)$ 相同. □

习题 4.4.11 求下列全纯函数在 $\mathbb{B}(0,1)$ 中的零点个数:

(1) $z^9-2z^6+z^2-8z-2$.

(2) $2z^5-z^3+3z^2-z+8$.

(3) $z^7-5z^4+z^2-2$.

(4) e^z-4z^n+1.

解答 记每问中的函数为 $f(z)$, $\gamma=\partial\mathbb{B}(0,1)$.

(1) 设 $g(z)=-8z$, 则当 $z\in\gamma$ 时, $|f(z)-g(z)|=\left|z^9-2z^6+z^2-2\right|\leqslant|z|^9+2|z|^6+|z|^2+2=6<8=|g(z)|$, 由 Rouché 定理知 f 和 g 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 1 个.

(2) 设 $g(z)=8$, 则当 $z\in\gamma$ 时, $|f(z)-g(z)|=\left|2z^5-z^3+3z^2-z\right|\leqslant2|z|^5+|z|^3+3|z|^2+|z|=7<8=|g(z)|$, 由 Rouché 定理知 f 和 g 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 0 个.

(3) 设 $g(z)=-5z^4$, 则当 $z\in\gamma$ 时, $|f(z)-g(z)|=\left|z^7+z^2-2\right|\leqslant|z|^7+|z|^2+2=4<5=|g(z)|$, 由 Rouché 定理知 f 和 g 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 4 个.

(4) 设 $g(z)=-4z^n$, 则当 $z\in\gamma$ 时, $|f(z)-g(z)|=|\mathrm{e}^z-1|\leqslant\mathrm{e}^{|z|}+1=\mathrm{e}+1<4=|g(z)|$, 由 Rouché 定理知 f 和 g 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 n 个. □

习题 4.4.12 若 $f\in\mathcal{H}(\mathbb{B}(0,1))\cap\mathcal{C}\left(\overline{\mathbb{B}(0,1)}\right)$, $f\left(\overline{\mathbb{B}(0,1)}\right)\subset\mathbb{B}(0,1)$, 则 $f(z)$ 在 $\mathbb{B}(0,1)$ 中有唯一的不动点.

证明 令 $g(z)=f(z)-z$, $h(z)=-z$, 则当 $z\in\partial\mathbb{B}(0,1)$ 时, $|g(z)-h(z)|=|f(z)|<1=|h(z)|$, 由 Rouché 定理知 g 和 h 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 1 个, 即 $f(z)$ 在 $\mathbb{B}(0,1)$ 中有唯一的不动点. □

习题 4.4.13 设 $a_1,a_2,\cdots,a_n\in\mathbb{B}(0,1)$, $f(z)=\prod_{k=1}^{n}\dfrac{a_k-z}{1-\overline{a_k}z}$. 证明:

(1) 若 $b\in\mathbb{B}(0,1)$, 则 $f(z)=b$ 在 $\mathbb{B}(0,1)$ 中恰有 n 个根.

(2) 若 $b\in\mathbb{B}(\infty,1)$, 则 $f(z)=b$ 在 $\mathbb{B}(\infty,1)$ 中恰有 n 个根.

证明 (1) 注意到 Blaschke 因子 $\dfrac{a-z}{1-\bar{a}z}$ ($|a|<1$) 有如下性质:

$$\left|\frac{a-z}{1-\bar{a}z}\right|<1 \iff |z|<1,\quad \left|\frac{a-z}{1-\bar{a}z}\right|=1 \iff |z|=1,\quad \left|\frac{a-z}{1-\bar{a}z}\right|>1 \iff |z|>1.$$

而 $f(z)=b \iff \prod_{k=1}^{n}(a_k-z)=b\prod_{k=1}^{n}(1-\overline{a_k}z)$ (这是 n 次方程, 因为 $|b\overline{a_1\cdots a_n}|<1$) 在 $\mathbb{C}$ 上恰有 n 个根. 此时 $|f(z)|=|b|<1$, 因此 $|z|<1$ (否则, 每项 $\left|\dfrac{a_k-z}{1-\overline{a_k}z}\right|\geqslant 1$, 进而 $|f(z)|\geqslant 1$), 即 $f(z)=b$ 在 $\mathbb{B}(0,1)$ 中恰有 n 个根.

(2) 即证 $f\left(\frac{1}{\bar{z}}\right)=b$ 在 $\mathbb{B}(0,1)$ 中恰有 n 个根, 这等价于证明 $\dfrac{1}{f\left(\frac{1}{\bar{z}}\right)}=\dfrac{1}{b}$ 在 $\mathbb{B}(0,1)$ 中恰有 n 个根. 而

$$\frac{1}{f\left(\frac{1}{\bar{z}}\right)}=\prod_{k=1}^{n}\frac{1-a_k\frac{1}{z}}{\overline{a_k}-\frac{1}{z}}=\prod_{k=1}^{n}\frac{a_k-z}{1-\overline{a_k}z}=f(z),$$

而此时 $\left|\frac{1}{b}\right|<1$, 因此由 (1) 即得证. □

习题 4.5.4 设 $f\in\mathcal{H}(\mathbb{B}(0,R))$. 证明: $M(r)=\max\limits_{|z|=r}|f(z)|$ 是 $[0,R)$ 上的增函数.

证明 不妨设 f 非常数. 由最大模原理, $M(r)=\max\limits_{|z|\leqslant r}|f(z)|$, 由此可见 $M(r)$ 为 $[0,R)$ 上的增函数. □

习题 4.5.5 利用最大模原理证明代数学基本定理.

证明 设 $P(z)\in\mathbb{C}[z]$, $\deg P=n$ ($n\geqslant 1$). 假设 $P(z)$ 在 $\mathbb{C}$ 中没有零点. 取 $R>0$ 使得当 $|z|\geqslant R$ 时有 $|P(z)|>|P(0)|$, 则 $|P(z)|$ 在闭圆盘 $\overline{\mathbb{B}(0,R)}$ 上的最小值在内部取到. 由于 $P(z)$ 在 $\mathbb{B}(0,R)$ 中无零点, 由最大模原理, $\left|\dfrac{1}{P(z)}\right|$ 在 $\mathbb{B}(0,R)$ 内取不到最大值, 即 $|P(z)|$ 在 $\mathbb{B}(0,R)$ 内取不到最小值, 矛盾. □

习题 4.5.10 设 $f\in\mathcal{H}(\mathbb{B}(0,R))$, $f(\mathbb{B}(0,R))\subset\mathbb{B}(0,M)$, $f(0)=0$. 证明:

(1) $|f(z)|\leqslant\dfrac{M}{R}|z|, |f'(0)|\leqslant\dfrac{M}{R}, \forall z\in\mathbb{B}(0,R)\setminus\{0\}$.

(2) 等号成立当且仅当 $f(z)=\dfrac{M}{R}\mathrm{e}^{\mathrm{i}\theta}z$ ($\theta\in\mathbb{R}$).

证明 考虑函数

$$g:\mathbb{B}(0,1)\to\mathbb{B}(0,1),\quad z\mapsto\frac{1}{M}f(Rz).$$

由于 $g\in\mathcal{H}(\mathbb{B}(0,1))$, $g(0)=0$, 由 Schwarz 引理可得

$$|g(z)|\leqslant|z|,\quad |g'(0)|\leqslant 1,\quad \forall z\in\mathbb{B}(0,1),$$

也即

$$|f(z)|\leqslant\frac{M}{R}|z|,\quad |f'(0)|\leqslant\frac{M}{R},\quad \forall z\in\mathbb{B}(0,R).$$

等号成立当且仅当 $g(z)=\mathrm{e}^{\mathrm{i}\theta}z$ ($\theta\in\mathbb{R}$) 即 $f(z)=\dfrac{M}{R}\mathrm{e}^{\mathrm{i}\theta}z$ ($\theta\in\mathbb{R}$). □

习题 4.5.11 设 $f\in\mathcal{H}(\mathbb{B}(0,1))$, $f(0)=0$, 并且存在 $A>0$, 使得 $\operatorname{Re} f(z)\leqslant A, \forall z\in\mathbb{B}(0,1)$. 证明:

$$|f(z)|\leqslant\frac{2A|z|}{1-|z|},\quad \forall z\in\mathbb{B}(0,1).$$

证明 设 $g(z)=\dfrac{z}{z-2A}$, 则 g 是从 $\{z\in\mathbb{C}:\operatorname{Re}z<A\}$ 到 $\mathbb{B}(0,1)$ 的共形变换 (分解如下), 且 $g(0)=0$.

$$\{z\in\mathbb{C}:\operatorname{Re}z<A\}\xrightarrow[0\mapsto -A]{z\mapsto z-A}\{z\in\mathbb{C}:\operatorname{Re}z<0\}\xrightarrow[-A\mapsto A\mathrm{i}]{z\mapsto -\mathrm{i}z}\mathbb{H}\xrightarrow[A\mathrm{i}\mapsto 0]{z\mapsto\frac{z-A\mathrm{i}}{z+A\mathrm{i}}}\mathbb{B}(0,1)$$

考虑 $h(z)=g\circ f(z)=\dfrac{f(z)}{f(z)-2A}$, 则 $h(0)=0$ 且 $|h(z)|\leqslant 1$, 由 Schwarz 引理可得 $|h(z)|\leqslant|z|$, 因此

$$\frac{|f(z)|}{|f(z)|+2A}\leqslant\frac{|f(z)|}{|f(z)-2A|}\leqslant|z|\implies|f(z)|\leqslant\frac{2A|z|}{1-|z|},\quad\forall z\in\mathbb{B}(0,1).$$

□

习题 4.5.12 (Carathéodory 不等式) 设 $f\in\mathcal{H}(\mathbb{B}(0,R))\cap\mathcal{C}\left(\overline{\mathbb{B}(0,R)}\right)$, $M(r)=\max\limits_{|z|=r}|f(z)|$, $A(r)=\max\limits_{|z|=r}\operatorname{Re}f(z)$ $(0\leqslant r\leqslant R)$. 证明:

$$M(r)\leqslant\frac{2r}{R-r}A(R)+\frac{R+r}{R-r}|f(0)|,\quad\forall r\in[0,R).$$

证明 设 $g(z)=f(Rz)-f(0)$, 则 $g(z)\in\mathcal{H}(\mathbb{B}(0,1))$ 且 $g(0)=0$. 对 $\mathbb{B}(0,1)$ 上的调和函数 $\operatorname{Re}g(z)$ 使用最大模原理可得

$$\max_{|z|\leqslant 1}\operatorname{Re}g(z)=\max_{|z|=1}\operatorname{Re}g(z)=A(R)-\operatorname{Re}f(0).$$

由习题 4.5.11 即得

$$|g(z)|\leqslant\frac{2[A(R)-\operatorname{Re}f(0)]\cdot|z|}{1-|z|}\leqslant\frac{2[A(R)+|f(0)|]\cdot|z|}{1-|z|},\quad\forall z\in\mathbb{B}(0,1).$$

由 $f(z)=g\left(\frac{z}{R}\right)+f(0)$ 即得

$$\begin{aligned}|f(z)|&\leqslant\left|g\left(\tfrac{z}{R}\right)\right|+|f(0)|\leqslant\frac{2[A(R)+|f(0)|]\cdot\left|\frac{z}{R}\right|}{1-\left|\frac{z}{R}\right|}+|f(0)|=\frac{2[A(R)+|f(0)|]\cdot|z|}{R-|z|}+|f(0)|\\&=\frac{2|z|}{R-|z|}A(R)+\frac{R+|z|}{R-|z|}|f(0)|,\quad\forall z\in\mathbb{B}(0,R).\end{aligned}$$

故

$$M(r)=\max_{|z|=r}|f(z)|\leqslant\frac{2r}{R-r}A(R)+\frac{R+r}{R-r}|f(0)|,\quad\forall r\in[0,R).$$

□

习题 4.5.18 设 $f\in\mathcal{H}(\mathbb{B}(0,1))$, $f(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$. 证明:

$$\frac{|f(0)|-|z|}{1-|f(0)||z|}\leqslant|f(z)|\leqslant\frac{|f(0)|+|z|}{1+|f(0)||z|}.$$

证明 记 $b=f(0)$, 对 $a\in\mathbb{B}(0,1)$, 记 $\phi_a(z)=\dfrac{z-a}{1-\bar{a}z}$, 则由 Schwarz-Pick 定理,

$$|\varphi_b(f(z))|\leqslant|\varphi_0(z)|\quad\text{即}\quad\left|\frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}\right|\leqslant|z|,\quad z\in\mathbb{B}(0,1).$$

另一方面, 由习题 1.1.6 (3),

$$\frac{||f(z)|-|f(0)||}{1-|f(0)||f(z)|} \leqslant \left|\frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}\right|.$$

由上述两个不等式即得

$$\frac{||f(z)|-|f(0)||}{1-|f(0)||f(z)|} \leqslant |z|,$$

也即

$$\begin{cases} |z|-|f(0)||f(z)||z| \geqslant |f(z)|-|f(0)|, \\ |z|-|f(0)||f(z)||z| \geqslant |f(0)|-|f(z)|. \end{cases}$$

整理即得

$$\frac{|f(0)|-|z|}{1-|f(0)||z|} \leqslant |f(z)| \leqslant \frac{|f(0)|+|z|}{1+|f(0)||z|}.$$

□

习题 4.5.19 设 $f \in \mathcal{H}(\mathbb{B}(0,1))$, $f(\mathbb{B}(0,1)) \subset \mathbb{B}(0,M)$. 证明:

$$M|f'(0)| \leqslant M^2-|f(0)|^2.$$

证明 记 $a=\dfrac{f(0)}{M}$, $g(z)=\dfrac{a-z}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{B}(0,1))$. 考虑 $h(z)=g\left(\frac{f(z)}{M}\right)$, 则 h 是从 $\mathbb{B}(0,1)$ 到 $\mathbb{B}(0,1)$ 的共形变换, 且 $h(0)=0$. 由 Schwarz 引理, $|h'(0)| \leqslant 1$. 注意到 $g^{-1}=g$, 因此 $M \cdot g \circ h = f$,

$$|f'(0)| = M|g'(0)| \cdot |h'(0)| \leqslant M|g'(0)| = M\big||a|^2-1\big| = M\big(1-|a|^2\big) = \frac{M^2-|f(0)|^2}{M},$$

得所欲证. □

习题 4.5.20 设 $f \in \mathcal{H}(\mathbb{B}(0,1))$, $f(0)=0$, $f(\mathbb{B}(0,1)) \subset \mathbb{B}(0,1)$. 证明: 若存在 $z_1, z_2 \in \mathbb{B}(0,1)$, 使得 $z_1 \neq z_2$, $|z_1|=|z_2|$, $f(z_1)=f(z_2)$, 则

$$|f(z_1)|=|f(z_2)| \leqslant |z_1|^2=|z_2|^2.$$

证明 令

$$F(z)=\frac{f(z_1)-f(z)}{1-\overline{f(z_1)}f(z)} \cdot \frac{1-\overline{z_1}z}{z_1-z} \cdot \frac{1-\overline{z_2}z}{z_2-z}.$$

注意到 z_1, z_2 均为 $F(z)$ 的可去奇点, 因此 $F(z) \in \mathcal{H}(\mathbb{B}(0,1))$, 由最大模原理, 及 $f(\mathbb{B}(0,1)) \subset \mathbb{B}(0,1)$, 有

$$\max_{|z|\leqslant 1}|F(z)| = \max_{|z|=1}|F(z)| = 1.$$

特别地,

$$|F(0)| = \left|\frac{f(z_1)}{z_1 z_2}\right| \leqslant 1 \implies |f(z_1)|=|f(z_2)| \leqslant |z_1 z_2| = |z_1|^2=|z_2|^2.$$

□

习题 4.5.21 设 $f \in \mathcal{H}(\mathbb{B}(0,1))$, $f(0)=0$, $f(\mathbb{B}(0,1)) \subset \mathbb{B}(0,1)$. 证明:

$$|z|\frac{|f'(0)|-|z|}{1-|f'(0)||z|} \leqslant |f(z)| \leqslant |z|\frac{|f'(0)|+|z|}{1+|f'(0)||z|}.$$

证明 令 $g(z)=\begin{cases}\dfrac{f(z)}{z}, & 0<|z|<1,\\ f'(0), & z=0.\end{cases}$ 由 Schwarz 引理知 $|g(z)|\leqslant 1$. 对 $g(z)$ 用习题 4.5.18 结论即可. □

习题 4.5.30 设 $f\in\mathcal{H}(\mathbb{B}(0,1))$, $f(0)=0$, 并且 $|\mathrm{Re}\, f(z)|<1, \forall z\in\mathbb{B}(0,1)$. 证明:

(1) $|\mathrm{Re}\, f(z)|\leqslant \dfrac{4}{\pi}\arctan|z|, \forall z\in\mathbb{B}(0,1)$.

(2) $|\mathrm{Im}\, f(z)|\leqslant \dfrac{2}{\pi}\log\left(\dfrac{1+|z|}{1-|z|}\right), \forall z\in\mathbb{B}(0,1)$.

证明 先构造共形变换 $g:\{z\in\mathbb{C}:|\mathrm{Re}\, f(z)|<1\}\to\mathbb{B}(0,1)$ 使得 $g(0)=0$, 分解如下:

$$\{z\in\mathbb{C}:|\mathrm{Re}\, z|<1\}\xrightarrow[0\mapsto 0]{z\mapsto\frac{\pi\mathrm{i}}{2}z}\{z\in\mathbb{C}:|\mathrm{Im}\, z|<\tfrac{\pi}{2}\}\xrightarrow[0\mapsto 1]{z\mapsto \mathrm{e}^z}\{z\in\mathbb{C}:\mathrm{Re}\, z>0\}$$

$$\downarrow\ 1\mapsto\mathrm{i},\ z\mapsto\mathrm{i}z$$

$$\mathbb{B}(0,1)\xleftarrow[\mathrm{i}\mapsto 0]{z\mapsto\frac{z-\mathrm{i}}{z+\mathrm{i}}}\{z\in\mathbb{C}:\mathrm{Im}\, z>0\}$$

复合结果为 $g(z)=\dfrac{\mathrm{e}^{\frac{\pi\mathrm{i}}{2}z}-1}{\mathrm{e}^{\frac{\pi\mathrm{i}}{2}z}+1}$. 考虑 $h(z)=g\circ f(z)=\dfrac{\mathrm{e}^{\frac{\pi\mathrm{i}}{2}f(z)}-1}{\mathrm{e}^{\frac{\pi\mathrm{i}}{2}f(z)}+1}$, 则 $h:\mathbb{B}(0,1)\to\mathbb{B}(0,1)$ 且 $h(0)=0$, 由 Schwarz 引理可得 $|h(z)|\leqslant|z|$. 而由 $f(0)=0$ 可解得

$$f(z)=\frac{2}{\pi\mathrm{i}}\log\frac{1+h(z)}{1-h(z)}\implies\begin{cases}\mathrm{Re}\, f(z)=\dfrac{2}{\pi}\arg\left(\dfrac{1+h(z)}{1-h(z)}\right),\\ \mathrm{Im}\, f(z)=-\dfrac{2}{\pi}\log\left|\dfrac{1+h(z)}{1-h(z)}\right|.\end{cases}$$

因此由

$$\log\left(\frac{1-|z|}{1+|z|}\right)\leqslant\log\left|\frac{1+h(z)}{1-h(z)}\right|\leqslant\log\left(\frac{1+|z|}{1-|z|}\right)$$

即得结论 (2). 由 $|\mathrm{Re}\, f(z)|<1$ 可得

$$\left|\arg\left(\frac{1+h(z)}{1-h(z)}\right)\right|<\frac{\pi}{2}.$$

因此

$$\frac{1+h(z)}{1-h(z)}=\frac{1+|h(z)|^2+2\mathrm{i}\,\mathrm{Im}\, h(z)}{|1-h(z)|^2}\implies\arg\left(\frac{1+h(z)}{1-h(z)}\right)=\arctan\left(\frac{2\,\mathrm{Im}\, h(z)}{1-|h(z)|^2}\right),$$

进而

$$|\mathrm{Re}\, f(z)|=\frac{2}{\pi}\left|\arctan\left(\frac{2\,\mathrm{Im}\, h(z)}{1-|h(z)|^2}\right)\right|\leqslant\frac{2}{\pi}\arctan\left(\frac{2|z|}{1-|z|^2}\right)\overset{\star}{=\!=}\frac{2}{\pi}\cdot 2\arctan|z|,$$

$\star$ 处用到了正切函数的二倍角公式及 $|z|<1$ 时 $\arctan|z|\in\left(0,\frac{\pi}{4}\right)$. 故结论 (1) 得证. □

习题 4.5.31 设 $f\in\mathcal{H}(\mathbb{B}(0,1)\cup\{1\})$, $f(0)=0$, $f(1)=1$, $f(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$. 证明: $f'(1)\geqslant 1$.

证明 设 $g(z)=\begin{cases}\dfrac{f(z)}{z}, & z\neq 0,\\ f'(0), & z=0.\end{cases}$ 由习题 4.3.1 即知 $g\in\mathcal{H}(\mathbb{B}(0,1))$. 由最大模原理,

$$\max_{|z|\leqslant 1}|g(z)|=\max_{|z|=1}|g(z)|=\max_{|z|=1}\left|\frac{f(z)}{z}\right|=\max_{|z|=1}|f(z)|=\max_{|z|\leqslant 1}|f(z)|,$$

因此 $g(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$ 且 $g(1)=1$. 由习题 2.3.3 即得 $g'(1)=f'(1)-f(1)\geqslant 0$, 故 $f'(1)\geqslant 1$. □

习题 4.5.32 设 P 是一个 k 次多项式, 在单位圆周上满足 $\left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant 1$. 证明: 对任意单位圆盘外的 z, 有 $|P(z)| \leqslant |z|^k$.

证明 设 $f(z) = \dfrac{P(z)}{z^k}$, 则 $f \in \mathcal{H}\left(\overline{\mathbb{B}(0,1)}^{\mathrm{c}}\right)$. 由最大模原理, $\max\limits_{|z|\geqslant 1} |f(z)| = \max\limits_{|z|=1} |f(z)| \leqslant 1$, 得所欲证. □

习题 5.2.2 下列初等全纯函数有哪些奇点? 指出其类别:

(2) $\dfrac{\mathrm{e}^{\frac{1}{1-z}}}{\mathrm{e}^z - 1}$.

(4) $\tan z$.

(6) $\mathrm{e}^{\cot \frac{1}{z}}$.

解答 (2) 1 阶极点: $2k\pi\mathrm{i}\ (k \in \mathbb{Z})$; 本性奇点: 1; 非孤立奇点: ∞.

(4) $\tan z = -\mathrm{i}\dfrac{\mathrm{e}^{2\mathrm{i}z} - 1}{\mathrm{e}^{2\mathrm{i}z} + 1}$. 1 阶极点: $\left(k + \frac{1}{2}\right)\pi\ (k \in \mathbb{Z})$; 非孤立奇点: ∞.

(6) $\mathrm{e}^{\cot \frac{1}{z}} = \exp\left(\mathrm{i}\dfrac{\mathrm{e}^{\frac{2\mathrm{i}}{z}} + 1}{\mathrm{e}^{\frac{2\mathrm{i}}{z}} - 1}\right)$. 本性奇点: $\dfrac{1}{k\pi}\ (k \in \mathbb{Z})$, ∞; 非孤立奇点: 0. □

习题 5.2.3 若 z_0 是全纯函数 $f : \mathbb{B}(z_0, r) \setminus \{z_0\} \to \mathbb{C} \setminus \{0\}$ 的本性奇点, 则 z_0 也是 $\dfrac{1}{f(z)}$ 的本性奇点.

证明 由 $f(z) \neq 0, \forall z \in \mathbb{B}(z_0, r) \setminus \{z_0\}$ 知 z_0 是 $\dfrac{1}{f(z)}$ 的孤立奇点. 由于 z_0 是 $f(z)$ 的本性奇点, 对任意 $A \in \overline{\mathbb{C}}$, 在任意 $\mathbb{B}(z_0, \delta) \setminus \{z_0\} \subset \mathbb{B}(z_0, r)$ 中存在一列互异的 $z_n \to z_0$ 使得 $f(z_n) \to \dfrac{1}{A}$, 进而 $\dfrac{1}{f(z_n)} \to A$, 即 z_0 是 $\dfrac{1}{f(z)}$ 的本性奇点. □

习题 5.2.4 设 $R(z)$ 是有理函数, $z_1, z_2, \cdots, z_n$ 是 $R(z)$ 在 $\overline{\mathbb{C}}$ 上的全部不同的极点. 证明: 若 z_0 是全纯函数 $f : \mathbb{B}(z_0, r) \setminus \{z_0\} \to \overline{\mathbb{C}} \setminus \{z_1, z_2, \cdots, z_n\}$ 的本性奇点, 则 z_0 也是 $R(f(z))$ 的本性奇点.

证明 由于 z_0 是 $f(z)$ 的本性奇点, 取互异的 $A, B \in \mathbb{C} \setminus \{z_1, z_2, \cdots, z_n\}$ 满足 $R(A) \neq R(B)$, 则存在两列点列 $a_n \to z_0$ 与 $b_n \to z_0$ 使得 $f(a_n) \to A$ 且 $f(b_n) \to B$. 此时 $R(f(a_n)) \to A$ 而 $R(f(b_n)) \to B$, 二者不等, 因此 z_0 是 $R(f(z))$ 的本性奇点. □

习题 5.2.8 设 f 在 $\mathbb{B}(0, R) \setminus \{0\}$ 上全纯. 若 $\operatorname{Re} f(z) > 0, \forall z \in \mathbb{B}(0, R) \setminus \{0\}$, 则 0 是 f 的可去奇点.

证明 由 $\operatorname{Re} f(z) > 0, \forall z \in \mathbb{B}(0, R) \setminus \{0\}$ 可见 0 不是 f 的本性奇点. 故只需证 0 不是 f 的极点. 用反证法, 若 0 是 f 的极点, 设 $g(z) = \dfrac{1}{f(z)}$, 则 $g(0) = 0$. 而对 $z \in \mathbb{B}(0, R) \setminus \{0\}$, 由 $\operatorname{Re} f(z) > 0$ 可知 $\operatorname{Re} g(z) > 0$, 由平均值公式, 当 $r \in (0, R)$ 时,

$$0 = \operatorname{Re} g(0) = \operatorname{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} g\left(r\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{d}\theta\right\} = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} g\left(r\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{d}\theta > 0,$$

矛盾. 故 0 不是 f 的极点, 从而 0 是 f 的可去奇点. □

习题 5.4.1 证明: 留数定理与 Cauchy 积分公式等价.

定理 1 (留数定理) 设 γ 是可求长 Jordan 曲线, 函数 $f(z)$ 在 γ 内部 D 中除去 $z_1, z_2, \cdots, z_n$ 外全纯, 且在 $\overline{D} \setminus \{z_1, z_2, \cdots, z_n\}$ 上连续, 则

$$\int\limits_{\gamma} f(z)\, \mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

定理 2 (Cauchy 积分公式) 设区域 D 是可求长 Jordan 曲线 γ 的内部, $f(z)\in\mathcal{H}(D)\cap\mathcal{C}(\overline{D})$, 则

(1) 在 D 内 $f(z)=\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_\gamma\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta$.

(2) $f(z)$ 在 D 内有各阶导数, 且在 D 内 $f^{(n)}(z)=\dfrac{n!}{2\pi\mathrm{i}}\displaystyle\int_\gamma\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta\ (n=1,2,\cdots)$.

证明 $\boxed{(1)\Rightarrow(2)}$ 对 $n\geqslant 0$, $\zeta=z$ 是 $\dfrac{f(\zeta)}{(\zeta-z)^{n+1}}$ 的 $n+1$ 阶极点, 由留数定理,

$$\int_\gamma\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta=2\pi\mathrm{i}\operatorname{Res}\Big(\tfrac{f(\zeta)}{(\zeta-z)^{n+1}},z\Big)=\frac{2\pi\mathrm{i}}{n!}\lim_{\zeta\to z}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[(\zeta-z)^{n+1}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\right]=\frac{2\pi\mathrm{i}f^{(n)}(z)}{n!}.$$

$\boxed{(2)\Rightarrow(1)}$ 由多连通域的 Cauchy 定理, 不妨设 $f(z)$ 在 D 中只有 1 个奇点 a, 并设 f 在 a 的邻域内有 Laurent 展开 $f(z)=\displaystyle\sum_{n=-\infty}^{+\infty}c_n(z-a)^n$. 由 Cauchy 积分公式,

$$\int_\gamma f(z)\,\mathrm{d}z=\int_\gamma\sum_{n=-\infty}^{+\infty}c_n(z-a)^n\,\mathrm{d}z=\sum_{n=-\infty}^{+\infty}\int_\gamma c_n(z-a)^n\,\mathrm{d}z=2\pi\mathrm{i}c_{-1}=2\pi\mathrm{i}\operatorname{Res}(f,a).$$

□

习题 5.4.2 若 a 是 $f\in\mathcal{H}(\mathbb{B}(a,R)\setminus\{a\})$ 的可去奇点, 其中 $a\neq\infty$, 则显然 $\operatorname{Res}(f,a)=0$. 举例说明, 若 ∞ 是 $f\in\mathcal{H}(\mathbb{B}(\infty,R))$ 的可去奇点, 则 $\operatorname{Res}(f,\infty)$ 可能不等于 0.

解答 设 $f(z)=1+\dfrac{1}{z}$, 则 ∞ 是 $f(z)\in\mathcal{H}(\mathbb{B}(\infty,R))$ $(R>0)$ 的可去奇点, 但

$$\operatorname{Res}(f,\infty)=-\frac{1}{2\pi\mathrm{i}}\int_{|z|=1}\left(1+\frac{1}{z}\right)\mathrm{d}z=-1.$$

□

习题 5.4.3 设 $f\in\mathcal{H}(\mathbb{B}(\infty,R))$. 证明:

(1) 若 ∞ 是 f 的可去奇点, 则 $\operatorname{Res}(f,\infty)=\lim\limits_{z\to\infty}z^2f'(z)$.

(2) 若 ∞ 是 f 的 m 阶极点, 则 $\operatorname{Res}(f,\infty)=\dfrac{(-1)^m}{(m+1)!}\lim\limits_{z\to\infty}z^{m+2}f^{(m+1)}(z)$.

证明 (1) 若 ∞ 是 f 的可去奇点, 可设

$$f(z)=\sum_{n=0}^{\infty}\frac{c_n}{z^n},\quad z\in\mathbb{B}(\infty,R).$$

于是

$$\operatorname{Res}(f,\infty)\xlongequal{\rho>R}-\frac{1}{2\pi\mathrm{i}}\int_{|z|=\rho}\sum_{n=0}^{\infty}\frac{c_n}{z^n}\,\mathrm{d}z=-c_1=\lim_{z\to\infty}\sum_{n=1}^{\infty}\frac{-nc_n}{z^{n-1}}=\lim_{z\to\infty}z^2f'(z).$$

(2) 若 ∞ 是 f 的 m 阶极点, 可设

$$f(z)=\sum_{n=-m}^{\infty}\frac{c_n}{z^n},\quad z\in\mathbb{B}(\infty,R).$$

于是

$$\operatorname{Res}(f,\infty) \xlongequal{\rho>R} -\frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=\rho} \sum_{n=-m}^{\infty} \frac{c_n}{z^n}\,\mathrm{d}z = -c_1.$$

而

$$f^{(m+1)}(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\left(\sum_{n=1}^{\infty} \frac{c_n}{z^n}\right) = (-1)^{m+1} \sum_{n=1}^{\infty} \frac{(n+m)!}{(n-1)!} \cdot \frac{c_n}{z^{n+m+1}},$$

因此

$$\frac{(-1)^m}{(m+1)!} \lim_{z\to\infty} z^{m+2} f^{(m+1)}(z) = -\frac{1}{(m+1)!} \lim_{z\to\infty} \sum_{n=1}^{\infty} \frac{(n+m)!}{(n-1)!} \cdot \frac{c_n}{z^{n-1}} = -c_1 = \operatorname{Res}(f,\infty).$$

□

习题 5.4.4 设 $f, g \in \mathcal{H}(\mathbb{B}(a,r))$, $f(a) \neq 0$, a 是 g 的 2 阶零点, 计算 $\operatorname{Res}\left(\frac{f}{g}, a\right)$.

解答 设

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n, \quad z \in \mathbb{B}(a,r),$$

其中

$$a_n = \frac{f^{(n)}(a)}{n!}, a_0 \neq 0, \quad b_n = \frac{g^{(n)}(a)}{n!}, b_0 = b_1 = 0, b_2 \neq 0.$$

设 $h(z) = \dfrac{g(z)}{(z-a)^2} = \displaystyle\sum_{n=2}^{\infty} b_n (z-a)^{n-2}$. 由于 $h(a) = b_2 \neq 0$, 在 a 的邻域内 $g(z) \neq 0$, 此时

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)^2 h(z)}$$

以 a 为 2 阶极点, 因此

$$\begin{aligned} \operatorname{Res}\left(\frac{f}{g}, a\right) &= \lim_{z\to a} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z-a)^2 \frac{f(z)}{g(z)}\right] = \lim_{z\to a} \left(\frac{f(z)}{h(z)}\right)' \\ &= \lim_{z\to a} \frac{f'(z)h(z) - f(z)h'(z)}{h^2(z)} = \frac{f'(a)h(a) - f(a)h'(a)}{h^2(a)}, \end{aligned}$$

代入 $h(a) = b_2 = \dfrac{g''(a)}{2}$ 与 $h'(a) = b_3 = \dfrac{g'''(a)}{6}$ 即得

$$\operatorname{Res}\left(\frac{f}{g}, a\right) = \frac{f'(a)\frac{g''(a)}{2} - f(a)\frac{g'''(a)}{6}}{\left(\frac{g''(a)}{2}\right)^2} = \frac{6f'(a)g''(a) - 2f(a)g'''(a)}{3[g''(a)]^2}.$$

□

习题 5.4.8 指出下列初等函数在 $\overline{\mathbb{C}}$ 中的全部孤立奇点, 并求出这些初等函数在它们各自孤立奇点处的留数:

(1) $\dfrac{1}{z^3 - z^5}$.

(2) $\dfrac{z^3 + z^2 + 2}{z(z^2-1)^2}$.

(3) $\dfrac{z^2 + z - 1}{z^2(z-1)}$.

(4) $\dfrac{z^{n-1}}{z^n+a^n}$ $(a\neq 0, n\in\mathbb{N})$.

(5) $\dfrac{1}{\sin z}$.

(6) $\sin\dfrac{z}{z+1}$.

(7) $\dfrac{\mathrm{e}^z}{z(z-1)}$.

(8) $\dfrac{\mathrm{e}^{\pi z}}{z^2+1}$.

解答 将每问中的函数记为 $f(z)$.

(1) 孤立奇点为 $0,1,-1,\infty$.

① 由 $\dfrac{1}{z^3-z^5}=\dfrac{1}{z^3(1-z^2)}=\dfrac{1}{z^3}\big(1+z^2+z^4+z^6+\cdots\big)$ 知 $\mathrm{Res}(f,0)=1$.

② 1 为 1 阶极点, 因此 $\mathrm{Res}(f,1)=\lim\limits_{z\to 1}\dfrac{z-1}{z^3-z^5}=\lim\limits_{z\to 1}\dfrac{-1}{z^3(1+z)}=-\dfrac{1}{2}$.

③ -1 为 1 阶极点, 因此 $\mathrm{Res}(f,-1)=\lim\limits_{z\to -1}\dfrac{z+1}{z^3-z^5}=\lim\limits_{z\to -1}\dfrac{1}{z^3(1-z)}=-\dfrac{1}{2}$.

④ $\mathrm{Res}(f,\infty)=-\left(1-\dfrac{1}{2}-\dfrac{1}{2}\right)=0$.

(2) 孤立奇点为 $0,1,-1,\infty$.

① 0 为 1 阶极点, 因此 $\mathrm{Res}(f,0)=\lim\limits_{z\to 0}\dfrac{z^3+z^2+2}{(z^2-1)^2}=2$.

② 1 为 2 阶极点, 因此 $\mathrm{Res}(f,1)=\dfrac{1}{1!}\lim\limits_{z\to 1}\left(\dfrac{z^3+z^2+2}{z(z+1)^2}\right)'=\lim\limits_{z\to 1}\dfrac{z^4+2z^3-5z^2-8z-2}{z^2(z+1)^4}=-\dfrac{3}{4}$.

③ -1 为 2 阶极点, 因此 $\mathrm{Res}(f,1)=\dfrac{1}{1!}\lim\limits_{z\to -1}\left(\dfrac{z^3+z^2+2}{z(z-1)^2}\right)'=-\dfrac{5}{4}$.

④ $\mathrm{Res}(f,\infty)=-\left(2-\dfrac{3}{4}-\dfrac{5}{4}\right)=0$.

(3) 孤立奇点为 $0,1,\infty$.

① 0 为 2 阶极点, 因此 $\mathrm{Res}(f,0)=\dfrac{1}{1!}\lim\limits_{z\to 0}\left(\dfrac{z^2+z-1}{z-1}\right)'=0$.

② 1 为 1 阶极点, 因此 $\mathrm{Res}(f,1)=\lim\limits_{z\to 1}\dfrac{z^2+z-1}{z^2}=1$.

③ $\mathrm{Res}(f,\infty)=-(0+1)=-1$.

(4) 孤立奇点为 $a(-1)^{\frac{1}{n}}=a\mathrm{e}^{\frac{\mathrm{i}(2k+1)\pi}{n}}$ $(k=0,1,\cdots,n-1)$ 及 ∞. 由于 $z_k=a\mathrm{e}^{\frac{\mathrm{i}(2k+1)\pi}{n}}$ 是 1 阶极点,

$$\mathrm{Res}(f,z_k)=\lim_{z\to z_k}\frac{z^{n-1}(z-z_k)}{z^n+a^n}=\lim_{z\to z_k}\frac{z^{n-1}}{\frac{z^n+a^n}{z-z_k}}=\frac{z_k^{n-1}}{(z^n+a^n)'|_{z=z_k}}=\frac{1}{n}.$$

由于 ∞ 为可去奇点, $\mathrm{Res}(f,\infty)=-\sum\limits_{k=0}^{n-1}\mathrm{Res}(f,z_k)=-1$.

(5) 孤立奇点为 $k\pi$ $(k\in\mathbb{Z})$. 由于 $k\pi$ 为 1 阶极点, $\mathrm{Res}(f,k\pi)=\lim\limits_{z\to k\pi}\dfrac{z-k\pi}{\sin z}=(-1)^k$.

(6) 孤立奇点为 $-1,\infty$.

① 由于 ∞ 是 f 的可去奇点, 由习题 5.4.3 (1),

$$\operatorname{Res}(f,\infty)=\lim_{z\to\infty}z^2f'(z)=\lim_{z\to\infty}\frac{z^2}{(1+z)^2}\cos\left(\frac{z}{1+z}\right)=\cos 1.$$

② $\operatorname{Res}(f,-1)=-\operatorname{Res}(f,\infty)=-\cos 1$.

(7) 孤立奇点为 $0,1,\infty$.

① 0 为 1 阶极点, 因此 $\operatorname{Res}(f,0)=\lim\limits_{z\to 0}\dfrac{\mathrm{e}^z}{z-1}=-1$.

② 1 为 1 阶极点, 因此 $\operatorname{Res}(f,1)=\lim\limits_{z\to 1}\dfrac{\mathrm{e}^z}{z}=\mathrm{e}$.

③ $\operatorname{Res}(f,\infty)=-(-1+\mathrm{e})=1-\mathrm{e}$.

(8) 孤立奇点为 $\mathrm{i},-\mathrm{i},\infty$.

① i 为 1 阶极点, 因此 $\operatorname{Res}(f,\mathrm{i})=\lim\limits_{z\to\mathrm{i}}\dfrac{\mathrm{e}^{\pi z}}{(z+\mathrm{i})}=\dfrac{\mathrm{i}}{2}$.

② $-\mathrm{i}$ 为 1 阶极点, 因此 $\operatorname{Res}(f,-\mathrm{i})=\lim\limits_{z\to-\mathrm{i}}\dfrac{\mathrm{e}^{\pi z}}{z-\mathrm{i}}=-\dfrac{\mathrm{i}}{2}$.

③ $\operatorname{Res}(f,\infty)=-\left(\dfrac{\mathrm{i}}{2}-\dfrac{\mathrm{i}}{2}\right)=0$. □

习题 5.4.9 设 $f,g\in\mathcal{H}(\mathbb{B}(0,R))\cap\mathcal{C}\left(\overline{\mathbb{B}(0,R)}\right)$, g 在 $\partial\mathbb{B}(0,R)$ 上无零点, g 在 $\mathbb{B}(0,R)$ 中的全部零点 $z_1,z_2,\cdots,z_n$ 都是 1 阶零点, 求

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=R}\frac{f(z)}{zg(z)}\,\mathrm{d}z.$$

解答 (1) 若 $z_1,z_2,\cdots,z_n\neq 0$.

① 若 $f(z_k)\neq 0$, 则 z_k 为 $\dfrac{f(z)}{zg(z)}$ 的 1 阶极点, $\operatorname{Res}\left(\dfrac{f(z)}{zg(z)},z_k\right)=\lim\limits_{z\to z_k}\dfrac{f(z)}{\frac{zg(z)}{z-z_k}}=\dfrac{f(z_k)}{z_kg'(z_k)}$.

② 若 $f(z_k)=0$, 则 z_k 为 $\dfrac{f(z)}{zg(z)}$ 的可去奇点, $\operatorname{Res}\left(\dfrac{f(z)}{zg(z)},z_k\right)=0$.

③ 对于充分小的 ε, 由 Cauchy 积分公式, $\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int\limits_{|z|=\varepsilon}\dfrac{f(z)}{zg(z)}\,\mathrm{d}z=\dfrac{f(0)}{g(0)}$.

故由留数定理, $\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int\limits_{|z|=R}\dfrac{f(z)}{zg(z)}\,\mathrm{d}z=\dfrac{f(0)}{g(0)}+\sum_{k=1}^{n}\dfrac{f(z_k)}{z_kg'(z_k)}$.

(2) 若 $z_1,z_2,\cdots,z_n$ 中有 0, 不妨设 $z_n=0$.

① 若 0 是 $f(z)$ 的 m 阶零点 $(m\geqslant 2)$, 则 0 是 $\dfrac{f(z)}{zg(z)}$ 的可去奇点, $\operatorname{Res}\left(\dfrac{f(z)}{zg(z)},0\right)=0$.

② 若 0 是 $f(z)$ 的 1 阶零点, 则 0 是 $\dfrac{f(z)}{zg(z)}$ 的 1 阶极点, $\operatorname{Res}\left(\dfrac{f(z)}{zg(z)},0\right)=\lim\limits_{z\to 0}\dfrac{f(z)}{g(z)}=\dfrac{f'(0)}{g'(0)}$.

③ 若 $f(0)\neq 0$, 由于 0 是 $zg(z)$ 的 2 阶零点, 由习题 5.4.4,

$$\operatorname{Res}\left(\frac{f(z)}{zg(z)},0\right)=\left.\frac{6f'(z)(zg(z))''-2f(z)(zg(z))'''}{3[(zg(z))'']^2}\right|_{z=0}=\frac{6f'(0)\cdot 2g'(0)-2f(0)\cdot 3g''(0)}{12[g'(0)]^2}$$

$$= \frac{f'(0)}{g'(0)} - \frac{f(0)g''(0)}{2[g'(0)]^2}.$$

故由留数定理,

$$\frac{1}{2\pi i}\int_{|z|=R} \frac{f(z)}{zg(z)}\,dz = \begin{cases} \displaystyle\sum_{k=1}^{n-1} \frac{f(z_k)}{z_k g'(z_k)}, & 0 \text{ 是 } f(z) \text{ 的 } m \text{ 阶零点}, m \geqslant 2, \\ \displaystyle\frac{f'(0)}{g'(0)} - \frac{f(0)g''(0)}{2[g'(0)]^2} + \sum_{k=1}^{n-1} \frac{f(z_k)}{z_k g'(z_k)}, & \text{其他}. \end{cases}$$

□

习题 5.4.12 设 D 是由有限条可求长简单闭曲线围成的域, $f(z)$ 在 D 上亚纯, 在 D 中的全部彼此不同的极点为 $w_1, w_2, \cdots, w_m$, 其相应的 Laurent 展开式的主要部分为 $f_1(z), f_2(z), \cdots, f_m(z)$, 并且在 $\overline{D} \setminus \{w_1, w_2, \cdots, w_m\}$ 上连续. 证明：对于任意 $z \in D$, 有

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z) - \sum_{j=1}^{m} f_j(z).$$

证明 设 $f_j(z) = \sum_{k=1}^{\infty} \frac{c_{-k}}{(z-w_j)^k}$. 由留数定理, $\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{j=1}^{m} \mathrm{Res}\left(\frac{f(\zeta)}{\zeta - z}, w_j\right) + \mathrm{Res}\left(\frac{f(\zeta)}{\zeta - z}, z\right)$.

(1) 若 $z \notin \{w_1, w_2, \cdots, w_m\}$, 则 z 是 $\frac{f(\zeta)}{\zeta - z}$ 的 1 阶极点, 从而

$$\mathrm{Res}\left(\frac{f(\zeta)}{\zeta - z}, z\right) = \lim_{\zeta \to z}(\zeta - z)\frac{f(\zeta)}{\zeta - z} = f(z),$$
$$\mathrm{Res}\left(\frac{f(\zeta)}{\zeta - z}, w_j\right) = \mathrm{Res}\left(\frac{f_j(\zeta)}{\zeta - z}, w_j\right) = \frac{c_{-1}}{w_j - z} = -f_j(z).$$

因此 $\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z) - \sum_{j=1}^{m} f_j(z)$.

(2) 若 $z \in \{w_1, w_2, \cdots, w_m\}$, 不妨设 $w_m = z$, 则

$$\begin{aligned} \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \sum_{j=1}^{m-1} \mathrm{Res}\left(\frac{f(\zeta)}{\zeta - z}, w_j\right) + \mathrm{Res}\left(\frac{f(\zeta)}{\zeta - z}, z\right) \\ &\overset{(1)}{=\!=} -\sum_{j=1}^{m-1} f_j(z) + \underbrace{\mathrm{Res}\left(\frac{f(\zeta) - f_m(\zeta)}{\zeta - z}, z\right)}_{z \text{ 是其 1 阶极点}} + \mathrm{Res}\left(\frac{f_m(\zeta)}{\zeta - z}, z\right) \\ &= -\sum_{j=1}^{m-1} f_j(z) + \lim_{\zeta \to z}(\zeta - z)\frac{f(\zeta) - f_m(\zeta)}{\zeta - z} + 0 \\ &= f(z) - \sum_{j=1}^{m} f_j(z). \end{aligned}$$

□

习题 5.5.1 利用留数定理和 Cauchy 积分公式计算下列积分：

(1) $\int_0^{+\infty} \frac{x^2+1}{x^4+1}\,dx$.

(4) $\int_0^{2\pi} \frac{1}{a + b\cos\theta}\,\mathrm{d}\theta\ (0 < b < a)$.

(9) $\int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x$.

(17) $\int_{-1}^{1} \frac{\sqrt[4]{(1-x)^3(1+x)}}{1+x^2}\,\mathrm{d}x$.

(21) $\int_0^{+\infty} \frac{\log x}{x^2 - 1}\,\mathrm{d}x$.

(24) $\int_0^{+\infty} \frac{\sin x}{\mathrm{e}^x - 1}\,\mathrm{d}x$.

(28) $\int_0^{+\infty} \mathrm{e}^{-ax^2}\cos\left(bx^2\right)\mathrm{d}x\ (a > 0)$.

(29) $\int_0^{\frac{\pi}{2}} \log\sin\theta\,\mathrm{d}\theta$.

解答 (1) 由于 $\gcd\left(x^2+1, x^4+1\right) = 1$, x^4+1 无实根, 在上半平面中有根 $a_1 = \zeta_8, a_2 = \zeta_8^3$, 且 $\deg\left(x^4+1\right) - \deg\left(x^2+1\right) = 2$, 因此

$$\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{k=1}^{2} \mathrm{Res}\left(\frac{z^2+1}{z^4+1}, a_k\right),$$

其中

$$\begin{aligned}
\mathrm{Res}\left(\frac{z^2+1}{z^4+1}, a_1\right) &= \lim_{z\to\zeta_8} \frac{\left(z^2+1\right)\left(z-\zeta_8\right)}{z^4+1} = \lim_{z\to\zeta_8} \frac{z^2+1}{\frac{z^4+1}{z-\zeta_8}} = \left.\frac{z^2+1}{\left(z^4+1\right)'}\right|_{z=\zeta_8} = -\frac{\mathrm{i}}{2\sqrt{2}},\\
\mathrm{Res}\left(\frac{z^2+1}{z^4+1}, a_2\right) &= \lim_{z\to\zeta_8^3} \frac{\left(z^2+1\right)\left(z-\zeta_8^3\right)}{z^4+1} = \lim_{z\to\zeta_8^3} \frac{z^2+1}{\frac{z^4+1}{z-\zeta_8^3}} = \left.\frac{z^2+1}{\left(z^4+1\right)'}\right|_{z=\zeta_8^3} = -\frac{\mathrm{i}}{2\sqrt{2}}.
\end{aligned}$$

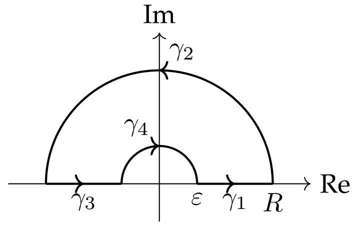
故

$$\int_0^{+\infty} \frac{x^2+1}{x^4+1}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

(4) 令 $z = \mathrm{e}^{\mathrm{i}\theta}$, 则 $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\mathrm{d}z = \mathrm{i}z\,\mathrm{d}\theta$, 从而

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a+b\cos\theta}\,\mathrm{d}\theta &= \int_{|z|=1} \frac{\mathrm{d}z}{\mathrm{i}z\left[a + \frac{b}{2}\left(z+\frac{1}{z}\right)\right]} = \frac{2}{b\mathrm{i}}\int_{|z|=1} \frac{\mathrm{d}z}{z^2 + \frac{2a}{b}z + 1}\\
&= \frac{2}{b\mathrm{i}}\cdot 2\pi\mathrm{i}\,\mathrm{Res}\left(\frac{1}{z^2+\frac{2a}{b}z+1}, \frac{-a+\sqrt{a^2-b^2}}{b}\right)\\
&= \frac{4\pi}{b}\cdot\frac{1}{\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right) - \left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)}\\
&= \frac{2\pi}{\sqrt{a^2-b^2}}.
\end{aligned}$$

(9) 选取如图积分路径.



设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ 所围区域为 D. 由 $\dfrac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\in\mathcal{H}(D)$ 知 $\displaystyle\int_{\partial D}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z=0$. 我们有

$$\mathrm{Re}\left\{\int_{\gamma_1}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z\right\}=\int_{\gamma_1}\mathrm{Re}\left\{\frac{\cos 2x+\mathrm{i}\sin 2x-1}{x^2}\right\}\mathrm{d}x\xrightarrow[R\to+\infty]{\varepsilon\to0^+}-2\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x,$$

$$\mathrm{Re}\left\{\int_{\gamma_3}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z\right\}=\int_{\gamma_3}\mathrm{Re}\left\{\frac{\cos 2x+\mathrm{i}\sin 2x-1}{x^2}\right\}\mathrm{d}x\xrightarrow[R\to+\infty]{\varepsilon\to0^+}-2\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x,$$

$$\left|\int_{\gamma_2}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z\right|\leqslant\int_0^{\pi}\frac{\left|\mathrm{e}^{2\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}}\right|+1}{R}\,\mathrm{d}\theta=\int_0^{\pi}\frac{\mathrm{e}^{-2R\sin\theta}+1}{R}\,\mathrm{d}\theta\leqslant\frac{2\pi}{R}\xrightarrow{R\to+\infty}0.$$

以及

$$\int_{\gamma_4}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z=\int_{\gamma_4}\sum_{k=1}^{\infty}\frac{(2\mathrm{i}z)^k}{k!}\cdot z^{-2}\,\mathrm{d}z=\int_{\gamma_4}\frac{2\mathrm{i}}{z}\,\mathrm{d}z+\int_{\gamma_4}\sum_{k=0}^{\infty}\frac{(2\mathrm{i})^{k+2}z^k}{(k+2)!}\,\mathrm{d}z,$$

其中

$$\int_{\gamma_4}\frac{2\mathrm{i}}{z}\,\mathrm{d}z=\int_{\pi}^{0}\frac{2\mathrm{i}}{\varepsilon\mathrm{e}^{\mathrm{i}\theta}}\cdot\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=2\pi,$$

$$\left|\int_{\gamma_4}\sum_{k=0}^{\infty}\frac{(2\mathrm{i})^{k+2}z^k}{(k+2)!}\,\mathrm{d}z\right|\overset{\varepsilon<1}{\leqslant}\pi\varepsilon\cdot\underbrace{\max_{|z|\leqslant1}\left|\sum_{k=0}^{\infty}\frac{(2\mathrm{i})^{k+2}z^k}{(k+2)!}\right|}_{<+\infty}\xrightarrow{\varepsilon\to0^+}0.$$

故令 $\varepsilon\to0^+, R\to+\infty$ 就得到

$$0=\mathrm{Re}\left\{\int_{\partial D}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z\right\}=-4\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x+2\pi,$$

即

$$\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x=\frac{\pi}{2}.$$

(17) 令 $f(z)=\dfrac{1}{1+z^2}, r=\dfrac{1}{4}, s=\dfrac{3}{4}$, 则 $r+s=1\in\mathbb{Z}$, $f(z)$ 在 $\mathbb{C}$ 中仅有极点 $a_1=\mathrm{i}, a_2=-\mathrm{i}$, 且 $\displaystyle\lim_{z\to\infty}z^{r+s+1}f(z)=\lim_{z\to\infty}\frac{z^2}{1+z^2}=1$, 由定理 5.5.14,

$$\int_{-1}^{1}(x+1)^r(1-x)^s f(x)\,\mathrm{d}x=-\frac{\pi}{\sin s\pi}+\frac{\pi}{\mathrm{e}^{-s\pi\mathrm{i}}\sin s\pi}\sum_{k=1}^{2}\mathrm{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2},a_k\right).$$

而

$$\operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2},\mathrm{i}\right)=\lim_{z\to\mathrm{i}}\frac{\sqrt[4]{(1-z)^3(1+z)}}{z+\mathrm{i}}=\frac{1}{2\mathrm{i}}\lim_{z\to\mathrm{i}}\sqrt[4]{(1-z)^3(1+z)},$$

$$\operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2},-\mathrm{i}\right)=\lim_{z\to-\mathrm{i}}\frac{\sqrt[4]{(1-z)^3(1+z)}}{z-\mathrm{i}}=\frac{1}{-2\mathrm{i}}\lim_{z\to-\mathrm{i}}\sqrt[4]{(1-z)^3(1+z)},$$

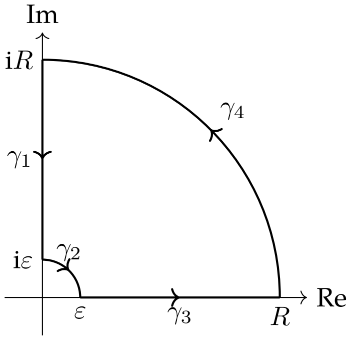
由习题 2.4.27 即得

$$\begin{aligned}&\operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2},\mathrm{i}\right)+\operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2},-\mathrm{i}\right)\\=&\frac{1}{2\mathrm{i}}\left(\lim_{z\to\mathrm{i}}\sqrt[4]{(1-z)^3(1+z)}-\lim_{z\to-\mathrm{i}}\sqrt[4]{(1-z)^3(1+z)}\right)=\frac{\sqrt{2}}{2\mathrm{i}}\left(\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}-\mathrm{e}^{\frac{5\pi}{8}\mathrm{i}}\right).\end{aligned}$$

故

$$\int_{-1}^{1}\frac{\sqrt[4]{(1-x)^3(1+x)}}{1+x^2}\,\mathrm{d}x=-\frac{\pi}{\sin\frac{3\pi}{4}}+\frac{\pi}{\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}\cdot\frac{1}{\sqrt{2}\mathrm{i}}\left(\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}-\mathrm{e}^{\frac{5\pi}{8}\mathrm{i}}\right)=\left(\sqrt{2+\sqrt{2}}-\sqrt{2}\right)\pi.$$

(21) 选取如图积分路径.



设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ 所围区域为 D. 由 $\dfrac{\log z}{z^2-1}\in\mathcal{H}(D)$ 知 $\displaystyle\int_{\partial D}\frac{\log z}{z^2-1}\,\mathrm{d}z=0$ (注意 1 是 $\dfrac{\log z}{z^2-1}$ 的可去奇点).

① 在 γ_1 上,

$$\begin{aligned}\operatorname{Re}\left\{\int_{\gamma_1}\frac{\log z}{z^2-1}\,\mathrm{d}z\right\}&=-\int_{\varepsilon}^{R}\operatorname{Im}\left\{\frac{\log(\mathrm{i}t)}{t^2+1}\right\}\mathrm{d}t=-\int_{\varepsilon}^{R}\frac{\frac{\pi}{2}}{t^2+1}\,\mathrm{d}t\xrightarrow[R\to+\infty]{\varepsilon\to0^+}-\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}t}{t^2+1}\\&=-\frac{\pi^2}{4}.\end{aligned}$$

② 在 γ_2 上, 由于 $\displaystyle\lim_{z\to0}\frac{z\log z}{z^2-1}=0$, 若记 $M(\varepsilon)=\displaystyle\max_{\gamma_2(\varepsilon)}\left|\frac{z\log z}{z^2-1}\right|$, 则 $\displaystyle\lim_{\varepsilon\to0^+}M(\varepsilon)=0$. 当 $z=\varepsilon\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z=\mathrm{i}z\,\mathrm{d}\theta$, 因此

$$\left|\int_{\gamma_2}\frac{\log z}{z^2-1}\,\mathrm{d}z\right|=\left|\int_{\gamma_2}\frac{\frac{z\log z}{z^2-1}}{z}\,\mathrm{d}z\right|\leqslant\int_0^{\frac{\pi}{2}}M(\varepsilon)=\frac{\pi}{2}M(\varepsilon)\xrightarrow{\varepsilon\to0^+}0.$$

③ 在 γ_3 上,

$$\int_{\gamma_3}\frac{\log z}{z^2-1}\,\mathrm{d}z=\int_{\varepsilon}^{R}\frac{\log x}{x^2-1}\,\mathrm{d}x\xrightarrow[R\to+\infty]{\varepsilon\to0^+}\int_0^{+\infty}\frac{\log x}{x^2-1}\,\mathrm{d}x.$$

④ 在 γ_4 上, 由于 $\lim\limits_{z\to\infty}\dfrac{\log z}{z^2-1}=0$, 若记 $M(R)=\max\limits_{\gamma_4(R)}\left|\dfrac{\log z}{z^2-1}\right|$, 则 $\lim\limits_{R\to+\infty}M(R)=0$. 当 $z=R\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z=\mathrm{i}z\,\mathrm{d}\theta$, 因此

$$\left|\int_{\gamma_4}\frac{\log z}{z^2-1}\,\mathrm{d}z\right|\leqslant\int_0^{\frac{\pi}{2}}M(R)\,\mathrm{d}\theta=\frac{\pi}{2}M(R)\xrightarrow{R\to+\infty}0.$$

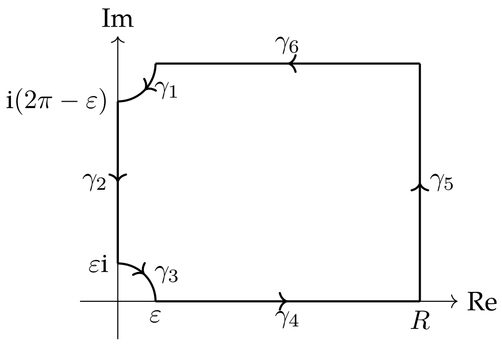
故

$$0=\mathrm{Re}\left\{\int_{\gamma}\frac{\log z}{z^2-1}\,\mathrm{d}z\right\}=-\frac{\pi^2}{4}+\int_0^{+\infty}\frac{\log x}{x^2-1}\,\mathrm{d}x,$$

即

$$\int_0^{+\infty}\frac{\log x}{x^2-1}\,\mathrm{d}x=\frac{\pi^2}{4}.$$

(24) 选取如图积分路径.



设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4\cup\gamma_5\cup\gamma_6$ 所围区域为 D. 由 $\dfrac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\in\mathcal{H}(D)$ 知 $\displaystyle\int_{\partial D}\frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z=0$. 我们有

① 在 γ_1 上, 由于 $\lim\limits_{z\to2\pi\mathrm{i}}(z-2\pi\mathrm{i})\dfrac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}=\lim\limits_{z\to2\pi\mathrm{i}}\dfrac{\mathrm{e}^{\mathrm{i}z}}{\frac{\mathrm{e}^z-1}{z-2\pi\mathrm{i}}}=\dfrac{\mathrm{e}^{\mathrm{i}z}}{(\mathrm{e}^z-1)'}\Big|_{z=2\pi\mathrm{i}}=\mathrm{e}^{-2\pi}$, 若记 $M(\varepsilon)=\max\limits_{\gamma_1(\varepsilon)}\left|\dfrac{\mathrm{e}^{\mathrm{i}z}(z-2\pi\mathrm{i})}{\mathrm{e}^z-1}-\mathrm{e}^{-2\pi}\right|$, 则 $\lim\limits_{\varepsilon\to0^+}M(\varepsilon)=0$. 当 $z=2\pi\mathrm{i}+\varepsilon\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z=\mathrm{i}(z-2\pi\mathrm{i})\,\mathrm{d}\theta$, 因此

$$\left|\int_{\gamma_1}\frac{\frac{\mathrm{e}^{\mathrm{i}z}(z-2\pi\mathrm{i})}{\mathrm{e}^z-1}-\mathrm{e}^{-2\pi}}{z-2\pi\mathrm{i}}\,\mathrm{d}z\right|\leqslant\int_{-\frac{\pi}{2}}^{0}M(\varepsilon)\,\mathrm{d}\theta=\frac{\pi}{2}M(\varepsilon)\xrightarrow{\varepsilon\to0^+}0,$$

即

$$\lim_{\varepsilon\to0^+}\int_{\gamma_1}\frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z=\lim_{\varepsilon\to0^+}\int_{\gamma_1}\frac{\mathrm{e}^{-2\pi}}{z-2\pi\mathrm{i}}\,\mathrm{d}z=-\frac{\pi\mathrm{i}}{2}\mathrm{e}^{-2\pi}.$$

② 在 γ_2 上,

$$\mathrm{Im}\left\{\int_{\gamma_2}\frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z\right\}=\mathrm{Im}\left\{\int_{2\pi-\varepsilon}^{\varepsilon}\frac{\mathrm{e}^{-t}}{\mathrm{e}^{\mathrm{i}t}-1}\mathrm{i}\,\mathrm{d}t\right\}\xrightarrow{\varepsilon\to0^+}-\int_0^{2\pi}\mathrm{Re}\left\{\frac{\mathrm{e}^{-t}}{\mathrm{e}^{\mathrm{i}t}-1}\right\}\mathrm{d}t$$

$$= \int_0^{2\pi} \frac{\mathrm{e}^{-t}(1-\cos t)}{2-2\cos t}\,\mathrm{d}t = \frac{1-\mathrm{e}^{-2\pi}}{2}.$$

③ 在 γ_3 上, 同 (1) 可得

$$\int\limits_{\gamma_3} \frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z \xrightarrow{\varepsilon\to 0^+} -\frac{\pi\mathrm{i}}{2}.$$

④ 在 γ_4 上,

$$\mathrm{Im}\left\{\int\limits_{\gamma_4} \frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z\right\} \xrightarrow[R\to+\infty]{\varepsilon\to 0^+} \int_0^{+\infty} \mathrm{Im}\left\{\frac{\cos x+\mathrm{i}\sin x}{\mathrm{e}^x-1}\right\}\mathrm{d}x = \int_0^{+\infty} \frac{\sin x}{\mathrm{e}^x-1}\,\mathrm{d}x.$$

⑤ 在 γ_5 上,

$$\left|\int\limits_{\gamma_5} \frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z\right| = \left|\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}(R+\mathrm{i}t)}}{\mathrm{e}^{R+\mathrm{i}t}-1}\,\mathrm{i}\,\mathrm{d}t\right| \leqslant \int_0^{2\pi} \frac{\mathrm{e}^{-t}}{\mathrm{e}^R-1}\,\mathrm{d}t \leqslant \frac{2\pi}{\mathrm{e}^R-1} \xrightarrow{R\to+\infty} 0.$$

⑥ 在 γ_6 上,

$$\mathrm{Im}\left\{\int\limits_{\gamma_6} \frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z\right\} = \mathrm{Im}\left\{\int_R^{\varepsilon} \frac{\mathrm{e}^{\mathrm{i}(x+2\pi\mathrm{i})}}{\mathrm{e}^{x+2\pi\mathrm{i}}-1}\,\mathrm{d}x\right\} \xrightarrow[R\to+\infty]{\varepsilon\to 0^+} -\int_0^{+\infty} \mathrm{Im}\left\{\frac{\mathrm{e}^{\mathrm{i}x-2\pi}}{\mathrm{e}^x-1}\right\}\mathrm{d}x$$
$$= -\mathrm{e}^{-2\pi}\int_0^{+\infty} \frac{\sin x}{\mathrm{e}^x-1}\,\mathrm{d}x.$$

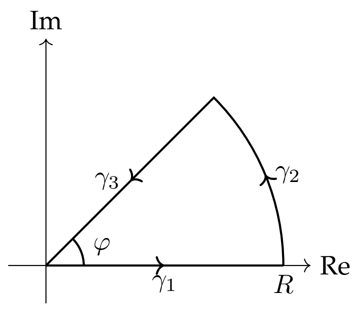
故

$$0 = \mathrm{Im}\left\{\int\limits_{\gamma} \frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z\right\} = -\frac{\pi}{2}\mathrm{e}^{-2\pi} + \frac{1-\mathrm{e}^{-2\pi}}{2} - \frac{\pi}{2} + \left(1-\mathrm{e}^{-2\pi}\right)\int_0^{+\infty} \frac{\sin x}{\mathrm{e}^x-1}\,\mathrm{d}x,$$

即

$$\int_0^{+\infty} \frac{\sin x}{\mathrm{e}^x-1}\,\mathrm{d}x = \frac{\pi}{2}\left(\frac{\mathrm{e}^{2\pi}+1}{\mathrm{e}^{2\pi}-1}\right) - \frac{1}{2}.$$

(28) $\displaystyle\int_0^{+\infty} \mathrm{e}^{-ax^2}\cos\left(bx^2\right)\mathrm{d}x = \mathrm{Re}\left\{\int_0^{+\infty} \mathrm{e}^{(-a+bi)x^2}\right\}$. 下证当 $\mathrm{Re}(c)>0$ 时, $\displaystyle\int_0^{+\infty} \mathrm{e}^{-cz^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{c}}$.



选取如图积分路径. 设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3$ 所围区域为 D. 由 $\mathrm{e}^{-cz^2}\in\mathcal{H}(D)$ 知 $\displaystyle\int\limits_{\partial D} \mathrm{e}^{-cz^2}\,\mathrm{d}z = 0$.

① 在 γ_1 上, $\displaystyle\int_{\gamma_1} \mathrm{e}^{-cz^2}\,\mathrm{d}z \xrightarrow{R\to+\infty} \int_0^{+\infty} \mathrm{e}^{-cx^2}\,\mathrm{d}x$.

② 在 γ_2 上, 由于 $\lim\limits_{z\to\infty} z\mathrm{e}^{-cz^2}=0$, 若记 $M(R)=\max\limits_{\gamma_2(R)}\left|z\mathrm{e}^{-cz^2}\right|$, 则 $\lim\limits_{R\to+\infty} M(R)=0$. 当 $z=R\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z=\mathrm{i}z\,\mathrm{d}\theta$, 因此

$$\left|\int_{\gamma_2} \mathrm{e}^{-cz^2}\,\mathrm{d}z\right| = \left|\int_{\gamma_2} \frac{z\mathrm{e}^{-cz^2}}{z}\,\mathrm{d}z\right| \leqslant \int_0^{\varphi} M(R)\,\mathrm{d}\theta = \varphi M(R) \xrightarrow{R\to+\infty} 0.$$

③ 在 $\gamma_3 : z=kt$ (待定 $k\in\mathbb{C}$ 于第一象限) 上,

$$\int_{\gamma_3} \mathrm{e}^{-cz^2}\,\mathrm{d}z \xrightarrow{R\to+\infty} \int_{+\infty}^0 \mathrm{e}^{-ck^2t^2} k\,\mathrm{d}t = -k\int_0^{+\infty} \mathrm{e}^{-ck^2t^2}\,\mathrm{d}t.$$

取 $k=\dfrac{1}{\sqrt{c}}$, 则

$$\int_{\gamma_3} \mathrm{e}^{-cz^2}\,\mathrm{d}z \xrightarrow{R\to+\infty} -\frac{1}{\sqrt{c}}\int_0^{+\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = -\frac{1}{\sqrt{c}}\cdot\frac{\sqrt{\pi}}{2}.$$

故

$$\int_0^{+\infty} \mathrm{e}^{-cx^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{c}}.$$

利用此结论即得

$$\int_0^{+\infty} \mathrm{e}^{-(a+b\mathrm{i})x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a-b\mathrm{i}}} = \frac{1}{2}\sqrt{\frac{\pi}{a^2+b^2}}\cdot\sqrt{a+b\mathrm{i}}.$$

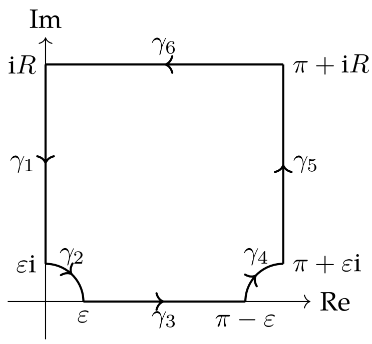
设 $\sqrt{a+b\mathrm{i}}=u+\mathrm{i}v$ $(u,v\in\mathbb{R})$, 则 $a=u^2-v^2$ 且 $b=2uv$, 从而

$$a=u^2-\left(\frac{b}{2u}\right)^2 \implies u^2=\frac{a+\sqrt{a^2+b^2}}{2} \xRightarrow{\text{不妨设 } b\geqslant 0} u=\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}},$$

故

$$\int_0^{+\infty} \mathrm{e}^{-ax^2}\cos(bx^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a^2+b^2}}\cdot u = \frac{\sqrt{2\pi}}{4}\sqrt{\frac{\sqrt{a^2+b^2}+a}{a^2+b^2}}.$$

(29) 选取如图积分路径.



设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4\cup\gamma_5\cup\gamma_6$ 所围区域为 D. 由于 $\sin z$ 在 $\mathbb{C}$ 中零点为 $k\pi$ $(k\in\mathbb{Z})$, 因此 $\log\sin z\in\mathcal{H}(D)$, $\displaystyle\int_{\partial D} \log\sin z\,\mathrm{d}z=0$.

① 在 $\gamma_1: z=\mathrm{i}t$ 上, $\sin(\mathrm{i}t)=\dfrac{\mathrm{i}(\mathrm{e}^t-\mathrm{e}^{-t})}{2}$, 因此

$$\operatorname{Re}\left\{\int_{\gamma_1}\log\sin z\,\mathrm{d}z\right\}=\operatorname{Re}\left\{\int_R^{\varepsilon}\log\sin(\mathrm{i}t)\mathrm{i}\,\mathrm{d}t\right\}=\int_{\varepsilon}^{R}\operatorname{Im}(\log\sin(\mathrm{i}t))\,\mathrm{d}t=\int_{\varepsilon}^{R}\frac{\pi}{2}\,\mathrm{d}t=\frac{\pi}{2}(R-\varepsilon).$$

② 在 γ_2 上, 由于 $\lim\limits_{z\to0}z\log\sin z=0$, 若记 $M(\varepsilon)=\max\limits_{\gamma_2(\varepsilon)}|z\log\sin z|$, 则 $\lim\limits_{\varepsilon\to0^+}M(\varepsilon)=0$. 当 $z=\varepsilon\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z=\mathrm{i}z\,\mathrm{d}\theta$, 因此

$$\left|\int_{\gamma_2}\log\sin z\,\mathrm{d}z\right|=\left|\int_{\gamma_2}\frac{z\log\sin z}{z}\,\mathrm{d}z\right|\leqslant\int_0^{\frac{\pi}{2}}M(\varepsilon)\,\mathrm{d}\theta=\frac{\pi}{2}M(\varepsilon)\xrightarrow{\varepsilon\to0^+}0.$$

③ 在 γ_3 上,

$$\int_{\gamma_3}\log\sin z\,\mathrm{d}z=\int_{\varepsilon}^{\pi-\varepsilon}\log\sin x\,\mathrm{d}x\xrightarrow{\varepsilon\to0^+}\int_0^{\pi}\log\sin x\,\mathrm{d}x.$$

④ 在 γ_4 上, 由于 $\lim\limits_{z\to\pi}(z-\pi)\log\sin z=0$, 同 (2) 可得

$$\left|\int_{\gamma_4}\frac{\log\sin z}{z}\right|\xrightarrow{\varepsilon\to0^+}0.$$

⑤ 在 $\gamma_5: z=\pi+\mathrm{i}t$ 上, $\sin(\pi+\mathrm{i}t)=-\dfrac{\mathrm{i}}{2}(\mathrm{e}^t-\mathrm{e}^{-t})$, 因此

$$\begin{aligned}\operatorname{Re}\left\{\int_{\gamma_5}\log\sin z\,\mathrm{d}z\right\}&=\operatorname{Re}\left\{\int_{\varepsilon}^{R}\log\sin(\pi+\mathrm{i}t)\mathrm{i}\,\mathrm{d}t\right\}=-\int_{\varepsilon}^{R}\operatorname{Im}(\log\sin(\pi+\mathrm{i}t))\,\mathrm{d}t\\&=-\int_{\varepsilon}^{R}-\frac{\pi}{2}\,\mathrm{d}t=\frac{\pi}{2}(R-\varepsilon).\end{aligned}$$

⑥ 在 $\gamma_6: z=t+\mathrm{i}R$ 上, 由

$$\sin(t+\mathrm{i}R)=\frac{1}{2}\left(\mathrm{e}^{-R}+\mathrm{e}^{R}\right)\sin t+\mathrm{i}\cdot\frac{1}{2}\left(\mathrm{e}^{R}-\mathrm{e}^{-R}\right)\cos t$$

可知

$$\begin{aligned}|\sin(t+\mathrm{i}R)|^2&=\frac{1}{4}\left(\mathrm{e}^{-R}+\mathrm{e}^{R}\right)^2\sin^2t+\frac{1}{4}\left(\mathrm{e}^{R}-\mathrm{e}^{-R}\right)^2\cos^2t\\&=\frac{1}{4}\left[\left(\mathrm{e}^{2R}+\mathrm{e}^{-2R}\right)\left(\sin^2t+\cos^2t\right)+2\left(\sin^2t-\cos^2t\right)\right]\\&=\frac{1}{4}\mathrm{e}^{2R}(1+\mu(R)),\end{aligned}$$

其中 $\lim\limits_{R\to+\infty}\mu(R)=0$. 于是

$$\log|\sin(t+\mathrm{i}R)|^2=\log\left(\frac{1}{4}\mathrm{e}^{2R}\right)+\log(1+\mu(R))\xrightarrow{R\to+\infty}2R-2\log2,$$

从而

$$\operatorname{Re}\left\{\int\limits_{\gamma_6}\log\sin z\,\mathrm{d}z\right\}=\operatorname{Re}\left\{\int_{\pi}^{0}\log\sin(t+\mathrm{i}R)\,\mathrm{d}t\right\}=-\int_0^{\pi}\log|\sin(t+\mathrm{i}R)|\,\mathrm{d}t$$

$$\xrightarrow{R\to+\infty}-\frac{1}{2}(2R-2\log 2)\pi=\pi(\log 2-R).$$

故

$$0=\operatorname{Re}\left\{\int\limits_{\gamma}\log\sin z\,\mathrm{d}z\right\}\xrightarrow[R\to+\infty]{\varepsilon\to0^+}\int_0^{\pi}\log\sin x\,\mathrm{d}x+\pi\log 2,$$

即

$$\int_0^{\frac{\pi}{2}}\log\sin x\,\mathrm{d}x=\frac{1}{2}\int_0^{\pi}\log\sin x\,\mathrm{d}x=-\frac{\pi}{2}\log 2.$$

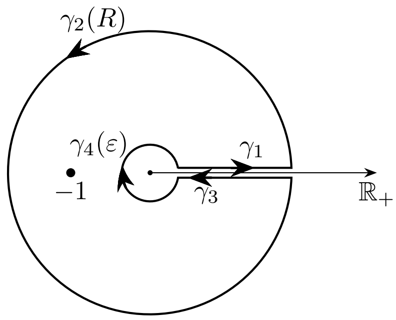
□

习题 5.5.2 设 $f(z)$ 是有理函数, 在 $[0,+\infty)$ 上无极点, 并且 ∞ 是 $f(z)$ 的零点. 证明:

$$\int_0^{+\infty}\frac{f(x)}{(\log x)^2+\pi^2}\,\mathrm{d}x=\sum_{k=1}^{n}\operatorname{Res}\left(\frac{f(z)}{\operatorname{Log} z-\pi\mathrm{i}},a_k\right),$$

其中 $a_1=-1,a_2,a_3,\cdots,a_n$ 是 $f(z)$ 在 $\mathbb{C}$ 中的全部彼此不同的极点, $\operatorname{Log} z=\log|z|+\mathrm{i}\operatorname{Arg} z, 0<\operatorname{Arg} z<2\pi, z\in\mathbb{C}\setminus[0,+\infty)$.

证明 选取如图“锁钥”路径.



设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ 包含 $f(z)$ 的全部极点. 由留数定理,

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}\frac{f(z)}{\operatorname{Log} z-\pi\mathrm{i}}\,\mathrm{d}z=\sum_{k=1}^{n}\operatorname{Res}\left(\frac{f(z)}{\operatorname{Log} z-\pi\mathrm{i}},a_k\right).$$

我们有

$$\int\limits_{\gamma_1}\frac{f(z)}{\operatorname{Log} z-\pi\mathrm{i}}\,\mathrm{d}z=\int_{\varepsilon}^{R}\frac{f(z)}{\log x-\pi\mathrm{i}}\,\mathrm{d}x\xrightarrow[\varepsilon\to0^+]{R\to+\infty}\int_0^{+\infty}\frac{f(x)}{\log x-\pi\mathrm{i}}\,\mathrm{d}x,$$

$$\left|\int\limits_{\gamma_2}\frac{f(z)}{\operatorname{Log} z-\pi\mathrm{i}}\,\mathrm{d}z\right|\leqslant\int\limits_{\gamma_2}\frac{|f(z)||\,\mathrm{d}z|}{|\operatorname{Log} z-\pi\mathrm{i}|}\leqslant\int\limits_{\gamma_2}\frac{|f(z)|}{\log R}|\,\mathrm{d}z|\leqslant\frac{2\pi R\max\limits_{|z|=R}|f(z)|}{\log R}\xrightarrow[f\text{ 有理},\ f(\infty)=0]{R\to+\infty}0,$$

$$\int\limits_{\gamma_3} \frac{f(z)}{\operatorname{Log} z - \pi\mathrm{i}}\,\mathrm{d}z = \int_R^{\varepsilon} \frac{f(x)}{\log x + 2\pi\mathrm{i} - \pi\mathrm{i}}\,\mathrm{d}x \xrightarrow[\varepsilon\to 0^+]{R\to+\infty} -\int_0^{+\infty} \frac{f(x)}{\log x + \pi\mathrm{i}}\,\mathrm{d}x,$$

$$\left|\int\limits_{\gamma_4} \frac{f(z)}{\operatorname{Log} z - \pi\mathrm{i}}\,\mathrm{d}z\right| \leqslant \int\limits_{\gamma_4} \frac{|f(z)||\,\mathrm{d}z|}{|\operatorname{Log} z - \pi\mathrm{i}|} \leqslant \frac{2\pi\varepsilon \max\limits_{|z|=\varepsilon}|f(z)|}{|\log\varepsilon|} \xrightarrow{\varepsilon\to 0^+} 0.$$

因此在 $\varepsilon \to 0^+, R \to +\infty$ 时就有

$$\frac{1}{2\pi\mathrm{i}}\left\{\int_0^{+\infty} \frac{f(x)}{\log x - \pi\mathrm{i}}\,\mathrm{d}x - \int_0^{+\infty} \frac{f(x)}{\log x + \pi\mathrm{i}}\,\mathrm{d}x\right\} = \sum_{k=1}^{n} \operatorname{Res}\left(\frac{f(z)}{\operatorname{Log} z - \pi\mathrm{i}}, a_k\right),$$

即

$$\int_0^{+\infty} \frac{f(x)}{(\log x)^2 + \pi^2}\,\mathrm{d}x = \sum_{k=1}^{n} \operatorname{Res}\left(\frac{f(z)}{\operatorname{Log} z - \pi\mathrm{i}}, a_k\right).$$

□

习题 6.2.6 证明: $\sum\limits_{n=0}^{\infty} z^{2^n}$ 的收敛圆周上的每个点皆为其和函数的奇异点.

证明 级数收敛半径为 1. 注意到对正整数 k, ℓ, 有

$$\sum_{n=0}^{\infty}\left(\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z\right)^{2^n} = \sum_{n=0}^{k-1}\left(\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z\right)^{2^n} + \sum_{n=k}^{\infty} z^{2^n},$$

因此在收敛圆周上 z 与 $\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z$ 同为奇异点或正则点, 而 1 显然是奇异点, 由 $\left\{\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}}\right\}_{k,\ell\geqslant 1}$ 在收敛圆周上稠密即知收敛圆周上的每个点皆为和函数的奇异点. □

习题 6.2.7 证明: $\sum\limits_{n=0}^{\infty} \dfrac{z^{2^n}}{2^n}$ 的收敛圆周上的每个点皆为其和函数的奇异点.

证明 级数收敛半径为 1. 注意到对正整数 k, ℓ, 有

$$\sum_{n=0}^{\infty}\left(\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z\right)^{2^n} = \sum_{n=0}^{k-1}\left(\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z\right)^{2^n} + \sum_{n=k}^{\infty} \frac{z^{2^n}}{2^n},$$

因此在收敛圆周上 z 与 $\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z$ 同为奇异点或正则点, 而 2 显然是奇异点, 由 $\left\{\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}}\right\}_{k,\ell\geqslant 1}$ 在收敛圆周上稠密即知收敛圆周上的每个点皆为和函数的奇异点. □

习题 6.2.8 设 $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ 的收敛半径为 1, $a_n \in \mathbb{R}$ $(n \geqslant 0)$, $S_n = \sum\limits_{k=0}^{n} a_k$. 证明: 若 $S_n \to \infty$ $(n \to \infty)$, 则 1 是 $f(z)$ 的奇异点. 举例说明, 仅有 $|S_n| \to \infty$ 不能保证 1 是 $f(z)$ 的奇异点.

证明 (1) 若 1 不是 f 的奇异点, 则 f 在 1 的某个邻域中全纯且 $f(1)$ 存在. 由于 f 限制在实轴上为实值函数, 故 $f(1) \in \mathbb{R}$. 考虑 $g(z) = \dfrac{f(z) - f(1)}{1 - z}$, 则由全纯函数的解析性知 g 在 1 处全纯. 由于 $f(z)$ 在单位圆周上必有奇异点, 因此 $g(z)$ 在单位圆周上必有非 1 的奇异点, 从而 $g(z)$ 的的幂级数的收敛半径仍为 1. 而 $g(z)$ 的幂级数为

$$g(z) = \left(\sum_{n=0}^{\infty} a_n z^n - f(1)\right)\left(\sum_{m=0}^{\infty} z^m\right) = \sum_{n=0}^{\infty}[S_n - f(1)]z^n,$$

由于 $S_n - f(1) \to \infty$, 当 n 充分大时 $S_n - f(1) > 0$, 由定理 6.2.4 知 1 是 $g(z)$ 的奇点, 矛盾.

(2) 考虑 $f(z)=\sum_{n=0}^{\infty}(-1)^n(n+1)z^n$, 其收敛半径为 1, $S_n=(-1)^n\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)$. 但当 $|z|<1$ 时,

$$f(z)=\sum_{n=0}^{\infty}\left[(-1)^n z^{n+1}\right]'=\left(z\sum_{n=0}^{\infty}(-z)^n\right)'=\left(\frac{z}{1+z}\right)'=\frac{1}{(1+z)^2},$$

由于 $\left.\frac{1}{(1+z)^2}\right|_{z=1}=\frac{1}{4}$, 因此 1 是 $f(z)$ 的正则点. □

习题 7.1.1 设 $\{f_n\}$ 是域 D 上的全纯函数列, 并且在 D 上内闭一致有界. 证明：若 $\lim_{n\to\infty}f_n(z)$ 在 D 上处处存在, 则 $\{f_n\}$ 在 D 上内闭一致收敛.

证明 记 $f(z)=\lim_{n\to\infty}f_n(z)$. 由于 $\{f_n\}$ 在 D 上内闭一致有界, 由 Montel 定理, $\{f_n\}$ 是正规族. 若 $\{f_n\}$ 在 D 上非内闭一致收敛, 则存在紧集 $K\subset D$ 与子列 $\{f_{n_k}\}$, 使得

$$\sup_{z\in K}|f_{n_k}(z)-f(z)|\geqslant\varepsilon>0,\quad\forall k.$$

于是该子列 $\{f_{n_k}\}$ 在 K 上无一致收敛子列, 这与 $\{f_n\}$ 是正规族矛盾. □

习题 7.1.2 设 $\{f_n\}$ 是域 D 上的全纯函数列, 并且在 D 上内闭一致有界, $A=\{x+\mathrm{i}y\in D:x,y\in\mathbb{Q}\}$. 证明：若 $\lim_{n\to\infty}f_n(z)$ 在 A 上处处存在, 则 $\{f_n\}$ 在 D 上内闭一致收敛.

证明 用反证法, 假设 $\{f_n\}$ 在 D 上非内闭一致收敛, 则存在紧集 $K\subset D$, 使得 $\{f_n\}$ 在 K 上非一致收敛. 由于 $\{f_n\}$ 在 D 上内闭一致有界, 由 Montel 定理, $\{f_n\}$ 是 D 上的正规族, 因此存在子列 $\{f_{n_k}\}$ 在 K 上一致收敛, 记极限函数为 f. 由于 $\{f_n\}$ 在 K 上不一致收敛, 存在子列 $\{f_{n_j}\}$ 使得

$$\sup_{z\in K}\left|f_{n_j}(z)-f(z)\right|\geqslant\varepsilon>0,\quad\forall j.$$

由于 $\{f_n\}$ 是正规族, 对于子列 $\{f_{n_j}\}$, 存在其子列 $\left\{f_{n_{j_\ell}}\right\}$ 在 K 上一致收敛, 记极限函数为 $\widetilde{f}$. 由于 $f,\widetilde{f}\in\mathcal{H}(K)$, 且 $f|_A=\widetilde{f}|_A$, $A\cap K$ 在 K 中稠密, 由全纯函数零点孤立性即知 $f=\widetilde{f}$. 于是 $f_{n_{j_\ell}}\rightrightarrows f$, 与

$$\sup_{z\in K}\left|f_{n_{j_\ell}}(z)-f(z)\right|\geqslant\varepsilon>0,\quad\forall\ell$$

矛盾. 故 $\{f_n\}$ 在 D 上内闭一致收敛. □

习题 7.1.4 设 $\mathcal{F}$ 是域 D 上的全纯函数族, $z_0\in D$. 证明：若

(1) $\operatorname{Re}f(z)\geqslant 0,\forall z\in D,f\in\mathcal{F}$;

(2) $f(z_0)=g(z_0),\forall f,g\in\mathcal{F}$,

则 $\mathcal{F}$ 是 D 上的正规族. 并举例说明条件 (2) 是不可去掉的.

证明 (1) 由 Montel 定理, 只需证 $\mathcal{F}$ 在 D 上内闭一致有界, 结合有限覆盖定理, 只需证 $\mathcal{F}$ 在 D 中任一圆盘上一致有界, 故不妨设 D 为单位圆盘, $z_0=0$, $f(0)=w$, 其中 $f\in\mathcal{F}$. 进一步地, 可不妨设 $|w|\leqslant 1$. 考虑 $g(z)=\frac{f(z)-1}{f(z)+1}$, 令 $h(z)=\frac{g(0)-g(z)}{1-\overline{g(0)}g(z)}$, 则 $h(0)=0$ 且 $|h(z)|\leqslant 1$, 由 Schwarz 引理知 $|h(z)|\leqslant|z|$. 由此可得 $\mathcal{F}$ 在 D 上内闭一致有界, 结论得证.

(2) 考虑 $f_n(z)=n$, 则 $\{f_n(z)\}_{n=0}^{\infty}$ 是 $D=\{z\in\mathbb{C}:\operatorname{Re}z>0\}$ 上的全纯函数列, 条件 (2) 显然不满足, 此时 $\{f_n(z)\}$ 在 D 上不一致有界, 因此不是正规族. □

习题 7.1.5 设 $\mathcal{F}$ 是域 D 上的正规全纯函数族, g 是整函数. 证明: $\{g\circ f:f\in\mathcal{F}\}$ 也是 D 上的正规族.

证明 $\{g\circ f:f\in\mathcal{F}\}$ 在 D 上显然内闭一致有界, 因此是 D 上的正规族. □

习题 7.1.6 设 D 是有界域, $0<M<+\infty$. 证明:

$$\mathcal{F}=\left\{f\in\mathcal{H}(D):\iint\limits_{D}|f(z)|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant M\right\}$$

是 D 上的正规族.

证明 对任意紧集 $K\subset D$, 由有限覆盖定理可知, 存在 $R>0$, 使得对任意 $z\in D$ 均有 $\mathbb{B}(z,R)\subset D$. 由定理 8.4.5 知, 对任意 $f\in\mathcal{F}$, $|f|^2$ 都是 D 上的次调和函数, 因此

$$|f(z)|^2\leqslant\frac{1}{\pi R^2}\int\limits_{\mathbb{B}(z,R)}|f(\zeta)|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{1}{\pi R^2}\int\limits_{D}|f(\zeta)|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{M}{\pi R^2},$$

因此 $f(z)$ 在 D 上内闭一致有界, 由 Montel 定理, $\mathcal{F}$ 是 D 上的正规族. □

习题 7.2.1 (推广的 Liouville 定理) 设 D 是异于 $\mathbb{C}$ 的单连通域. 证明: 若 f 是整函数, 并且 $f(\mathbb{C})\subset D$, 则 f 是常值函数.

证明 由 Riemann 映射定理, 可取双全纯变换 $g:D\to\mathbb{B}(0,1)$, 则 $g\circ f$ 为有界整函数, 由 Liouville 定理, $g\circ f$ 为常值函数, 从而 f 为常值函数. □

习题 7.2.2 设 D 是异于 $\mathbb{C}$ 的单连通域, $a\in D$. 证明: 若 f 将 D 双全纯地映为 $\mathbb{B}(0,1)$, 并且 $f(a)=0$, $f'(a)>0$, 则

$$\min_{z\in\partial D}|z-a|\leqslant\frac{1}{f'(a)}\leqslant\max_{z\in\partial D}|z-a|.$$

称 $\dfrac{1}{f'(a)}$ 为 D 在 a 处的映射半径.

证明 令 $F(w)=\begin{cases}\dfrac{f^{-1}(w)-a}{w}, & w\in\mathbb{B}(0,1)\setminus\{0\},\\ \dfrac{1}{f'(a)}, & w=0.\end{cases}$ 由 Morera 定理易知 $F\in\mathcal{H}(\mathbb{B}(0,1))$. 由最大模原理,

$$\min_{|w|=1}\left|f^{-1}(w)-a\right|=\min_{|w|=1}|F(w)|\leqslant|F(0)|\leqslant\max_{|w|=1}|F(w)|=\max_{|w|=1}\left|f^{-1}(w)-a\right|.$$

由边界对应定理, f^{-1} 将 $\partial\mathbb{B}(0,1)$ 一一地映为 ∂D, 因此上式可改写为

$$\min_{z\in\partial D}|z-a|\leqslant\frac{1}{f'(a)}\leqslant\max_{z\in\partial D}|z-a|.$$

□

习题 7.2.3 设 D 是异于 $\mathbb{C}$ 的单连通域, $a\in D$, f 将 D 双全纯地映为 $\mathbb{B}(0,1)$, 并且 $f(a)=0, f'(a)>0$. 证明: 若 g 将 D 双全纯地映为 $\mathbb{B}(0,1)$, $p=g^{-1}(0)$, 则

$$g(z)=\frac{g'(a)}{|g'(a)|}\cdot\frac{f(z)-f(p)}{1-\overline{f(p)}f(z)}.$$

证明 由于 $g\circ f^{-1}\in \mathrm{Aut}(\mathbb{D})$, 故它具有形式 $g\circ f^{-1}(z)=\mathrm{e}^{\mathrm{i}\theta}\dfrac{z-z_0}{1-\overline{z_0}z}$, 其中 $|z_0|<1,\theta\in\mathbb{R}$ 待定. 由于 $g\circ f^{-1}(f(p))=g(p)=0$, 因此 $z_0=f(p)$. 由

$$\frac{g'(a)}{f'(a)}=\left(g\circ f^{-1}\right)'(0)=\mathrm{e}^{\mathrm{i}\theta}\left(\frac{z-f(p)}{1-\overline{f(p)}z}\right)'\bigg|_{z=0}=\mathrm{e}^{\mathrm{i}\theta}\left(1-|f(p)|^2\right)$$

及 $f'(a)>0,|f(p)|<1$ 可知 $\mathrm{e}^{\mathrm{i}\theta}=\dfrac{g'(a)}{|g'(a)|}$. 故

$$g\circ f^{-1}(z)=\frac{g'(a)}{|g'(a)|}\cdot\frac{z-f(p)}{1-\overline{f(p)}z}\xrightarrow{z\to f(z)}g(z)=\frac{g'(a)}{|g'(a)|}\cdot\frac{f(z)-f(p)}{1-\overline{f(p)}f(z)}.$$

□

习题 7.2.4 设 D 为异于 $\mathbb{C}$ 的凸域, $a\in D$, $\mathcal{F}=\{f\in\mathcal{H}(D):f(a)=0,f'(a)>0\}$. 证明: $\mathcal{F}$ 中满足 $f(D)=\mathbb{B}(0,1)$ 和 $\mathrm{Re}\,f'(z)\geqslant 0\ (\forall z\in D)$ 的 f 最多只有一个.

证明 对任意 $f\in\mathcal{F}$, 若 $\mathrm{Re}\,f'(z)\geqslant 0,\forall z\in D$, 我们证明 f 必为单叶函数. 用反证法, 假设存在不同的两点 $z_1,z_2\in D$, 使得 $f(z_1)=f(z_2)$, 由于 D 为凸域 (从而为单连通域), 我们有

$$0=f(z_2)-f(z_1)=\int_{[z_1,z_2]}f'(\zeta)\,\mathrm{d}\zeta=\int_0^1 f'(z_1+t(z_2-z_1))(z_2-z_1)\,\mathrm{d}t,$$

因此

$$\int_0^1 f'(z_1+t(z_2-z_1))\,\mathrm{d}t=0\implies\int_0^1\mathrm{Re}\,f'(z_1+t(z_2-z_1))\,\mathrm{d}t=0$$
$$\xRightarrow{\mathrm{Re}\,f'(z)\geqslant 0}\mathrm{Re}\,f'(z)=0,\quad\forall z\in[z_1,z_2].$$

同习题 2.2.2 (1) 可知 $f'(z)$ 在 $[z_1,z_2]$ 上为常数, 再由全纯函数零点孤立性可知 $f'(z)$ 在 D 上为常数, 从而在 D 上 $\mathrm{Re}\,f'(z)\equiv 0$, 这与 $\mathrm{Re}\,f'(a)=f'(a)>0$ 矛盾. 故 f 为单叶函数, 结合 $f(D)=\mathbb{B}(0,1)$, 由 Riemann 映射定理, f 唯一. □

习题 7.2.7 设 D 是异于 $\mathbb{C}$ 的单连通域, $a\in D$, R 为 D 在 a 处的映射半径 (定义见习题 7.2.2). 证明: 若 $F\in\mathcal{H}(D),F(a)=0,F'(a)=1$, 则

$$\iint_D\left|F'(z)\right|^2\mathrm{d}x\,\mathrm{d}y\geqslant\pi R^2.$$

等号成立当且仅当 F 是将 D 映为 $\mathbb{B}(0,R)$ 的双全纯映射.

证明 令 f 为从 $\mathbb{B}(0,1)$ 到 D 的双全纯映射, 满足 $f(0)=a,f'(0)>0$. 由于 f 作为 $\mathbb{R}^2$ 上映射的 Jacobi 行列式为 $|f'|^2$, 且由定理 8.4.5, $|(F\circ f)'|^2$ 为次调和函数, 我们有

$$\begin{aligned}\iint_D\left|F'(z)\right|^2\mathrm{d}x\,\mathrm{d}y&=\iint_{\mathbb{B}(0,1)}\left|F'\circ f(w)\right|^2\left|f'(w)\right|^2\mathrm{d}x\,\mathrm{d}y=\iint_{\mathbb{B}(0,1)}\left|(F\circ f)'(w)\right|^2\mathrm{d}x\,\mathrm{d}y\\&\geqslant\pi\left|(F\circ f)'(0)\right|^2=\pi\left|F'(a)f'(0)\right|^2=\frac{\pi}{\left|(f^{-1})'(a)\right|^2}=\pi R^2.\end{aligned}$$

等号成立当且仅当 $|(F\circ f)'|$ 为常值函数, 由习题 2.2.2, $(F\circ f)'$ 为常值函数, 结合 $F\circ f(0)=F(a)=0$ 即

知 $F\circ f(z)=cz$, 其中 $c\in\mathbb{C}$, 故 $F(z)=cf^{-1}(z)$ 是将 D 映为 $\mathbb{B}(0,R)$ 的双全纯映射. □

习题 7.3.1 利用 Schwarz 对称原理和边界对应定理证明：将 $\mathbb{B}(0,1)$ 映为自身的双全纯映射一定是分式线性变换.

证明 任取将 $\mathbb{B}(0,1)$ 映为自身的双全纯映射 f, 由边界对应定理, f 可延拓为 $\overline{\mathbb{B}(0,1)}$ 上的连续函数, 且将 $\partial\mathbb{B}(0,1)$ 一一地映为 $\partial\mathbb{B}(0,1)$. 于是 $f(z)$ 可延拓为 $\widetilde{f}(z)=\begin{cases} f(z), & |z|\leqslant 1,\\ \dfrac{1}{\overline{f\left(\frac{1}{\bar z}\right)}}, & |z|>1.\end{cases}$ 由于 f 在 $\mathbb{B}(0,1)$ 上有且仅有 1 个零点, $\widetilde{f}$ 在 $\partial\mathbb{B}(0,1)$ 上连续, 由 Painlevé 原理可知 $\widetilde{f}$ 为 $\overline{\mathbb{C}}$ 上的亚纯函数, 进而 $\widetilde{f}\in\mathrm{Aut}(\overline{\mathbb{C}})$. 由定理 5.3.5, $\widetilde{f}$ 为分式线性变换, 从而 f 为分式线性变换. □

习题 7.3.3 设 D 是由简单闭曲线所围成的单连通域, $z_1,z_2,z_3\in\partial D$ 是彼此不同的三点, 按 ∂D 的正向排列. 证明：若 $w_1,w_2,w_3\in\partial\mathbb{B}(0,1)$ 是彼此不同的三点, 按 $\partial\mathbb{B}(0,1)$ 的正向排列, 则存在唯一的 φ, 将 D 双全纯地映为 $\mathbb{B}(0,1)$, 将 $\overline{D}$ 同胚地映为 $\overline{\mathbb{B}(0,1)}$, 并且 $f(z_k)=w_k, k=1,2,3$.

证明 **(存在性)** 由 Riemann 映射定理与边界对应定理, 存在函数 f, 将 D 双全纯地映为 $\mathbb{B}(0,1)$, 并将 $\overline{D}$ 同胚地映为 $\overline{\mathbb{B}(0,1)}$, 再取分式线性变换 g 使得 $g(f(z_i))=w_i$ $(i=1,2,3)$, 由分式线性变换的保圆性即知 $\varphi:=g\circ f$ 为所求.

(唯一性) 设函数 φ_1,φ_2 均满足题意, 则 $\varphi_1\circ\varphi_2^{-1}$ 是 $\mathbb{B}(0,1)$ 的全纯自同构 (从而为分式线性变换), 且 $\varphi_1\circ\varphi_2^{-1}(w_i)=w_i$ $(i=1,2,3)$, 由于三点可确定一个分式线性变换, $\varphi_1\circ\varphi_2^{-1}=\mathrm{Id}$, 即 $\varphi_1=\varphi_2$. □

习题 7.3.5 设 $f\in\mathcal{H}(\mathbb{B}(0,1)), f(0)=0, f'(0)=a>0$. 证明：若 $f(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$, 则 f 在 $\mathbb{B}\left(0,\dfrac{a}{1+\sqrt{1-a^2}}\right)$ 上双全纯.

证明 由 Schwarz 引理知 $a=f'(0)\in(0,1)$. 设 $f(z)$ 在 $\mathbb{B}(0,\rho)$ 上非单叶函数, 则存在不同的两点 $z_1,z_2\in\mathbb{B}(0,\rho)$ 使得 $f(z_1)=f(z_2)$. 由于 z_1,z_2 均为 $f(z)-f(z_1)$ 的零点, 由定理 4.4.1,

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=\rho}\frac{f'(z)}{f(z)-f(z_1)}\,\mathrm{d}z\geqslant 2.$$

记 $\gamma_\rho=f(\partial\mathbb{B}(0,\rho))$, 则 γ_ρ 不是简单闭曲线, 否则

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=\rho}\frac{f'(z)}{f(z)-f(z_1)}\,\mathrm{d}z=\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=\rho}\mathrm{d}\,\mathrm{Log}(f(z)-f(z_1))\xlongequal{w=f(z)}\frac{1}{2\pi}\Delta_{\gamma_\rho}\mathrm{Arg}(w-f(z_1))=1,$$

与前一式矛盾. 因此 γ_ρ 自交, 即存在不同的两点 $\zeta_1,\zeta_2\in\partial\mathbb{B}(0,\rho)$, 使得 $f(\zeta_1)=f(\zeta_2)$. 由习题 4.5.20 即得 $|f(\zeta_1)|\leqslant\rho^2$. 而由习题 4.5.21,

$$|\zeta_1|\frac{a-|\zeta_1|}{1-a|\zeta_1|}\leqslant|f(\zeta_1)|\implies\rho\cdot\frac{a-\rho}{1-a\rho}\leqslant|f(\zeta_1)|\leqslant\rho^2\implies\rho\geqslant\frac{1-\sqrt{1-a^2}}{a}=\frac{a}{1+\sqrt{1-a^2}}.$$

故 f 在 $\mathbb{B}\left(0,\dfrac{a}{1+\sqrt{1-a^2}}\right)$ 上双全纯. □

补充题 1 求分式线性变换 $T\in\mathrm{Aut}(\mathbb{D})$, 使得 $T(1)=\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}$ 且 $T(a)=\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$, 其中 $|a|=1$.

解答 注意到 $\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}$ 与 $\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$ 为对径点, 故先求分式线性变换 $w\in\mathrm{Aut}(\mathbb{D})$ 使得 $w(1)=1, w(-1)=a$. 设

$w(z)=\mathrm{e}^{\mathrm{i}\theta}\dfrac{z-z_0}{1-\overline{z_0}z}$, 其中 $|z_0|<1,\theta\in\mathbb{R}$ 待定. 我们有

$$\begin{cases}\mathrm{e}^{\mathrm{i}\theta}\dfrac{1-z_0}{1-\overline{z_0}}=1,\\ \mathrm{e}^{\mathrm{i}\theta}\dfrac{-1-z_0}{1+\overline{z_0}}=a\end{cases} \implies \overline{z_0}=1-\mathrm{e}^{\mathrm{i}\theta}(1-z_0)\xRightarrow{\text{回代}} z_0=\frac{a-1}{a+1}-\frac{2a}{a+1}\mathrm{e}^{-\mathrm{i}\theta}$$
$$\xRightarrow{\text{回代}}\mathrm{e}^{\mathrm{i}\theta}=\frac{a+1}{\bar{a}+1}\xRightarrow{\text{回代}} z_0=\frac{a-3}{a+1}.$$

由 $w^{-1}:1\mapsto 1,a\mapsto -1$ 知

$$T(z)=\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}w^{-1}(z)=\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}\cdot\frac{z+\mathrm{e}^{\mathrm{i}\theta}z_0}{\mathrm{e}^{\mathrm{i}\theta}+\overline{z_0}z}=\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}\cdot\frac{(\bar{a}+1)z+(a-3)}{(\bar{a}-3)z+(a+1)}.$$

□

补充题 2 对 $t>0$ 定义 $\vartheta(t)=\sum\limits_{n=-\infty}^{\infty}\mathrm{e}^{-\pi n^2 t}$, 证明：$\vartheta(t)=t^{-\frac{1}{2}}\vartheta\left(\frac{1}{t}\right)$.

证明 令 $f(z)=\mathrm{e}^{-\pi z^2 t}$, 则

$$\begin{aligned}\hat{f}(\xi)&=\int_{\mathbb{R}}\mathrm{e}^{-\pi x^2 t}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x=\mathrm{e}^{-\frac{\pi\xi^2}{t}}\int_{\mathbb{R}}\mathrm{e}^{-\pi t\left(x+\frac{\mathrm{i}\xi}{t}\right)^2}\,\mathrm{d}x\\&=\mathrm{e}^{-\frac{\pi\xi^2}{t}}\int_{\mathbb{R}}\mathrm{e}^{-\pi t x^2}\,\mathrm{d}x\xlongequal{t>0}2\mathrm{e}^{-\frac{\pi\xi^2}{t}}\cdot\frac{1}{2\sqrt{t}}=\frac{1}{\sqrt{t}}\mathrm{e}^{-\frac{\pi\xi^2}{t}}.\end{aligned}$$

由于 $f\in\mathfrak{F}$, 由 Poisson 求和公式得

$$\vartheta(t)=\sum_{n=-\infty}^{\infty}f(n)=\sum_{n=-\infty}^{\infty}\hat{f}(n)=\frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi n^2}{t}}=t^{-\frac{1}{2}}\vartheta\left(\frac{1}{t}\right).$$

□

补充题 3 设 $t>0,a\in\mathbb{R}$. 证明：$\sum\limits_{n=-\infty}^{\infty}\dfrac{\mathrm{e}^{-2\pi\mathrm{i}an}}{\cosh\left(\frac{\pi n}{t}\right)}=\sum\limits_{n=-\infty}^{\infty}\dfrac{t}{\cosh(\pi(n+a)t)}$.

证明 由于 $\dfrac{1}{\cosh\pi x}$ 是 Fourier 变换的不动点,

$$\int_{\mathbb{R}}\frac{\mathrm{e}^{-2\pi\mathrm{i}x\xi}}{\cosh\pi x}\,\mathrm{d}x=\frac{1}{\cosh\pi\xi},$$

由此可知 $f(z)=\dfrac{\mathrm{e}^{-2\pi\mathrm{i}az}}{\cosh\left(\frac{\pi z}{t}\right)}$ 的 Fourier 变换为

$$\hat{f}(\xi)=\int_{\mathbb{R}}\frac{\mathrm{e}^{-2\pi\mathrm{i}x(a+\xi)}}{\cosh\left(\frac{\pi x}{t}\right)}\,\mathrm{d}x\xlongequal{x=ty}t\int_{\mathbb{R}}\frac{\mathrm{e}^{-2\pi\mathrm{i}y[t(a+\xi)]}}{\cosh(\pi y)}\,\mathrm{d}y=\frac{t}{\cosh(\pi(\xi+a)t)}.$$

由

$$|f(x)|=\left|\frac{\mathrm{e}^{-2\pi\mathrm{i}ax}}{\cosh\left(\frac{\pi x}{t}\right)}\right|=\frac{2}{\mathrm{e}^{\frac{\pi x}{t}}+\mathrm{e}^{-\frac{\pi x}{t}}}\leqslant 2\mathrm{e}^{-\frac{\pi|x|}{t}}$$

可见 $f\in\mathfrak{F}$, 故由 Poisson 求和公式得

$$\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}an}}{\cosh\left(\frac{\pi n}{t}\right)}=\sum_{n=-\infty}^{\infty}\frac{t}{\cosh(\pi(n+a)t)}.$$

□

补充题 4 补充用 Phragmén-Lindelöf 定理实现 Paley-Wiener 定理证明中 Step 3 的细节：

$$\begin{cases} |f(x)| \leqslant 1, \\ |f(z)| \leqslant \mathrm{e}^{2\pi M|z|} \end{cases} \implies |f(z)| \leqslant \mathrm{e}^{2\pi M|y|}.$$

证明 通过乘恰当的旋转因子可知, Phragmén-Lindelöf 定理中的角状区域可换为第一象限. 令 $F(z) = f(z)\mathrm{e}^{2\pi \mathrm{i} M z}$, 注意到 $F(z)$ 在第一象限的边界上有上界 1：

$$|F(x)| = |f(x)| \leqslant 1, \quad \forall x \in \mathbb{R}_+,$$
$$|F(\mathrm{i}y)| = |f(\mathrm{i}y)|\mathrm{e}^{-2\pi M y} \leqslant \mathrm{e}^{2\pi M|y|}\mathrm{e}^{-2\pi M y} = 1, \quad \forall y \in \mathbb{R}_+,$$

又 $|F(z)| = |f(z)|\left|\mathrm{e}^{2\pi \mathrm{i} M z}\right| \leqslant \mathrm{e}^{4\pi M|z|}$, 由 Phragmén-Lindelöf 定理知, 在第一象限中有 $|F(z)| \leqslant 1$, 即 $|f(z)| \leqslant \left|\mathrm{e}^{-2\pi \mathrm{i} M z}\right| = \left|\mathrm{e}^{-2\pi \mathrm{i} M (x+\mathrm{i}y)}\right| = \mathrm{e}^{2\pi M y}$. 对余下三个象限类似讨论可得结论成立. □

Stein 4.4.1 Suppose f is continuous and of moderate decrease, and $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Show that $f = 0$ by completing the following outline:

(1) For each fixed real number t consider the two functions

$$A(z) = \int_{-\infty}^{t} f(x)\mathrm{e}^{-2\pi \mathrm{i} z(x-t)}\,\mathrm{d}x \quad \text{and} \quad B(z) = -\int_{t}^{\infty} f(x)\mathrm{e}^{-2\pi \mathrm{i} z(x-t)}\,\mathrm{d}x.$$

Show that $A(\xi) = B(\xi)$ for all $\xi \in \mathbb{R}$.

(2) Prove that the function F equal to A in the closed upper half-plane, and B in the lower half-plane, is entire and bounded, thus constant. In fact, show that $F = 0$.

(3) Deduce that

$$\int_{-\infty}^{t} f(x)\,\mathrm{d}x = 0,$$

for all t, and conclude that $f = 0$.

Proof (1) We have

$$A(\xi) - B(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi \mathrm{i}\xi(x-t)}\,\mathrm{d}x = \hat{f}(\xi) = 0.$$

(2) By Symmetry principle, the function $F(z) \coloneqq \begin{cases} A(z), & \operatorname{Im} z \geqslant 0, \\ B(z), & \operatorname{Im} z < 0 \end{cases}$ is an entire function. Since f is of moderate decrease, we see that

$$|A(z)| \leqslant \int_{-\infty}^{t} |f(x)|\mathrm{e}^{2\pi \operatorname{Im}(z)(x-t)}\,\mathrm{d}x \leqslant \int_{-\infty}^{t} \frac{A}{1+x^2}\,\mathrm{d}x \leqslant \pi A$$

is bounded in the closed upper-half plane. Similarly,

$$|B(z)| \leqslant \int_{t}^{\infty} |f(x)|\mathrm{e}^{2\pi \operatorname{Im}(z)(x-t)}\,\mathrm{d}x \leqslant \int_{t}^{\infty} \frac{A}{1+x^2}\,\mathrm{d}x \leqslant \pi A$$

is bounded in the lower-half plane. So F is both entire and bounded, thus constant by Liouville's theorem. Let $z = \mathrm{i}s$ for $s \geqslant 0$, we have

$$A(\mathrm{i}s) = \int_{-\infty}^{t} f(x)\mathrm{e}^{2\pi s(x-t)}\,\mathrm{d}x \xrightarrow{s\to\infty} 0$$

by DCT. So $F = 0$.

(3) Take $z = 0$ we find $\int_{-\infty}^{t} f(x)\,\mathrm{d}x = F(0) = 0$ for all t, hence $f = 0$. □

Stein 4.4.3 Show, by contour integration, that if $a > 0$ and $\xi \in \mathbb{R}$ then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2 + x^2}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = \mathrm{e}^{-2\pi a|\xi|},$$

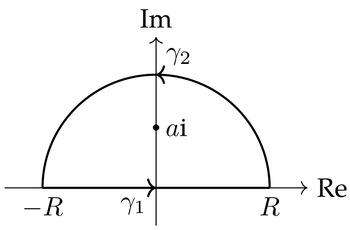
and check that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi a|\xi|}\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi = \frac{1}{\pi}\frac{a}{a^2 + x^2}.$$

Proof Let $f(z) = \dfrac{a}{a^2 + z^2}\mathrm{e}^{-2\pi\mathrm{i}z\xi}$.

(1) If $\xi = 0$ then LHS $= \dfrac{1}{\pi}\int_{\mathbb{R}} \dfrac{a}{a^2 + x^2}\,\mathrm{d}x = 1 =$ RHS.

(2) For $\xi < 0$, choose upper semicircle contour, from the residue formula we get



$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f, a\mathrm{i}) = 2\pi\mathrm{i}\lim_{z\to a\mathrm{i}} \frac{a}{z + a\mathrm{i}}\mathrm{e}^{-2\pi\mathrm{i}z\xi} = \pi\mathrm{e}^{-2\pi a|\xi|}.$$

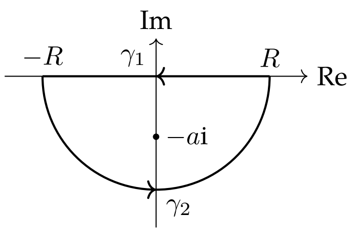
Since when $R \to +\infty$,

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leqslant \int_0^{\pi} \left|\frac{a}{a^2 + R^2\mathrm{e}^{2\mathrm{i}\theta}}\mathrm{e}^{2\pi R\xi\sin\theta}\right|\mathrm{d}\theta \leqslant \frac{\pi a}{R^2 - a^2} \to 0,$$

it follows that

$$\int_{\mathbb{R}} \frac{a}{a^2 + x^2}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = \pi\mathrm{e}^{-2\pi a|\xi|}.$$

(3) For $\xi > 0$, choose lower semicircle contour, like in (2) we get



$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f, -a\mathrm{i}) = 2\pi\mathrm{i}\lim_{z\to -a\mathrm{i}} \frac{a}{z-a\mathrm{i}}\mathrm{e}^{-2\pi\mathrm{i}z\xi} = -\pi\mathrm{e}^{-2\pi a|\xi|}.$$

When $R \to +\infty$,

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leqslant \int_{-\pi}^{0}\left|\frac{a}{a^2+R^2\mathrm{e}^{2\mathrm{i}\theta}}\mathrm{e}^{2\pi R\xi\sin\theta}\right|\mathrm{d}\theta \leqslant \frac{\pi a}{R^2-a^2} \to 0,$$

hence

$$\int_{\mathbb{R}} \frac{a}{a^2+x^2}\,\mathrm{d}x = -\left(-\pi\mathrm{e}^{-2\pi a|\xi|}\right) = \pi\mathrm{e}^{-2\pi a|\xi|}.$$

For the second part of the exercise, notice $f \in \mathfrak{F}$, so Fourier inversion implies the result. □

Stein 4.4.7 The Poisson summation formula applied to specific examples often provides interesting identities.

(1) Let τ be fixed with $\mathrm{Im}(\tau) > 0$. Apply the Poisson summation formula to

$$f(z) = (\tau + z)^{-k},$$

where k is an integer $\geqslant 2$, to obtain

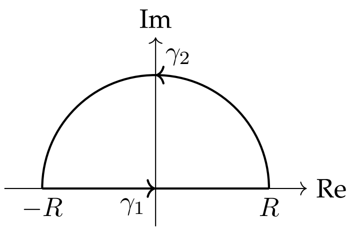
$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}\mathrm{e}^{2\pi\mathrm{i}m\tau}.$$

(2) Set $k = 2$ in the above formula to show that if $\mathrm{Im}(\tau) > 0$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(3) Can one conclude that the above formula holds true whenever τ is any complex number that is not an integer?

Proof (1) ① For $\xi \leqslant 0$, choose upper semicircle contour.



Since $(\tau + z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}$ is holomorphic in the upper half-plane, we have

$$\int_{\gamma_1} (\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z + \int_{\gamma_2} (\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z = 0.$$

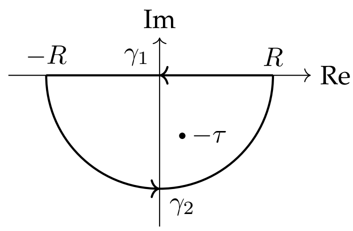
When $R\to+\infty$,

$$\left|\int_{\gamma_2}(\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z\right|=\left|\int_0^{\pi}\frac{\mathrm{e}^{-2\pi\mathrm{i}\xi R\mathrm{e}^{\mathrm{i}\theta}}R\mathrm{i}}{(\tau+R\mathrm{e}^{\mathrm{i}\theta})^k}\,\mathrm{d}\theta\right|\leqslant\int_0^{\pi}\frac{R\left|\mathrm{e}^{-2\pi\mathrm{i}\xi R\mathrm{e}^{\mathrm{i}\theta}}\right|}{(R-|\tau|)^k}\,\mathrm{d}\theta$$
$$\leqslant\frac{\pi R^2\mathrm{e}^{2\pi\xi R\sin\theta}}{(R-|\tau|)^k}\overset{\xi\leqslant 0}{\leqslant}\frac{\pi R^2}{(R-|\tau|)^k}\xrightarrow{k\geqslant 2}0,$$

hence when $\xi\leqslant 0$ we get

$$\hat{f}(\xi)=\int_{\mathbb{R}}(\tau+x)^{-k}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x=0.$$

② For $\xi>0$, choose lower semicircle contour.



The residue at $-\tau$ is

$$\operatorname{Res}\big((\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi},-\tau\big)=\frac{1}{(k-1)!}\big(\mathrm{e}^{-2\pi\mathrm{i}z\xi}\big)^{(k-1)}\bigg|_{z=-\tau}=\frac{(-2\pi\mathrm{i}\xi)^{k-1}}{(k-1)!}\mathrm{e}^{2\pi\mathrm{i}\tau\xi},$$

thus

$$\int_{\gamma_1}(\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z+\int_{\gamma_2}(\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z=-\frac{(-2\pi\mathrm{i})^k\xi^{k-1}}{(k-1)!}\mathrm{e}^{2\pi\mathrm{i}\tau\xi}.$$

When $R\to+\infty$,

$$\left|\int_{\gamma_2}(\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z\right|=\left|\int_{-\pi}^{0}\frac{\mathrm{e}^{-2\pi\mathrm{i}\xi R\mathrm{e}^{\mathrm{i}\theta}}R\mathrm{i}}{(\tau+R\mathrm{e}^{\mathrm{i}\theta})^k}\,\mathrm{d}\theta\right|\leqslant\int_{-\pi}^{0}\frac{R\left|\mathrm{e}^{-2\pi\mathrm{i}\xi R\mathrm{e}^{\mathrm{i}\theta}}\right|}{(R-|\tau|)^k}\,\mathrm{d}\theta$$
$$\leqslant\frac{\pi R^2\mathrm{e}^{2\pi\xi R\sin\theta}}{(R-|\tau|)^k}\overset{\xi>0}{\leqslant}\frac{\pi R^2}{(R-|\tau|)^k}\xrightarrow{k\geqslant 2}0,$$

hence when $\xi>0$ we get

$$\hat{f}(\xi)=\int_{\mathbb{R}}(\tau+x)^{-k}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x=\frac{(-2\pi\mathrm{i})^k\xi^{k-1}}{(k-1)!}\mathrm{e}^{2\pi\mathrm{i}\tau\xi}.$$

Since $f\in\mathfrak{F}$, by Poisson summation formula we have

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^k}=\frac{(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}\mathrm{e}^{2\pi\mathrm{i}m\tau}.$$

(2) Set $k=2$ in the above formula, we get

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^2}=-4\pi^2\sum_{m=1}^{\infty}m\mathrm{e}^{2\pi\mathrm{i}m\tau}.$$

To finish the proof, notice that when $\mathrm{Im}(\tau)>0$ we have $\left|\mathrm{e}^{2\pi\mathrm{i}\tau}\right|=\mathrm{e}^{-2\pi\,\mathrm{Im}(\tau)}<1$, hence

$$\begin{aligned}\sum_{m=1}^{\infty}m\mathrm{e}^{2\pi\mathrm{i}m\tau}&=\frac{1}{2\pi\mathrm{i}}\sum_{m=1}^{\infty}\frac{\partial}{\partial\tau}\left(\mathrm{e}^{2\pi\mathrm{i}m\tau}\right)=\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial\tau}\left(\sum_{m=1}^{\infty}\mathrm{e}^{2\pi\mathrm{i}m\tau}\right)=\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial\tau}\left(\frac{\mathrm{e}^{2\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2\pi\mathrm{i}\tau}}\right)\\&=\frac{\mathrm{e}^{2\pi\mathrm{i}\tau}}{\left(1-\mathrm{e}^{2\pi\mathrm{i}\tau}\right)^2}=\frac{1}{\left(\mathrm{e}^{\pi\mathrm{i}\tau}-\mathrm{e}^{-\pi\mathrm{i}\tau}\right)^2}=\frac{1}{-4\sin^2(\pi\tau)}.\end{aligned}$$

(3) For the case that $\mathrm{Im}(\tau)<0$, by replacing τ with $-\tau$, we see the formula in (2) still holds. When τ is a real number that is not an integer, the same formula holds by the isolating property of the zeros of a holomorphic function. □

Stein 4.4.9 Here are further results similar to the Phragmén-Lindelöf theorem.

(1) Let F be a holomorphic function in the right half-plane that extends continuously to the boundary, that is, the imaginary axis. Suppose that $|F(\mathrm{i}y)|\leqslant 1$ for all $y\in\mathbb{R}$, and

$$|F(z)|\leqslant C\mathrm{e}^{c|z|^\gamma}$$

for some $c,C>0$ and $\gamma<1$. Prove that $|F(z)|\leqslant 1$ for all z in the right half-plane.

(2) More generally, let S be a sector whose vertex is the origin, and forming an angle of $\frac{\pi}{\beta}$. Let F be a holomorphic function in S that is continuous on the closure of S, so that $|F(z)|\leqslant 1$ on the boundary of S and

$$|F(z)|\leqslant C\mathrm{e}^{c|z|^\alpha}\quad\text{for all}\quad z\in S$$

for some $c,C>0$ and $0<\alpha<\beta$. Prove that $|F(z)|\leqslant 1$ for all $z\in S$.

Proof We prove (2) directly. Let $F_\varepsilon(z)=F(z)\mathrm{e}^{-\varepsilon z^r}$, where $r\in(\alpha,\beta)\cap\mathbb{Q}$ and $\varepsilon>0$. Then

$$|F_\varepsilon(z)|=|F(z)|\mathrm{e}^{-\varepsilon|z|^r\cos(r\arg z)}\leqslant C\mathrm{e}^{c|z|^\alpha-\varepsilon|z|^r\cos(r\arg z)}.$$

Without loss of generality, we consider the sector

$$S=\left\{z\in\mathbb{C}:-\frac{\pi}{2\beta}<\arg z<\frac{\pi}{2\beta}\right\},$$

then $r\arg z\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\cos(r\arg z)>0$. Hence $|F_\varepsilon(z)|\to 0$ as $|z|\to\infty$, and we can conclude that $|F_\varepsilon(z)|$ achieves its maximum on $\overline{S}$ at some point $z_0\neq\infty$. Using the maximum modulus principle on some region with compact closure that contains z_0, we see that z_0 must lie on the boundary of S. Thus $|F_\varepsilon(z)|\leqslant|F_\varepsilon(z_0)|\leqslant 1$, and by letting $\varepsilon\to 0$ we get $|F(z)|\leqslant 1$ for all $z\in S$. □

Stein 4.4.11 One can give a neater formulation of the result in Exercise 10 by proving the following fact.

Suppose $f(z)$ is an entire function of strict order 2, that is,

$$f(z)=O\left(\mathrm{e}^{c_1|z|^2}\right)$$

for some $c_1>0$. Suppose also that for x real,

$$f(x)=O\left(\mathrm{e}^{-c_2|x|^2}\right)$$

for some $c_2>0$. Then

$$|f(x+\mathrm{i}y)|=O\left(\mathrm{e}^{-ax^2+by^2}\right)$$

for some $a,b>0$. The converse is obviously true.

Proof For $z=x+\mathrm{i}y$, if $x^2\leqslant y^2$, then

$$c_1|z|^2=c_1\left(x^2+y^2\right)\leqslant 2c_1y^2\leqslant 3c_1y^2-c_1x^2,$$

and so we already have

$$|f(z)|=O\left(\mathrm{e}^{c_1|z|^2}\right)=O\left(\mathrm{e}^{-c_1x^2+3c_1y^2}\right),$$

which is the desired result. So we may assume $x^2>y^2$. By symmetry, we can only focus on the sector $S=\left\{z\in\mathbb{C}:0<\arg z<\frac{\pi}{4}\right\}$. Let

$$g_\varepsilon(z)=f(z)\mathrm{e}^{[c_2-\varepsilon+\mathrm{i}(c_1+\varepsilon)]z^2}\quad\text{for }\varepsilon>0,$$

then $|g_\varepsilon(z)|\leqslant\mathrm{e}^{c|z|^2}$ in S for some $c>0$. And on the boundary of S, we have

$$\begin{aligned}|g_\varepsilon(x)|&=|f(x)|\mathrm{e}^{(c_2-\varepsilon)x^2}\leqslant C_2\mathrm{e}^{-c_2x^2}\mathrm{e}^{(c_2-\varepsilon)x^2}=C_2\mathrm{e}^{-\varepsilon x^2},\quad x\geqslant 0,\\ \left|g_\varepsilon\left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\right|&=|f(x)|\mathrm{e}^{-(c_1+\varepsilon)R^2}\leqslant C_1\mathrm{e}^{c_1R^2}\mathrm{e}^{-(c_1+\varepsilon)R^2}=C_1\mathrm{e}^{-\varepsilon R^2},\quad R\geqslant 0.\end{aligned}$$

So $|g_\varepsilon(z)|\leqslant C\mathrm{e}^{-\varepsilon|z|^2}$ on the boundary of S for some $C>0$. Now we can apply Exercise 4.4.9 (2), where we take $\alpha=2$ and $\beta=4$, to the function $g_\varepsilon(z)\mathrm{e}^{\varepsilon|z|^2}$ (recall $\left|g_\varepsilon(z)\mathrm{e}^{\varepsilon|z|^2}\right|\leqslant\mathrm{e}^{(c+\varepsilon)|z|^2}$), and conclude that

$$\left|g_\varepsilon(z)\mathrm{e}^{\varepsilon|z|^2}\right|\leqslant C\quad\text{for all }z\in S.$$

Then let $\varepsilon\to 0$ we get

$$|f(z)|\cdot\left|\mathrm{e}^{(c_2+\mathrm{i}c_1)(x+\mathrm{i}y)^2}\right|=|f(z)|\mathrm{e}^{c_2\left(x^2-y^2\right)-2c_1xy}\leqslant C\quad\text{for }x+\mathrm{i}y\in S.$$

Hence

$$|f(x)|\leqslant C\mathrm{e}^{-c_2\left(x^2-y^2\right)+2c_1xy}\overset{\lambda>0}{\leqslant}C\mathrm{e}^{-c_2\left(x^2-y^2\right)+c_1\lambda x^2+\frac{c_1}{\lambda}y^2}=C\mathrm{e}^{-(c_2-c_1\lambda)x^2+\left(c_2+\frac{c_1}{\lambda}\right)y^2},$$

choosing $\lambda<\dfrac{c_2}{c_1}$ we complete the proof with $a=c_2-c_1\lambda$ and $b=c_2+\dfrac{c_1}{\lambda}$. □

Stein 4.4.12 The principle that a function and its Fourier transform cannot both be too small at infinity is illustrated by the following theorem of Hardy.

If f is a function on $\mathbb{R}$ that satisfies

$$f(x) = O\left(\mathrm{e}^{-\pi x^2}\right) \quad \text{and} \quad \hat{f}(\xi) = O\left(\mathrm{e}^{-\pi \xi^2}\right),$$

then f is a constant multiple of $\mathrm{e}^{-\pi x^2}$. As a result, if $f(x) = O\left(\mathrm{e}^{-\pi A x^2}\right)$, and $\hat{f}(\xi) = O\left(\mathrm{e}^{-\pi B \xi^2}\right)$, with $AB > 1$ and $A, B > 0$, then f is identically zero.

(1) If f is even, show that $\hat{f}$ extends to an even entire function. Moreover, if $g(z) = \hat{f}\left(z^{\frac{1}{2}}\right)$, then g satisfies

$$|g(x)| \leqslant c\mathrm{e}^{-\pi x} \quad \text{and} \quad |g(z)| \leqslant c\mathrm{e}^{\pi R \sin^2 \frac{\theta}{2}} \leqslant c\mathrm{e}^{\pi |z|}$$

when $x \in \mathbb{R}$ and $z = R\mathrm{e}^{\mathrm{i}\theta}$ with $R \geqslant 0$ and $\theta \in \mathbb{R}$.

(2) Apply the Phragmén-Lindelöf principle to the function

$$F(z) = g(z)\mathrm{e}^{\gamma z} \quad \text{where } \gamma = \mathrm{i}\pi \frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{2\beta}}}{\sin \frac{\pi}{2\beta}}$$

and the sector $0 \leqslant \theta \leqslant \frac{\pi}{\beta} < \pi$, and let $\beta \to 1$ to deduce that $\mathrm{e}^{\pi z} g(z)$ is bounded in the closed upper half-plane. The same result holds in the lower half-plane, so by Liouville's theorem $\mathrm{e}^{\pi z} g(z)$ is constant, as desired.

(3) If f is odd, then $\hat{f}(0) = 0$, and apply the above argument to $\dfrac{\hat{f}(z)}{z}$ to deduce that $f = \hat{f} = 0$. Finally, write an arbitrary f as an appropriate sum of an even function and an odd function.

Proof (1) Since $\hat{f}(\xi) = O\left(\mathrm{e}^{-\pi \xi^2}\right)$, $\hat{f}$ can be extended to an entire function by Theorem 3.1. Moreover, when f is even,

$$\hat{f}(-\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{2\pi \mathrm{i} x\xi} \,\mathrm{d}x = \int_{\mathbb{R}} f(-x)\mathrm{e}^{-2\pi \mathrm{i} x\xi} \,\mathrm{d}x = \hat{f}(\xi)$$

for all $\xi \in \mathbb{R}$, which implies that $\hat{f}(z) - \hat{f}(-z)$ is identically zero in the whole complex plane. So $\hat{f}$ extends to an even entire function. For $g(z) = \hat{f}\left(z^{\frac{1}{2}}\right)$, we have

$$|g(x)| = \left|\hat{f}\left(x^{\frac{1}{2}}\right)\right| \leqslant c\mathrm{e}^{-\pi x}$$

and

$$\begin{aligned}
\left|\hat{f}\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| &= \left|\int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi \mathrm{i} x R(\cos\theta + \mathrm{i}\sin\theta)} \,\mathrm{d}x\right| \leqslant \int_{\mathbb{R}} |f(x)|\mathrm{e}^{2\pi x R \sin\theta} \,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}} c\mathrm{e}^{-\pi x^2 + 2\pi x R \sin\theta} \,\mathrm{d}x = c\mathrm{e}^{\pi R^2 \sin^2\theta} \int_{\mathbb{R}} \mathrm{e}^{-\pi(x - R\sin\theta)^2} \,\mathrm{d}x \\
&= c\mathrm{e}^{\pi R^2 \sin^2\theta},
\end{aligned}$$

and so

$$\left|g\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| = \left|f\left(R^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\left(\frac{\theta}{2} + k\pi\right)}\right)\right| \leqslant c\mathrm{e}^{\pi R \sin^2 \frac{\theta}{2}} \leqslant c\mathrm{e}^{\pi R}.$$

(2) First we show that

$$\left|F\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| = \left|g\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| \cdot \left|\mathrm{e}^{\mathrm{i}\frac{\pi R}{\sin\frac{\pi}{2\beta}}\mathrm{e}^{\mathrm{i}\left(\theta - \frac{\pi}{2\beta}\right)}}\right| = \left|g\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{e}^{-\frac{\pi R}{\sin\frac{\pi}{2\beta}}\sin\left(\theta - \frac{\pi}{2\beta}\right)}$$

$$\overset{(1)}{\leqslant} c\mathrm{e}^{\pi R(1-\varepsilon_\theta)}, \quad \text{where } \varepsilon_\theta = \frac{\sin\left(\theta - \frac{\pi}{2\beta}\right)}{\sin\frac{\pi}{2\beta}}.$$

For $\beta > 1$, consider the sector $S = \left\{z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{\beta}\right\}$, on its boundary we have

$$|F(x)| = |g(x)| \cdot \left|\mathrm{e}^{\mathrm{i}\frac{\pi x}{\sin\frac{\pi}{2\beta}}\left(\cos\frac{\pi}{2\beta} - \mathrm{i}\sin\frac{\pi}{2\beta}\right)}\right| = |g(x)|\mathrm{e}^{\pi x} \overset{(1)}{\leqslant} c\mathrm{e}^{-\pi x} \cdot \mathrm{e}^{\pi x} = c, \quad \forall x \geqslant 0,$$

$$\left|F\left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{\beta}}\right)\right| \leqslant c\mathrm{e}^{\pi R(1-1)} = c, \quad \forall R \geqslant 0.$$

Hence $|F(z)| \leqslant 1$ on the boundary of S. Note that $|\varepsilon_\theta| \leqslant 1$ for $0 \leqslant \theta \leqslant \frac{\pi}{\beta}$, so

$$\left|F\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant c\mathrm{e}^{2\pi R}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{\beta}.$$

Since $\beta > 1$, we can apply result in Exercise 4.4.9 (2) to $\dfrac{F(z)}{c}$ to get $|F(z)| \leqslant c$ for all z in S. Let $\beta \to 1$, then $\gamma \to \pi$ and we conclude that $|g(z)\mathrm{e}^{\pi z}| \leqslant c$ for all z in the upper half-plane. The same result holds in the lower half-plane, so by Liouville's theorem $\mathrm{e}^{\pi z}g(z)$ is constant.

(3) If f is odd, then $\hat{f}(0) = 0$, $\hat{f}$ extends to an odd entire function by the same argument in (1), and $\dfrac{\hat{f}(z)}{z}$ is even. Let $h(z) = \hat{f}\left(z^{\frac{1}{2}}\right)z^{-\frac{1}{2}}$ and we get the same bound as in (1), then follow the same argument in (2) to conclude that $h(z)$ is constant for all $z \in \mathbb{C}$. Hence from $\hat{f}(0) = 0$ we see $\hat{f} \equiv 0$ and then $f \equiv 0$ by Fourier inversion.

Finally, for an arbitrary f, by decomposing f into even and odd parts, we see that f is a constant multiple of $\mathrm{e}^{-\pi x^2}$. □

Stein 4.5.3 In this problem, we investigate the behavior of certain bounded holomorphic functions in an infinite strip. The particular result described here is sometimes called the three-lines lemma.

(1) Suppose $F(z)$ is holomorphic and bounded in the strip $0 < \mathrm{Im}(z) < 1$ and continuous on its closure. If $|F(z)| \leqslant 1$ on the boundary lines, then $|F(z)| \leqslant 1$ throughout the strip.

(2) For the more general F, let $\sup\limits_{x\in\mathbb{R}} |F(x)| = M_0$ and $\sup\limits_{x\in\mathbb{R}} |F(x+\mathrm{i})| = M_1$. Then,

$$\sup_{x\in\mathbb{R}} |F(x+\mathrm{i}y)| \leqslant M_0^{1-y}M_1^y, \quad \text{if } 0 \leqslant y \leqslant 1.$$

(3) As a consequence, prove that $\log \sup\limits_{x\in\mathbb{R}} |F(x+\mathrm{i}y)|$ is a convex function of y when $0 \leqslant y \leqslant 1$.

Proof (1) Let $F_\varepsilon(z) = F(z)\mathrm{e}^{-\varepsilon z^2}$ for some $\varepsilon > 0$, then

$$|F_\varepsilon(z)| = |F(z)|\mathrm{e}^{-\varepsilon\left(x^2-y^2\right)} \to 0 \quad \text{as } x \to \infty.$$

Hence $|F_\varepsilon(z)|$ achieves its maximum in the strip at some point $z_0 \neq \infty$. Using the maximum modulus principle on some region with compact closure that contains z_0, we see that z_0 must lie on the boundary of the strip. Thus $|F_\varepsilon(z)| \leqslant |F_\varepsilon(z_0)| \leqslant 1$, and by letting $\varepsilon \to 0$ we get $|F(z)| \leqslant 1$ throughout strip.

(2) Let $G(z)=M_0^{-\mathrm{i}z-1}M_1^{\mathrm{i}z}F(z)$, then $G(z)$ satisfies the conditions of (1), i.e.

$$|G(x)|=\left|M_0^{-\mathrm{i}x-1}\right|\cdot\left|M_1^{\mathrm{i}x}\right|\cdot|F(x)|=M_0^{-1}|F(x)|\leqslant 1,\quad \forall x\in\mathbb{R},$$
$$|G(x+\mathrm{i})|=\left|M_0^{-\mathrm{i}(x+\mathrm{i})-1}\right|\cdot\left|M_1^{\mathrm{i}(x+\mathrm{i})}\right|\cdot|F(x+\mathrm{i})|=M_1^{-1}|F(x+\mathrm{i})|\leqslant 1,\quad \forall x\in\mathbb{R}.$$

By (1), we have $|G(z)|\leqslant 1$ throughout the strip, i.e.

$$|G(z)|=\left|M_0^{-\mathrm{i}(x+\mathrm{i}y)-1}\right|\cdot\left|M_1^{\mathrm{i}(x+\mathrm{i}y)}\right|\cdot|F(z)|=M_0^{y-1}M_1^{-y}|F(x+\mathrm{i}y)|\leqslant 1,$$

which implies the desired result.

(3) Set $M(y)=\sup\limits_{x\in\mathbb{R}}|F(x+\mathrm{i}y)|$ for $y\in[0,1]$. For $0\leqslant y_1<y_2\leqslant 1$, by scaling we see the result in (2) applies to the strip $y_1<\operatorname{Im}z<y_2$, i.e., for all $y\in[y_1,y_2]$,

$$\log M(y)\leqslant\log\Big(M(y_1)^{\frac{y_2-y}{y_2-y_1}}M(y_2)^{\frac{y-y_1}{y_2-y_1}}\Big)=\frac{y_2-y}{y_2-y_1}\log M(y_1)+\frac{y-y_1}{y_2-y_1}\log M(y_2),$$

which implies the convexity of $\log M(y)$. □

补充题 5 设 $|w|\leqslant 1$, 估计使 $|1-\mathrm{e}^w|\leqslant c|w|$ 成立的常数 c.

解答 记 $f(w)=\dfrac{1-\mathrm{e}^w}{w}$, 由于 0 是可去奇点, 因此 $f\in\mathcal{H}(\mathbb{B}(0,1))$, 作幂级数展开可得

$$\mathrm{e}^w-1=\sum_{n=1}^{\infty}\frac{w^n}{n!}\implies|f(w)|=\left|\sum_{n=0}^{\infty}\frac{w^n}{(n+1)!}\right|\leqslant\sum_{n=0}^{\infty}\frac{1}{(n+1)!}=\mathrm{e}-1.$$

因此可取 $c=\mathrm{e}-1$ (代入 $w=1$ 可知这是最佳常数). □

Stein 5.6.1 Give another proof of Jensen's formula in the unit disc using the functions (called Blaschke factors)

$$\psi_\alpha(z)=\frac{\alpha-z}{1-\bar{\alpha}z}.$$

Proof Let Ω be an open set that contains the closure of a disc D_R and suppose that f is holomorphic in Ω, $f(0)\neq 0$, and f vanishes nowhere on the circle C_R. Let $z_1,\cdots,z_N$ denote the zeros of f inside the disc (counted with multiplicities), we want to show that

$$\log|f(0)|=\sum_{k=1}^{N}\log\left(\frac{|z_k|}{R}\right)+\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(R\mathrm{e}^{\mathrm{i}\theta})\right|\mathrm{d}\theta.$$

(1) First, we observe that if f_1 and f_2 are two functions satisfying the hypotheses and the conclusion of the theorem, then so does their product f_1f_2.

(2) By setting $\widetilde{f}(z)=f(Rz)$, what we want to prove can be reduced to the specific case when $R=1$. Note that the function

$$g(z)=\frac{f(z)}{\psi_{z_1}(z)\cdots\psi_{z_N}(z)}$$

initially defined on $\Omega\setminus\{z_1,\cdots,z_N\}$, is bounded near each z_j. Therefore each z_j is a removable singularity, and hence we can write

$$f(z)=\psi_{z_1}(z)\cdots\psi_{z_N}(z)g(z)$$

where g is holomorphic in Ω and nowhere vanishing in $\overline{\mathbb{B}(0,1)}$. By (1) above, it suffices to prove Jensen's formula for functions like g that vanish nowhere, and for Blaschke factors.

(3) The case of functions that vanish nowhere follows from the mean value theorem for holomorphic functions. So it remains to show the result for Blaschke factors. We have

$$\log|\psi_\alpha(0)| = \log|\alpha| = \log|\alpha| + \frac{1}{2\pi}\int_0^{2\pi} \log\left|\psi_\alpha(\mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\theta$$

since $|\psi_\alpha(z)| = 1$ for $z \in \partial\mathbb{B}(0,1)$. □

Stein 5.6.3 Show that if τ is fixed with $\operatorname{Im}(\tau) > 0$, then the Jacobi theta function

$$\Theta(z \mid \tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi \mathrm{i} n^2 \tau} \mathrm{e}^{2\pi \mathrm{i} n z}$$

is of order 2 as a function of z.

Proof We have

$$\begin{aligned}
|\Theta(z \mid \tau)| &\leqslant \sum_{n=-\infty}^{\infty} \left|\mathrm{e}^{\pi i n^2 \tau} \mathrm{e}^{2\pi i n z}\right| \leqslant \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 \operatorname{Im}(\tau) + 2\pi n |z|} \\
&= \underbrace{\sum_{n < \frac{4|z|}{\operatorname{Im}(\tau)}} \mathrm{e}^{-\pi n^2 \operatorname{Im}(\tau) + 2\pi n |z|}}_{2\pi n|z| \leqslant \frac{8\pi|z|^2}{\operatorname{Im}(\tau)} \text{ when } n < \frac{4|z|}{\operatorname{Im}(\tau)}} + \underbrace{\sum_{n \geqslant \frac{4|z|}{\operatorname{Im}(\tau)}} \mathrm{e}^{-\pi n^2 \operatorname{Im}(\tau) + 2\pi n |z|}}_{-n^2 \operatorname{Im}(\tau) + 2n|z| \leqslant -\frac{n^2 \operatorname{Im}(\tau)}{2} \text{ when } n \geqslant \frac{4|z|}{\operatorname{Im}(\tau)}} \\
&\leqslant \mathrm{e}^{\frac{8\pi|z|^2}{\operatorname{Im}(\tau)}} \sum_{n < \frac{4|z|}{\operatorname{Im}(\tau)}} \mathrm{e}^{-\pi n^2 \operatorname{Im}(\tau)} + \sum_{n \geqslant \frac{4|z|}{\operatorname{Im}(\tau)}} \mathrm{e}^{-\frac{\pi n^2 \operatorname{Im}(\tau)}{2}} \\
&\overset{\mathrm{e}^{-x} \leqslant \frac{1}{x+1}}{\leqslant} \mathrm{e}^{\frac{8\pi|z|^2}{\operatorname{Im}(\tau)}} \sum_{n < \frac{4|z|}{\operatorname{Im}(\tau)}} \frac{1}{\pi n^2 \operatorname{Im}(\tau) + 1} + \sum_{n \geqslant \frac{4|z|}{\operatorname{Im}(\tau)}} \frac{1}{\frac{\pi n^2 \operatorname{Im}(\tau)}{2} + 1} \\
&\leqslant C_1 \mathrm{e}^{\frac{8\pi|z|^2}{\operatorname{Im}(\tau)}} + C_2.
\end{aligned}$$

It remains to show that the order is at least 2. We use repeatedly Proposition 1.1 (iii) in Chapter 10, which is about the quasi-periodicity of $\Theta(z \mid \tau)$, to see that

$$\Theta(x + m\tau \mid \tau) = \mathrm{e}^{-2\pi \mathrm{i} m x - \pi \mathrm{i} m^2 \tau} \Theta(x \mid \tau).$$

Then take $x = 0$ to get

$$|\Theta(m\tau \mid \tau)| = \mathrm{e}^{\pi m^2 \operatorname{Im}(\tau)} \Theta(0 \mid \tau) = A\mathrm{e}^{B|m\tau|^2},$$

which shows that the order of $\Theta(z \mid \tau)$ is at least 2. □

Stein 5.6.5 Show that if $\alpha > 1$, then

$$F_\alpha(z) = \int_{-\infty}^{\infty} \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{2\pi \mathrm{i} z t} \, \mathrm{d}t$$

is an entire function of growth order $\dfrac{\alpha}{\alpha - 1}$.

Proof By Fubini's theorem we have

$$\int_\gamma F_\alpha(z)\,\mathrm{d}z = \int_\gamma \int_{-\infty}^{\infty} \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{2\pi \mathrm{i} z t}\,\mathrm{d}t\,\mathrm{d}z = \int_{-\infty}^{\infty} \int_\gamma \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{2\pi \mathrm{i} z t}\,\mathrm{d}z\,\mathrm{d}t = \int_{-\infty}^{\infty} 0\,\mathrm{d}t = 0$$

for all closed curves γ, hence from Morera's theorem we see that $F_\alpha(z)$ is an entire function. To approximate the order of $F_\alpha(z)$, we first set $A = 4\pi$ and observe that

◇ If $|t|^{\alpha-1} \leqslant A|z|$, then

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| \leqslant 2\pi|z|A^{\frac{1}{\alpha-1}}|z|^{\frac{1}{\alpha-1}} = 2\pi A^{\frac{1}{\alpha-1}}|z|^{\frac{\alpha}{\alpha-1}}.$$

◇ If $|t|^{\alpha-1} > A|z|$, then

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| = |t|\left(-\frac{|t|^{\alpha-1}}{2} + 2\pi|z|\right) \leqslant |t|\left(-\frac{A|z|}{2} + 2\pi|z|\right) = |t||z|\left(2\pi - \frac{A}{2}\right) = 0.$$

So we can conclude that

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| \leqslant c|z|^{\frac{\alpha}{\alpha-1}} \tag{5.6.5–1}$$

for some constant $c > 0$. Denote ρ the order of growth of $F_\alpha(z)$.

(1) We first show that $\rho \leqslant \dfrac{\alpha}{\alpha-1}$. Using (5.6.5–1) we have

$$\begin{aligned} |F_\alpha(z)| &\leqslant \int_{\mathbb{R}} \mathrm{e}^{-|t|^\alpha + 2\pi|z||t|}\,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\frac{|t|^\alpha}{2}} \mathrm{e}^{-\frac{|t|^\alpha}{2} + 2\pi|z||t|}\,\mathrm{d}t \\ &\leqslant \mathrm{e}^{c|z|^{\frac{\alpha}{\alpha-1}}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{|t|^\alpha}{2}}\,\mathrm{d}t = 2\mathrm{e}^{c|z|^{\frac{\alpha}{\alpha-1}}} \int_0^{+\infty} \mathrm{e}^{-\frac{t^\alpha}{2}}\,\mathrm{d}t \\ &\leqslant 2\mathrm{e}^{c|z|^{\frac{\alpha}{\alpha-1}}} \left(1 + \int_1^{+\infty} \mathrm{e}^{-\frac{t}{2}}\,\mathrm{d}t\right) = 2c\left(1 + \mathrm{e}^{-\frac{1}{2}}\right)\mathrm{e}^{c|z|^{\frac{\alpha}{\alpha-1}}}, \end{aligned}$$

hence $\rho \leqslant \dfrac{\alpha}{\alpha-1}$.

(2) Next we show that $\rho \geqslant \dfrac{\alpha}{\alpha-1}$. For simplicity we consider $G_\alpha(z) = F_\alpha\left(\frac{z}{2\pi \mathrm{i}}\right) = \int_{\mathbb{R}} \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{zt}\,\mathrm{d}t$ and it has the same order of growth as $F_\alpha(z)$. Suppose to the contrary that $\rho < \dfrac{\alpha}{\alpha-1}$, and that

$$|G_\alpha(z)| \leqslant A\mathrm{e}^{B|z|^\rho}, \quad \forall z \in \mathbb{C}$$

for some positive constants A and B. For $R \in \mathbb{R}_{>0}$, we have

$$G_\alpha(R) = \int_{\mathbb{R}} \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{Rt}\,\mathrm{d}t > \int_0^{+\infty} \mathrm{e}^{-t^\alpha} \mathrm{e}^{Rt}\,\mathrm{d}t > \int_0^{\frac{R^{\frac{1}{\alpha-1}}}{2}} \mathrm{e}^{-t^\alpha} \mathrm{e}^{Rt}\,\mathrm{d}t > \mathrm{e}^{-\frac{R^\rho}{2^\alpha}} \int_0^{\frac{R^{\frac{1}{\alpha-1}}}{2}} \mathrm{e}^{Rt}\,\mathrm{d}t.$$

Therefore we have

$$G_\alpha(R) > \mathrm{e}^{-\frac{R^{\frac{\alpha}{\alpha-1}}}{2^\alpha}} \frac{1}{R}\left(\mathrm{e}^{\frac{R^{\frac{\alpha}{\alpha-1}}}{2}} - 1\right) = \frac{1}{R}\left(\mathrm{e}^{\left(\frac{1}{2} - \frac{1}{2^\alpha}\right)R^{\frac{\alpha}{\alpha-1}}} - 1\right).$$

But we know that

$$G_\alpha(R) \leqslant Ae^{BR^\rho} \implies \frac{1}{R}\left(e^{\left(\frac{1}{2}-\frac{1}{2^\alpha}\right)R^{\frac{\alpha}{\alpha-1}}}-1\right) < Ae^{BR^\rho},$$

which does not hold for large R by our assumption that $\rho < \dfrac{\alpha}{\alpha-1}$.

Now we conclude that $F_\alpha(z)$ is an entire function of growth order $\dfrac{\alpha}{\alpha-1}$. □

Stein 5.6.7 Establish the following properties of infinite products.

(1) Show that if $\sum_{n=1}^{\infty}|a_n|^2$ converges, then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges to a non-zero limit if and only if $\sum_{n=1}^{\infty}a_n$ converges.

(2) Find an example of a sequence of complex numbers $\{a_n\}$ such that $\sum_{n=1}^{\infty}a_n$ converges but $\prod_{n=1}^{\infty}(1+a_n)$ diverges.

(3) Also find an example such that $\prod_{n=1}^{\infty}(1+a_n)$ converges and $\sum_{n=1}^{\infty}a_n$ diverges.

Solution (1) If $\sum_{n=1}^{\infty}|a_n|^2$ converges, then $\lim_{n\to\infty} a_n = 0$, hence

$$\lim_{n\to\infty}\frac{a_n-\log(1+a_n)}{a_n^2}=\frac{1}{2}.$$

By the limit comparison test, we see that $\sum_{n=1}^{\infty}[a_n-\log(1+a_n)]$ converges, then

$$\prod_{n=1}^{\infty}(1+a_n) \text{ converges to a non-zero limit} \iff \sum_{n=1}^{\infty}\log(1+a_n) \text{ converges} \iff \sum_{n=1}^{\infty}a_n \text{ converges.}$$

(2) Let $a_n=\dfrac{(-1)^n}{\sqrt{n}}$, then $\sum_{n=2}^{\infty}a_n$ converges by the Leibniz's test for alternating series, but

$$\prod_{n=2}^{\infty}a_n=\prod_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2k}}\right)\left(1-\frac{1}{\sqrt{2k+1}}\right)=:\prod_{k=1}^{\infty}b_k,$$

since $b_k<\left(1+\frac{1}{\sqrt{2k+1}}\right)\left(1-\frac{1}{\sqrt{2k+1}}\right)=1-\frac{1}{2k+1}$, we have $1-b_k>\frac{1}{2k+1}$ and hence $\sum_{k=1}^{\infty}(1-b_k)$ diverges. Note that $b_k\to 1$, therefore

$$\lim_{k\to\infty}-\frac{\log b_k}{1-b_k}=1.$$

Hence $\sum_{k=1}^{\infty} -\log b_k$ diverges by the limit comparison test, and it follows that

$$\sum_{k=1}^{\infty} \log b_k \text{ diverges } \Longrightarrow \prod_{k=1}^{\infty} b_k = \prod_{n=2}^{\infty} a_n \text{ diverges.}$$

(3) Let

$$a_n = \begin{cases} -\dfrac{1}{\sqrt{k}}, & n = 2k-1, \\ \dfrac{1}{\sqrt{k}} + \dfrac{1}{k} + \dfrac{1}{k\sqrt{k}}, & n = 2k. \end{cases}$$

Then

$$\sum_{n=1}^{2N} a_n = \sum_{k=1}^{N} (a_{2k-1} + a_{2k}) = \sum_{k=1}^{N} \frac{1}{\sqrt{k}} + \sum_{k=1}^{N} \frac{1}{k\sqrt{k}} \xrightarrow{N\to\infty} +\infty,$$

but

$$\begin{aligned} \prod_{n=2}^{2N} (1 + a_n) &= (1 + a_2) \prod_{k=2}^{N} (1 + a_{2k-1})(1 + a_{2k}) = 4 \prod_{k=2}^{N} \left(1 - \frac{1}{\sqrt{k}}\right)\left(1 + \frac{1}{\sqrt{k}}\right)\left(1 + \frac{1}{k}\right) \\ &= 4 \prod_{k=2}^{N} \frac{k-1}{k} \cdot \frac{k+1}{k} = 4 \cdot \frac{N+1}{2N} \xrightarrow{N\to\infty} 2. \end{aligned}$$

□

Stein 5.6.9 Prove that if $|z| < 1$, then

$$(1+z)\left(1+z^2\right)\left(1+z^4\right)\left(1+z^8\right)\cdots = \prod_{k=0}^{\infty} \left(1 + z^{2^k}\right) = \frac{1}{1-z}.$$

Proof If we denote $P_n = \prod_{k=0}^{n-1} \left(1 + z^{2^n}\right)$, then

$$(1-z)P_n = (1-z)(1+z)\left(1+z^2\right)\cdots\left(1 + z^{2^{n-1}}\right) = 1 - z^{2n}.$$

Hence $P_n = \dfrac{1 - z^{2n}}{1-z}$ and by taking the limit as $n \to \infty$ we get the desired result when $|z| < 1$. □

Stein 5.6.10 Find the Hadamard products for:

(1) $\mathrm{e}^z - 1$;

(2) $\cos \pi z$.

Solution (1) Since $\mathrm{e}^z - 1$ has growth order 1 and $\mathrm{e}^z - 1 = 0 \iff z = 2\pi \mathrm{i} n$ for $n \in \mathbb{Z}$, by Hadamard's factorization theorem we see it has the form

$$\mathrm{e}^z - 1 = \mathrm{e}^{Az+B} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{2\pi \mathrm{i} n}\right)\left(1 + \frac{z}{2\pi \mathrm{i} n}\right) \mathrm{e}^{\frac{z}{2\pi \mathrm{i} n} - \frac{z}{2\pi \mathrm{i} n}} = \mathrm{e}^{Az+B} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Then

$$\mathrm{e}^{\frac{z}{2}} - \mathrm{e}^{-\frac{z}{2}} = \mathrm{e}^{\left(A - \frac{1}{2}\right)z + B} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Since LHS is odd we get $A = \frac{1}{2}$, and from

$$1 = \lim_{z\to 0} \frac{\mathrm{e}^z - 1}{z} = \lim_{z\to 0} \mathrm{e}^B \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

we see that $B = 0$. So we have

$$\mathrm{e}^z - 1 = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

(2) Since $\cos \pi z$ has growth order 1 and $\cos \pi z = 0 \iff z = n + \frac{1}{2}$ for $n \in \mathbb{Z}$, by Hadamard's factorization theorem we see it has the form

$$\cos \pi z = \mathrm{e}^{Az+B} \prod_{n\in\mathbb{Z}} \left(1 - \frac{z}{n+\frac{1}{2}}\right) \mathrm{e}^{\frac{z}{n+\frac{1}{2}}} = \mathrm{e}^{Az+B} \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

Since LHS is even we get $A = 0$, and by letting $z = 0$ we see that $B = 0$. So we have

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

□

Stein 5.6.13 Show that the equation $\mathrm{e}^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.

Proof Suppose to the contrary that $\mathrm{e}^z - z = 0$ has only finitely many solutions, then since $\mathrm{e}^z - z$ is entire and has growth order 1, by Hadamard's factorization theorem we have $\mathrm{e}^z - z = \mathrm{e}^{Az+B} P(z)$ for some polynomial $P(z)$. Then $P(z) = \dfrac{\mathrm{e}^z - z}{\mathrm{e}^{Az+B}} = O\left(\mathrm{e}^{(1-A)z}\right)$, which is possible only when $P(z)$ is constant and $A = 1$, and hence $z = \mathrm{e}^z\left(1 - \mathrm{e}^B C\right)$ for some constant C, which is impossible. □

Stein 5.6.14 Deduce from Hadamard's theorem that if F is entire and of growth order ρ that is non-integral, then F has infinitely many zeros.

Proof Let $k = \lfloor \rho \rfloor$, then $k < \rho < k+1$. Suppose to the contrary that F has only finitely many zeros, then by Hadamard's factorization theorem we have $F(z) = \mathrm{e}^{P(z)} Q(z)$ for some polynomials with $\deg P \leqslant k$. However, this implies that F has growth order at most k, which is a contradiction. □

Stein 5.7.1 Prove that if f is holomorphic in the unit disc, bounded and not identically zero, and $z_1, z_2, \cdots, z_n, \cdots$ are its zeros ($|z_k| < 1$), then

$$\sum_n (1 - |z_n|) < \infty.$$

Proof Without loss of generality, we may assume that $f(0) \neq 0$ (otherwise just factor out z^m) and the number of zeros is infinite. Fix $k \in \mathbb{N}$ and consider $r \in (0, 1)$ such that $\mathfrak{n}(r) > k$ and f vanishes nowhere on the circle $|z| = r$, where $\mathfrak{n}(r)$ denotes the number of zeros of f (counted with their multiplicities) inside the disc $|z| < r$. Recall Jensen's formula:

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta - \log|f(0)| = \sum_{n=1}^{\mathfrak{n}(r)} \log\left(\frac{r}{|z_n|}\right).$$

The boundedness of f implies that there exists $M > 0$ such that

$$|f(0)| \prod_{n=1}^{k} \frac{r}{|z_n|} \leqslant |f(0)| \prod_{n=1}^{\mathfrak{n}(r)} \frac{r}{|z_n|} = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta \right\} \leqslant M.$$

Let $r \to 1^-$ to see that

$$\prod_{n=1}^{k} |z_n| \geqslant \frac{|f(0)|}{M} \quad \text{for all } k \in \mathbb{N}.$$

Then by taking $k \to \infty$ we find

$$\prod_{n=1}^{\infty} |z_n| \geqslant \frac{|f(0)|}{M} > 0.$$

Therefore, by taking the logarithm we have

$$\sum_{n=1}^{\infty} (-\log|z_n|) < \infty$$

and $\lim\limits_{n\to\infty} |z_n| = 1$. Hence $\lim\limits_{n\to\infty} \dfrac{-\log|z_n|}{1-|z_n|} = 1$ and by the comparison test we get

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

□

Stein 6.3.1 Prove that

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)\cdots(s+n)}$$

whenever $s \neq 0, -1, -2, \cdots$.

Proof By Theorem 1.7 in Chapter 6 we have

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}} \quad \text{or} \quad \Gamma(s) = e^{-\gamma s} s^{-1} \prod_{n=1}^{\infty} \frac{n}{n+s} e^{\frac{s}{n}}.$$

Using the definition of Euler's constant γ one can write

$$\begin{aligned} \Gamma(s) &= \lim_{N\to\infty} \exp\left\{ -s \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right) \right\} s^{-1} \prod_{n=1}^{N} \frac{n}{n+s} e^{\frac{s}{n}} \\ &= \lim_{N\to\infty} e^{s \log N} \frac{N!}{s(s+1)\cdots(s+N)} = \lim_{N\to\infty} \frac{N^s N!}{s(s+1)\cdots(s+N)}, \end{aligned}$$

which is the desired result. □

Stein 6.3.2 Prove that

$$\prod_{n=1}^{\infty} \frac{n(n+a+b)}{(n+a)(n+b)} = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}$$

whenever a and b are positive. Using the product formula for $\sin \pi s$, give another proof that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Proof Using the formula proved in Exercise 6.3.1 we have

$$\begin{aligned}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)} &= \lim_{n\to\infty}\frac{\frac{n^{a+1}n!}{(a+1)(a+2)\cdots(a+1+n)}\cdot\frac{n^{b+1}n!}{(b+1)(b+2)\cdots(b+1+n)}}{\frac{n^{a+b+1}n!}{(a+b+1)(a+b+2)\cdots(a+b+1+n)}}\\ &= \lim_{n\to\infty}\frac{n\cdot n!(a+b+1)(a+b+2)\cdots(a+b+1+n)}{(a+1)(a+2)\cdots(a+1+n)(b+1)(b+2)\cdots(b+1+n)}\\ &= \lim_{N\to\infty}\frac{N}{N+1}\prod_{n=1}^{N+1}\frac{n(n+a+b)}{(n+a)(n+b)} = \prod_{n=1}^{\infty}\frac{n(n+a+b)}{(n+a)(n+b)}.\end{aligned}$$

In fact, the requirement that a and b are positive is unnecessary, we only need $a+1, b+1, a+b+1 \neq 0, -1, -2, \cdots$. Now for $s \in (0,1)$, set $a = s$ and $b = -s$, then

$$\Gamma(s)\Gamma(1-s) = \frac{1}{s}\cdot\frac{\Gamma(1+s)\Gamma(1-s)}{\Gamma(1)} = \frac{1}{s}\prod_{n=1}^{\infty}\frac{n^2}{n^2-s^2} = \frac{\pi}{\sin \pi s}$$

by the product formula above. The desired idntity then holds on all of $\mathbb{C}$ by analytic continuation. □

Stein 6.3.3 Show that Wallis's product formula can be written as

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}(2n+1)^{\frac{1}{2}}.$$

As a result, prove the following identity:

$$\Gamma(s)\Gamma\big(s+\tfrac{1}{2}\big) = \sqrt{\pi}2^{1-2s}\Gamma(2s).$$

Proof By Wallis's product formula we have

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n)^2-1} = \lim_{n\to\infty}\frac{2^{2n}(n!)^2(2n+1)}{[(2n+1)!!]^2} = \lim_{n\to\infty}\frac{2^{4n}(n!)^4(2n+1)}{[(2n+1)!]^2},$$

which implies the desired result. Now use the formula proved in Exercise 6.3.1 to get

$$\begin{aligned}\Gamma(s)\Gamma\big(s+\tfrac{1}{2}\big) &= \lim_{n\to\infty}\frac{n^s n!}{s(s+1)\cdots(s+n)}\cdot\frac{n^{s+\frac{1}{2}}n!}{\big(s+\frac{1}{2}\big)\big(s+\frac{3}{2}\big)\cdots\big(s+\frac{1}{2}+n\big)}\\ &= \lim_{n\to\infty}\frac{n^{2s+\frac{1}{2}}(n!)^2 2^{2n+2}}{(2s)(2s+2)\cdots(2s+2n)(2s+1)(2s+3)\cdots(2s+2n+1)}\\ &= \lim_{n\to\infty}\left(\frac{2^{2n}(n!)^2\sqrt{2n+1}}{(2n+1)!}\right)\left(\frac{(2n)^{2s}(2n)!}{(2s)(2s+1)\cdots(2s+2n)}\right)\frac{\sqrt{n}(2n+1)2^{2-2s}}{\sqrt{2n+1}(2s+2n+1)}\\ &= \sqrt{\frac{\pi}{2}}\cdot\Gamma(2s)\cdot\frac{1}{\sqrt{2}}\cdot 2^{2-2s}\\ &= \sqrt{\pi}2^{1-2s}\Gamma(2s).\end{aligned}$$

□

Remark The identity $\Gamma(s)\Gamma\big(s+\frac{1}{2}\big) = \sqrt{\pi}2^{1-2s}\Gamma(2s)$ can be derived in another way using Exercise

6.3.13. Let $f(s)=\dfrac{\Gamma(s)\Gamma\big(s+\frac{1}{2}\big)}{\Gamma(2s)}$, then

$$\frac{\mathrm{d}^2\log f(s)}{\mathrm{d}s^2}=\sum_{n=0}^{\infty}\left[\frac{1}{(s+n)^2}+\frac{1}{\big(s+n+\frac{1}{2}\big)^2}-\frac{4}{(2s+n)^2}\right]=\sum_{n=0}^{\infty}\left[\frac{1}{\big(s+\frac{n}{2}\big)^2}-\frac{1}{\big(s+\frac{n}{2}\big)^2}\right]=0,$$

Hence $\log f(s)=As+B$ for some constant A,B, and so $f(s)=\mathrm{e}^{As+B}$. Substituting $s=1$ and $s=\frac{1}{2}$ one gets $A=-2\log 2$ and $B=\log 2+\log\sqrt{\pi}$, then $f(s)=\sqrt{\pi}2^{1-2s}$.

Stein 6.3.4 Prove that if we take

$$f(z)=\frac{1}{(1-z)^{\alpha}},\quad \text{for } |z|<1$$

(defined in terms of the principal branch of the logarithm), where α is a fixed complex number, then

$$f(z)=\sum_{n=0}^{\infty}a_n(\alpha)z^n$$

with

$$a_n(\alpha)\sim\frac{1}{\Gamma(\alpha)}n^{\alpha-1}\quad \text{as } n\to\infty.$$

Proof From the Taylor series of $f(z)$ we get

$$a_n(\alpha)=\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}\sim\frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n\cdot n!}\quad \text{as } n\to\infty,$$

and our proof is complete by the formula of $\Gamma(\alpha)$ in Exercise 6.3.1. □

Stein 6.3.5 Use the fact that $\Gamma(s)\Gamma(1-s)=\dfrac{\pi}{\sin\pi s}$ to prove that

$$\big|\Gamma\big(\tfrac{1}{2}+\mathrm{i}t\big)\big|=\sqrt{\frac{2\pi}{\mathrm{e}^{\pi t}+\mathrm{e}^{-\pi t}}},\quad \text{whenever } t\in\mathbb{R}.$$

Proof By the definition of $\Gamma(s)$ we have $\Gamma(\bar{s})=\overline{\Gamma(s)}$, so substituting $s=\frac{1}{2}+\mathrm{i}t$ we get

$$\frac{\pi}{\sin\pi\big(\frac{1}{2}+\mathrm{i}t\big)}=\Gamma\big(\tfrac{1}{2}+\mathrm{i}t\big)\Gamma\big(\tfrac{1}{2}-\mathrm{i}t\big)=\big|\Gamma\big(\tfrac{1}{2}+\mathrm{i}t\big)\big|^2.$$

Then the desired result follows from

$$\frac{\pi}{\sin\pi\big(\frac{1}{2}+\mathrm{i}t\big)}=\frac{\pi}{\cos(\mathrm{i}\pi t)}=\frac{2\pi}{\mathrm{e}^{\pi t}+\mathrm{e}^{-\pi t}}.$$

□

Stein 6.3.6 Show that

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}-\frac{1}{2}\log n\to\frac{\gamma}{2}+\log 2,$$

where γ is Euler's constant.

Proof By the definition of γ we have

$$\left(\sum_{k=1}^{n}\frac{1}{2k-1}-\frac{1}{2}\log n\right)-\frac{1}{2}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)=\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\xrightarrow{n\to\infty}\log 2.$$

□

Stein 6.3.7 The Beta function is defined for $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$ by

$$\mathrm{B}(\alpha,\beta)=\int_0^1(1-t)^{\alpha-1}t^{\beta-1}\,\mathrm{d}t.$$

(1) Prove that $\mathrm{B}(\alpha,\beta)=\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(2) Show that $\mathrm{B}(\alpha,\beta)=\displaystyle\int_0^\infty\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}\,\mathrm{d}u$.

Proof (1) A change of variables gives

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta)&=\int_0^\infty\int_0^\infty t^{\alpha-1}s^{\beta-1}\mathrm{e}^{-t-s}\,\mathrm{d}t\,\mathrm{d}s\\&\xlongequal[t=u(1-r)]{s=ur}\int_0^\infty\int_0^1(ur)^{\beta-1}[u(1-r)]^{\alpha-1}\mathrm{e}^{-u}u\,\mathrm{d}r\,\mathrm{d}u\\&=\int_0^\infty u^{\alpha+\beta-1}\mathrm{e}^{-u}\,\mathrm{d}u\int_0^1(1-r)^{\alpha-1}r^{\beta-1}\,\mathrm{d}r\\&=\Gamma(\alpha+\beta)\mathrm{B}(\alpha,\beta).\end{aligned}$$

(2) Substituting $t=\dfrac{1}{1+u}$ in the integral we get

$$\mathrm{B}(\alpha,\beta)=\int_0^\infty\left(\frac{u}{1+u}\right)^{\alpha-1}\left(\frac{1}{1+u}\right)^{\beta-1}\frac{1}{(1+u)^2}\,\mathrm{d}u=\int_0^\infty\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}\,\mathrm{d}u.$$

□

Stein 6.3.8 The Bessel functions arise in the study of spherical symmetries and the Fourier transform. Prove that the following power series identity holds for Bessel functions of real order $\nu>-\frac{1}{2}$:

$$J_\nu(x)=\frac{\left(\frac{x}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}}\int_{-1}^1\mathrm{e}^{\mathrm{i}xt}\left(1-t^2\right)^{\nu-\frac{1}{2}}\,\mathrm{d}t=\left(\frac{x}{2}\right)^\nu\sum_{m=0}^\infty\frac{(-1)^m\left(\frac{x^2}{4}\right)^m}{m!\Gamma(\nu+m+1)}$$

whenever $x>0$. In particular, the Bessel function J_ν satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2J_\nu}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}J_\nu}{\mathrm{d}x}+\left(1-\frac{\nu^2}{x^2}\right)J_\nu=0.$$

Proof Expand the exponential $\mathrm{e}^{\mathrm{i}xt}$ in a power series and switch the order of summation and integration to get

$$J_\nu(x)=\frac{\left(\frac{x}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}}\sum_{n=0}^\infty\frac{(\mathrm{i}x)^n}{n!}\int_{-1}^1t^n\left(1-t^2\right)^{\nu-\frac{1}{2}}\,\mathrm{d}t$$

$$
\begin{aligned}
&= \frac{\left(\frac{x}{2}\right)^{\nu}}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(\mathrm{i}x)^{2m}}{(2m)!} \cdot 2\int_0^1 t^{2m}\left(1-t^2\right)^{\nu-\frac{1}{2}}\,\mathrm{d}t \\
&= \frac{\left(\frac{x}{2}\right)^{\nu}}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(\mathrm{i}x)^{2m}}{(2m)!} \underbrace{\int_0^1 \left(t^2\right)^{m-\frac{1}{2}}\left(1-t^2\right)^{\nu-\frac{1}{2}}\,\mathrm{d}t^2}_{\mathrm{B}\left(m+\frac{1}{2},\nu+\frac{1}{2}\right)} \\
&\xlongequal{\text{Exercise 6.3.7}} \frac{\left(\frac{x}{2}\right)^{\nu}}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(\mathrm{i}x)^{2m}}{(2m)!} \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(m+\nu+1)} \\
&= \frac{\left(\frac{x}{2}\right)^{\nu}}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+\nu+1)}.
\end{aligned}
$$

Note that

$$
\Gamma\left(m+\tfrac{1}{2}\right) = \left(m-\tfrac{1}{2}\right)\left(m-\tfrac{3}{2}\right)\cdots\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{(2m-1)!!}{2^m}\sqrt{\pi} = \frac{(2m)!}{2^{2m}m!}\sqrt{\pi},
$$

substituting this into the above formula we proved the identity. To verify that $J_\nu(x)$ solves the linear ODE, we only need to check termwise, also ignore those coefficients that only depend on ν:

$$
\frac{(-1)^m x^{\nu+2m}}{4^m m!\Gamma(\nu+m+1)} \xrightarrow{\cdot\left(1-\frac{\nu^2}{x^2}\right)} \left(x^2-\nu^2\right)\frac{(-1)^m x^{\nu+2m-2}}{4^m m!\Gamma(\nu+m+1)}
$$

$$
\frac{(-1)^m x^{\nu+2m}}{4^m m!\Gamma(\nu+m+1)} \xrightarrow{\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}} \frac{(-1)^m(\nu+2m)x^{\nu+2m-2}}{4^m m!\Gamma(\nu+m+1)}
$$

$$
\frac{(-1)^m x^{\nu+2m}}{4^m m!\Gamma(\nu+m+1)} \xrightarrow{\frac{\mathrm{d}^2}{\mathrm{d}x^2}} \frac{(-1)^m(\nu+2m)(\nu+2m-1)x^{\nu+2m-2}}{4^m m!\Gamma(\nu+m+1)}
$$

Take each coefficient before $x^{\nu+2m-2}$ and add them together:

$$
\begin{aligned}
&\frac{(-1)^{m-1}}{4^{m-1}(m-1)!\Gamma(\nu+m)} - \frac{(-1)^m\nu^2}{4^m m!\Gamma(\nu+m+1)} + \frac{(-1)^m(\nu+2m)}{4^m m!\Gamma(\nu+m+1)} + \frac{(-1)^m(\nu+2m)(\nu+2m-1)}{4^m!\Gamma(\nu+m+1)} \\
=&\frac{(-1)^m}{4^m m!\Gamma(\nu+m+1)}\left[-4m(\nu+m)-\nu^2+(\nu+2m)+(\nu+2m)(\nu+2m-1)\right]=0,
\end{aligned}
$$

which is the desired result. □

Stein 6.3.9 The hypergeometric series $F(\alpha,\beta,\gamma;z)$ is defined by

$$
F(\alpha,\beta,\gamma;z) = 1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)}z^n.
$$

Show that

$$
F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,\mathrm{d}t.
$$

Here $\alpha>0, \beta>0, \gamma>\beta$, and $|z|<1$.

Show as a result that the hypergeometric function, initially defined by a power series convergent in the unit disc, can be continued analytically to the complex plane slit along the half-line $[1,\infty)$.

Note that

$$\log(1-z) = -zF(1,1,2;z),$$
$$e^z = \lim_{\beta\to\infty} F\left(1,\beta,1;\tfrac{z}{\beta}\right),$$
$$(1-z)^{-\alpha} = F(\alpha,1,1;z).$$

Proof Since $|z|<1$, we can expand $(1-zt)^{-\alpha}$ as a power series to get

$$\begin{aligned}
&\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,\mathrm{d}t\\
=&\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\sum_{n=0}^{\infty}(-1)^n\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}(-zt)^n\,\mathrm{d}t\\
=&\sum_{n=0}^{\infty}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}\underbrace{\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}\,\mathrm{d}t}_{\mathrm{B}(\beta+n,\gamma-\beta)}z^n\\
=&\sum_{n=0}^{\infty}\frac{\Gamma(\gamma)\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\beta)\Gamma(\gamma-\beta)\Gamma(\gamma+n)}\cdot\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}z^n\\
=&\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)}z^n\\
=&F(\alpha,\beta,\gamma;z).
\end{aligned}$$

The desired continuation follows directly from the above formula. □

Stein 6.3.10 An integral of the form

$$F(z) = \int_0^{\infty} f(t)t^{z-1}\,\mathrm{d}t$$

is called a Mellin transform, and we shall write $\mathcal{M}(f)(z) = F(z)$. For example, the gamma function is the Mellin transform of the function e^{-t}.

(1) Prove that

$$\mathcal{M}(\cos)(z) = \int_0^{\infty}\cos(t)t^{z-1}\,\mathrm{d}t = \Gamma(z)\cos\left(\pi\tfrac{z}{2}\right) \quad \text{for } 0<\mathrm{Re}(z)<1,$$

and

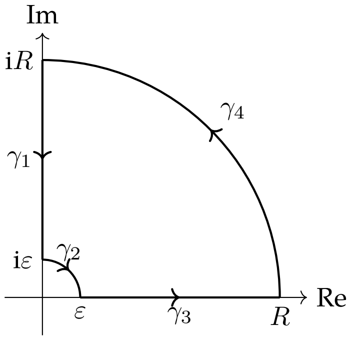
$$\mathcal{M}(\sin)(z) = \int_0^{\infty}\sin(t)t^{z-1}\,\mathrm{d}t = \Gamma(z)\sin\left(\pi\tfrac{z}{2}\right) \quad \text{for } 0<\mathrm{Re}(z)<1.$$

(2) Show that the second of the above identities is valid in the larger strip $-1<\mathrm{Re}(z)<1$, and that as a consequence, one has

$$\int_0^{\infty}\frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2} \quad\text{and}\quad \int_0^{\infty}\frac{\sin x}{x^{\frac{3}{2}}}\,\mathrm{d}x = \sqrt{2\pi}.$$

This generalizes the calculation in Exercise 2 of Chapter 2.

Proof (1) Consider the integral of $f(w) = \mathrm{e}^{-w} w^{z-1}$ around the contour illustrated below.



Since

$$\left|\int_{\gamma_2} \mathrm{e}^{-w} w^{z-1}\,\mathrm{d}w\right| = \left|\int_0^{\frac{\pi}{2}} \mathrm{i}\varepsilon \mathrm{e}^{-\varepsilon \mathrm{e}^{\mathrm{i}\theta}} \left(\varepsilon \mathrm{e}^{\mathrm{i}\theta}\right)^{z-1} \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\right| \leqslant \varepsilon^{\mathrm{Re}(z)} \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\varepsilon\cos\theta - \mathrm{Im}(z)\theta}\,\mathrm{d}\theta \xrightarrow[0<\mathrm{Re}(z)<1]{\varepsilon\to 0^+} 0$$

and similarly

$$\left|\int_{\gamma_4} \mathrm{e}^{-w} w^{z-1}\,\mathrm{d}w\right| = \left|\int_0^{\frac{\pi}{2}} \mathrm{i}R \mathrm{e}^{-R \mathrm{e}^{\mathrm{i}\theta}} \left(R \mathrm{e}^{\mathrm{i}\theta}\right)^{z-1} \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\right| \leqslant R^{\mathrm{Re}(z)} \int_0^{\frac{\pi}{2}} \mathrm{e}^{-R\cos\theta - \mathrm{Im}(z)\theta}\,\mathrm{d}\theta \xrightarrow[0<\mathrm{Re}(z)<1]{R\to +\infty} 0$$

by DCT, letting $\varepsilon \to 0^+$ and $R \to +\infty$ we have

$$\Gamma(z) = \int_0^\infty \mathrm{i}\mathrm{e}^{-\mathrm{i}t} (\mathrm{i}t)^{z-1}\,\mathrm{d}t = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}z} \int_0^\infty \mathrm{e}^{-\mathrm{i}t} t^{z-1}\,\mathrm{d}t \iff \int_0^\infty \mathrm{e}^{-\mathrm{i}t} t^{z-1}\,\mathrm{d}t = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}z}\Gamma(z).$$

Similarly, by choosing the contour that reflects over the real axis one gets

$$\int_0^\infty \mathrm{e}^{\mathrm{i}t} t^{z-1}\,\mathrm{d}t = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}z}\Gamma(z).$$

Then it follows that

$$\int_0^\infty \cos(t) t^{z-1}\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}}{2} t^{z-1}\,\mathrm{d}t = \Gamma(z) \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{2}z} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}z}}{2} = \Gamma(z)\cos\left(\pi \tfrac{z}{2}\right)$$

and

$$\int_0^\infty \sin(t) t^{z-1}\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}} t^{z-1}\,\mathrm{d}t = \Gamma(z) \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{2}z} - \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}z}}{2\mathrm{i}} = \Gamma(z)\sin\left(\pi \tfrac{z}{2}\right).$$

(2) Integrating by parts we have

$$\begin{aligned}\int_0^\infty \sin(t) t^{z-1}\,\mathrm{d}t &= \int_0^1 \sin(t) t^{z-1}\,\mathrm{d}t + \int_1^\infty \sin(t) t^{z-1}\,\mathrm{d}t \\ &= \underbrace{\int_0^1 \frac{\sin t}{t} \cdot t^z\,\mathrm{d}t}_{\text{holomorphic when } \mathrm{Re}(z) > -1} + \underbrace{\cos 1 + (z-1)\int_1^\infty \cos(t) t^{z-2}\,\mathrm{d}t}_{\text{holomorphic when } \mathrm{Re}(z) < 1},\end{aligned}$$

hence $\mathcal{M}(\sin)(z)$ is well-defined for $-1 < \mathrm{Re}(z) < 1$, and the second identity gets valid by analytic continuation and the identity theorem for holomorphic functions. Taking $z=0$ and $z=-\frac{1}{2}$ we get

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \mathcal{M}(\sin)(0) = \lim_{z\to 0}\Gamma(z)\sin\left(\pi\tfrac{z}{2}\right) = \lim_{z\to 0}\frac{\pi}{2}z\Gamma(z) = \frac{\pi}{2}\Gamma(1) = \frac{\pi}{2}$$

and

$$\int_0^\infty \frac{\sin x}{x^{\frac{3}{2}}}\,\mathrm{d}x = \mathcal{M}(\sin)\left(-\tfrac{1}{2}\right) = \Gamma\left(-\tfrac{1}{2}\right)\sin\left(-\tfrac{\pi}{4}\right) = -2\sqrt{\pi}\left(-\tfrac{1}{\sqrt{2}}\right) = \sqrt{2\pi}.$$

□

Stein 6.3.11 Let $f(z)=\mathrm{e}^{az}\mathrm{e}^{-\mathrm{e}^z}$ where $a>0$. Observe that in the strip $\left\{x+\mathrm{i}y : |y|<\frac{\pi}{2}\right\}$ the function $f(x+\mathrm{i}y)$ is exponentially decreasing as $|x|$ tends to infinity. Prove that

$$\hat{f}(\xi) = \Gamma(a-2\pi\mathrm{i}\xi), \quad \text{for all } \xi\in\mathbb{R}.$$

Proof Since

$$|f(x+\mathrm{i}y)| = \left|\mathrm{e}^{a(x+\mathrm{i}y)-\mathrm{e}^{x+\mathrm{i}y}}\right| = \mathrm{e}^{ax-\mathrm{e}^x\cos y} \quad \text{and } \cos y>0 \text{ when } |y|<\frac{\pi}{2},$$

we see that $f(x+\mathrm{i}y)$ is exponentially decreasing as $|x|\to\infty$. Using a substitution $t=\mathrm{e}^x$ we have

$$\hat{f}(\xi) = \int_{-\infty}^\infty \mathrm{e}^{ax-\mathrm{e}^x-2\pi\mathrm{i}\xi x}\,\mathrm{d}x = \int_0^\infty t^{a-2\pi\mathrm{i}\xi-1}\mathrm{e}^{-t}\,\mathrm{d}t = \Gamma(a-2\pi\mathrm{i}\xi).$$

□

Stein 6.3.12 This exercise gives two simple observations about $1/\Gamma$.

(1) Show that $\dfrac{1}{|\Gamma(s)|}$ is not $O\left(\mathrm{e}^{c|s|}\right)$ for any $c>0$.

(2) Show that there is no entire function $F(s)$ with $F(s)=O\left(\mathrm{e}^{c|s|}\right)$ that has simple zeros at $s=0,-1,-2,\cdots,-n,\cdots$, and that vanishes nowhere else.

Proof (1) Using $s\Gamma(s)=\Gamma(s+1)$, for $k\in\mathbb{N}$, we have

$$\Gamma\left(-k-\tfrac{1}{2}\right) = \frac{\Gamma\left(-k+\frac{1}{2}\right)}{-k-\frac{1}{2}} = \cdots = \frac{\sqrt{\pi}}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-k-\frac{1}{2}\right)},$$

hence

$$\left|\frac{1}{\Gamma\left(-k-\frac{1}{2}\right)}\right| = \frac{\frac{3}{2}\cdot\frac{5}{2}\cdots\left(k+\frac{1}{2}\right)}{2\sqrt{\pi}} \geqslant \frac{k!}{2\sqrt{\pi}}.$$

If $\dfrac{1}{|\Gamma(s)|}$ is $O\left(\mathrm{e}^{c|s|}\right)$ for some $c>0$, then there exists $C>0$ such that

$$k! \leqslant C\mathrm{e}^{c\left(k+\frac{1}{2}\right)} \quad \text{for all } k\in\mathbb{N},$$

which is impossible since $\lim\limits_{k\to\infty} k!\mathrm{e}^{-c\left(k+\frac{1}{2}\right)} = +\infty$.

(2) Suppose that $F(s)$ is such a function with growth order $\leqslant 1$, then by Hadamard's factorization theorem we have

$$F(s) = \mathrm{e}^{As+B}s\prod_{n=1}^\infty\left(1+\frac{s}{n}\right)\mathrm{e}^{-\frac{s}{n}}.$$

Combine this with the Weierstrass product for $\Gamma(s)$ in Theorem 1.7 of Chapter 6 we get

$$\frac{1}{\Gamma(s)} = F(s)\mathrm{e}^{(\gamma-A)s-B},$$

but this contradicts (1) by our assumption on $F(s)$. □

Stein 6.3.13 Prove that

$$\frac{\mathrm{d}^2\log\Gamma(s)}{\mathrm{d}s^2} = \sum_{n=0}^{\infty}\frac{1}{(s+n)^2}$$

whenever s is a positive number. Show that if the left-hand side is interpreted as $(\Gamma'/\Gamma)'$, then the above formula also holds for all complex numbers s with $s \neq 0, -1, -2, \cdots$.

Proof By Theorem 1.7 in Chapter 6 we have

$$\Gamma(s) = \mathrm{e}^{-\gamma s}s^{-1}\prod_{n=1}^{\infty}\frac{n}{n+s}\mathrm{e}^{\frac{s}{n}}$$

for $s \neq 0, -1, -2, \cdots$. Then

$$\log\Gamma(s) = -\gamma s - \log s + \sum_{n=1}^{\infty}\left(\frac{s}{n} + \log\frac{n}{n+s}\right)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\Gamma(s) = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+s}\right),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\log\Gamma(s) = \frac{1}{s^2} + \sum_{n=1}^{\infty}\frac{1}{(n+s)^2} = \sum_{n=0}^{\infty}\frac{1}{(s+n)^2}.$$

Since Γ'/Γ is holomorphic on $\mathbb{C}\setminus\{0,-1,-2,\cdots\}$, its derivative is also holomorphic on this domain, hence the above formula holds for all complex numbers s with $s \neq 0, -1, -2, \cdots$. □

Stein 6.3.14 This exercise gives an asymptotic formula for $\log n!$. A more refined asymptotic formula for $\Gamma(s)$ as $s \to \infty$ (Stirling's formula) is given in Appendix A.

(1) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x+1}\log\Gamma(t)\,\mathrm{d}t = \log x, \quad \text{for } x > 0,$$

and as a result

$$\int_x^{x+1}\log\Gamma(t)\,\mathrm{d}t = x\log x - x + c.$$

(2) Show as a consequence that $\log\Gamma(n) \sim n\log n$ as $n \to \infty$. In fact, prove that $\log\Gamma(n) \sim n\log n + O(n)$ as $n \to \infty$.

Proof (1) For $x > 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x+1}\log\Gamma(t)\,\mathrm{d}t = \log\Gamma(x+1) - \log\Gamma(x) = \log\frac{\Gamma(x+1)}{\Gamma(x)} = \log x,$$

and by integrating both sides we get the second formula.

(2) Since $\log\Gamma(t)$ is monotonically increasing when $t \geqslant 1$, we have

$$\log\Gamma(n) \leqslant \int_n^{n+1} \log\Gamma(t)\,\mathrm{d}t \leqslant \log\Gamma(n+1) = \log n + \log\Gamma(n).$$

This implies that

$$(n-1)\log n - n + c \leqslant \log\Gamma(n) \leqslant n\log n - n + c,$$

which gives the desired result. □

Stein 6.3.15 Prove that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x.$$

Proof For $x > 0$ we have

$$\frac{1}{\mathrm{e}^x - 1} = \sum_{n=1}^\infty \mathrm{e}^{-nx}.$$

Substituting this into the integral and using Fubini's theorem we get

$$\int_0^\infty \frac{x^{s-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty x^{s-1}\mathrm{e}^{-nx}\,\mathrm{d}x \xlongequal{t=nx} \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t = \zeta(s)\Gamma(s).$$ □

Stein 6.3.16 Use the previous exercise to give another proof that $\zeta(s)$ is continuable in the complex plane with only singularity as a simple pole at $s = 1$.

Proof Use Exercise 6.3.15 to write

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x + \frac{1}{\Gamma(s)}\int_1^\infty \frac{x^{s-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x.$$

The second integral defines an entire function because of exponential decay near infinity, while

$$\int_0^1 \frac{x^{s-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x = \int_0^1 x^{s-2}\sum_{m=0}^\infty \frac{B_m}{m!}x^m\,\mathrm{d}x = \sum_{m=0}^\infty \frac{B_m}{m!}\int_0^1 x^{s+m-2}\,\mathrm{d}x = \sum_{m=0}^\infty \frac{B_m}{m!(s+m-1)},$$

where B_m denotes the m-th Bernoulli number defined by

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{m=0}^\infty \frac{B_m}{m!}x^m.$$

Since $\dfrac{z}{\mathrm{e}^z - 1}$ is holomorphic for $|z| < 2\pi$, and the right-hand side above has the same radius of convergence as $\displaystyle\sum_{m=0}^\infty \frac{B_m}{m!(s+m-1)}z^m$ when $s \neq 1$, we conclude that $\displaystyle\sum_{m=0}^\infty \frac{B_m}{m!(s+m-1)}$ converges for all $s \in \mathbb{C}\setminus\{1\}$. And from $B_0 = 1$ we see that $s = 1$ becomes a simple pole of $\zeta(s)$. □

Stein 6.3.17 Let f be an infinitely differentiable function on $\mathbb{R}$ that has compact support, or more generally, let f belong to the Schwartz space $\mathcal{S}$. Consider

$$I(s) = \frac{1}{\Gamma(s)}\int_0^\infty f(x)x^{-1+s}\,\mathrm{d}x.$$

(1) Observe that $I(s)$ is holomorphic for $\mathrm{Re}(s) > 0$. Prove that I has an analytic continuation as an entire function in the complex plane.

(2) Prove that $I(0) = f(0)$, and more generally

$$I(-n) = (-1)^n f^{(n)}(0) \quad \text{for all } n \geqslant 0.$$

Proof (1) Write

$$I(s) = \frac{1}{\Gamma(s)} \int_0^1 f(x) x^{-1+s} \,\mathrm{d}x + \frac{1}{\Gamma(s)} \int_1^\infty f(x) x^{-1+s} \,\mathrm{d}x.$$

The first integral defines a holomorphic function when $\mathrm{Re}(s) > 0$, and the second is holomorphic since $f \in \mathcal{S}$. Next, we integrate by parts to get

$$I(s) = -\frac{1}{s\Gamma(s)} \int_0^\infty f'(x) x^s \,\mathrm{d}x = \cdots = \frac{(-1)^k}{\Gamma(s+k)} \int_0^\infty f^{(k)}(x) x^{s+k-1} \,\mathrm{d}x.$$

This shows that $I(s)$ can be analytically continued to $\mathrm{Re}(s) > -k$ for any positive integer k. Therefore, we have obtained the desired continuation of $I(s)$.

(2) Taking $s = -n$ and $k = n+1$ in the above formula gives

$$I(-n) = \frac{(-1)^{n+1}}{\Gamma(1)} \int_0^\infty f^{(n+1)}(x) \,\mathrm{d}x = (-1)^{n+1} f^{(n)}(x)\Big|_0^\infty = (-1)^n f^{(n)}(0).$$

□

Stein 6.4.1 This problem provides further estimates for ζ and ζ' near $\mathrm{Re}(s) = 1$.

(1) Use Proposition 2.5 and its corollary to prove

$$\zeta(s) = \sum_{1 \leqslant n < N} n^{-s} - \frac{N^{1-s}}{1-s} + \sum_{n \geqslant N} \delta_n(s)$$

for every integer $N \geqslant 2$, whenever $\mathrm{Re}(s) > 0$.

(2) Show that $|\zeta(1+\mathrm{i}t)| = O(\log|t|)$, as $|t| \to \infty$ by using the previous result with $N =$ greatest integer in $|t|$.

(3) The second conclusion of Proposition 2.7 can be similarly refined.

(4) Show that if $t \neq 0$ and t is fixed, then the partial sums of the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{1+\mathrm{i}t}}$ are bounded, but the series does not converge.

Proof (1) For $\mathrm{Re}(s) > 0$ we have

$$\begin{aligned}
\zeta(s) &= \sum_{1 \leqslant n < N} \frac{1}{n^s} + \sum_{n \geqslant N} \frac{1}{n^s} \\
&= \sum_{1 \leqslant n < N} \frac{1}{n^s} + \int_N^\infty \frac{1}{x^s} \,\mathrm{d}x + \sum_{n \geqslant N} \underbrace{\int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) \mathrm{d}x}_{\delta_n(s)} \\
&= \sum_{1 \leqslant n < N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + \sum_{n \geqslant N} \delta_n(s).
\end{aligned}$$

(2) For any $t \in \mathbb{R} \setminus \{0\}$, take $N = \lfloor |t| \rfloor$ in the formula above to get

$$\begin{aligned}
|\zeta(1+\mathrm{i}t)| &= \left| \sum_{1 \leqslant n < N} n^{-1-\mathrm{i}t} - \frac{N^{-\mathrm{i}t}}{-\mathrm{i}t} + \sum_{n \geqslant N} \delta_n(1+\mathrm{i}t) \right| \\
&\leqslant \left| \sum_{1 \leqslant n < N} n^{-1-\mathrm{i}t} \right| + \left| \frac{N^{-\mathrm{i}t}}{-\mathrm{i}t} \right| + \left| \sum_{n \geqslant N} \delta_n(1+\mathrm{i}t) \right| \\
&\leqslant \sum_{1 \leqslant n < N} n^{-1} + \frac{1}{|t|} + \sum_{n \geqslant N} \frac{|1+\mathrm{i}t|}{n^2} \\
&\leqslant \sum_{1 \leqslant n < N} n^{-1} + \frac{1}{|t|} + \sqrt{1+t^2} \sum_{n \geqslant N} \frac{1}{n(n-1)} \\
&\sim \log|t| + \frac{1}{|t|} + \frac{\sqrt{1+t^2}}{|t|-1} \\
&\sim \log|t| + 1 \quad \text{as } |t| \to \infty.
\end{aligned}$$

This shows that $|\zeta(1+\mathrm{i}t)| = O(\log|t|)$ as $|t| \to \infty$.

(3) For

$$\delta_n(s) = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^2} \right) \mathrm{d}x,$$

we have

$$\delta_n'(s) = \int_n^{n+1} \left(-\frac{\log n}{n^s} + \frac{\log x}{x^s} \right) \mathrm{d}x.$$

For that $x \in [n, n+1]$ in the integrand and $|s| \geqslant 1$, we have

$$\begin{aligned}
\left| \frac{\log x}{x^s} - \frac{\log n}{n^s} \right| = \left| \int_x^n \left(\frac{\log u}{u^s} \right)' \mathrm{d}u \right| \leqslant \int_n^x \left| \frac{1 - s\log u}{u^{s+1}} \right| \mathrm{d}u \leqslant \frac{1+|s|\log x}{n^{\mathrm{Re}(s)+1}}(x-n) \\
\leqslant \frac{|s| + |s|\log(n+1)}{n^{\mathrm{Re}(s)+1}} \leqslant \frac{4|s|\log n}{n^{\mathrm{Re}(s)+1}},
\end{aligned}$$

where the scalar 4 here comes from the inequality $1 + \log(n+1) \leqslant 4\log n$ when $n \geqslant 2$. Therefore,

$$|\delta_n'(s)| \leqslant \frac{4|s|\log n}{n^{\mathrm{Re}(s)+1}}.$$

Differentiating both sides of the formula in (1) we get

$$\begin{aligned}
\zeta'(s) &= -\sum_{1 \leqslant n < N} \frac{\log n}{n^s} - \frac{(s-1)N^{1-s}\log N - N^{1-s}}{(s-1)^2} + \sum_{n \geqslant N} \delta_n'(s) \\
&= -\sum_{1 \leqslant n < N} \frac{\log n}{n^s} + \frac{N^{1-s}\log N}{1-s} - \frac{N^{1-s}}{(1-s)^2} + \sum_{n \geqslant N} \delta_n'(s).
\end{aligned}$$

Now take $s = 1 + \mathrm{i}t$ and use the facts that

- ◇ $\dfrac{\log x}{x}$ is decreasing when $x \geqslant \mathrm{e}$,
- ◇ $\dfrac{\log x}{x^2}$ is decreasing when $x \geqslant \sqrt{\mathrm{e}}$,

we have

$$
\begin{aligned}
|\zeta'(1+\mathrm{i}t)| &\leqslant \sum_{1\leqslant n<N}\left|\frac{\log n}{n^{1+\mathrm{i}t}}\right| + \left|\frac{N^{-\mathrm{i}t}\log N}{-\mathrm{i}t}\right| + \left|\frac{N^{-\mathrm{i}t}}{(-\mathrm{i}t)^2}\right| + \sum_{n\geqslant N}|\delta_n'(1+\mathrm{i}t)| \\
&\leqslant \sum_{1\leqslant n<N}\frac{\log n}{n} + \frac{\log N}{|t|} + \frac{1}{t^2} + 4\sqrt{1+t^2}\sum_{n\geqslant N}\frac{\log n}{n^2} \\
&\leqslant \frac{\log 2}{2} + \frac{\log 3}{3} + \int_3^{N-1}\frac{\log x}{x}\,\mathrm{d}x + \frac{\log N}{|t|} + \frac{1}{t^2} + 4\sqrt{1+t^2}\int_{N-1}^{\infty}\frac{\log x}{x^2}\,\mathrm{d}x.
\end{aligned}
$$

Integrating by parts we get

$$
\int_3^{N-1}\frac{\log x}{x}\,\mathrm{d}x = \frac{1}{2}(\log x)^2\Big|_3^{N-1} = \frac{[\log(N-1)]^2-(\log 3)^2}{2}
$$

and

$$
\int_{N-1}^{\infty}\frac{\log x}{x^2}\,\mathrm{d}x = \int_{N-1}^{\infty}\frac{1}{x^2}\,\mathrm{d}x - \frac{\log x}{x}\Big|_{N-1}^{\infty} = \frac{1+\log(N-1)}{N-1}.
$$

Therefore, taking $N=\lfloor |t| \rfloor$ we have

$$
\begin{aligned}
|\zeta'(1+\mathrm{i}t)| &\leqslant \frac{\log 2}{2} + \frac{\log 3}{3} + \frac{[\log(N-1)]^2-(\log 3)^2}{2} + \frac{\log N}{|t|} + \frac{1}{t^2} + 4\sqrt{1+t^2}\frac{1+\log(N-1)}{N-1} \\
&\sim \frac{\log 2}{2} + \frac{\log 3}{3} - \frac{(\log 3)^2}{2} + 4 + \frac{1}{2}(\log|t|)^2 + 4\log|t| \quad \text{as } |t|\to\infty,
\end{aligned}
$$

which implies that $\zeta'(1+\mathrm{i}t) = O\left(\log^2|t|\right)$ as $|t|\to\infty$.

(4) By the formula proved in (1) we see the partial sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^{1+\mathrm{i}t}}$ can be expressed as

$$
\sum_{1\leqslant n<N}\frac{1}{n^{1+\mathrm{i}t}} = \zeta(1+\mathrm{i}t) - \frac{N^{-\mathrm{i}t}}{\mathrm{i}t} - \sum_{n\geqslant N}\delta_n(1+\mathrm{i}t). \tag{6.4.1–1}
$$

Hence

$$
\left|\sum_{1\leqslant n<N}\frac{1}{n^{1+\mathrm{i}t}}\right| \leqslant |\zeta(1+\mathrm{i}t)| + \frac{1}{|t|} + \sum_{n\geqslant N}\frac{|1+\mathrm{i}t|}{n^2} \leqslant |\zeta(1+\mathrm{i}t)| + \frac{1}{|t|} + \frac{\pi^2\sqrt{1+t^2}}{6},
$$

which shows that the partial sums are uniformly bounded for any fixed nonzero t. To see the series does not converge, again we use (6.4.1–1) to write

$$
\sum_{1\leqslant n<N}\frac{1}{n^{1+\mathrm{i}t}} = \zeta(1+\mathrm{i}t) - \frac{N^{-\mathrm{i}t}}{\mathrm{i}t} + O\left(\tfrac{1}{N}\right) \quad \text{as } N\to\infty
$$

since we have seen in (2) that

$$
\left|\sum_{n\geqslant N}\delta_n(1+\mathrm{i}t)\right| \leqslant \sum_{n\geqslant N}\frac{|1+\mathrm{i}t|}{n^2} \leqslant \sum_{n\geqslant N}\frac{\sqrt{1+t^2}}{n(n-1)} = \frac{\sqrt{1+t^2}}{N-1}.
$$

Note that our conclusion follows from the fact that $N^{-\mathrm{i}t}$ is an oscillating term as $N\to\infty$. To prove this, one just needs to show that $\sin(t\log N)$ does not converge as $N\to\infty$. We prove by contradiction: suppose that $\lim\limits_{N\to\infty}\sin(t\log N)=\lambda$, then

$$\lim_{n\to\infty}\sin[t\log(kN)]=\lambda \quad \text{for any } k\in\mathbb{N}_+.$$

Observe that

$$\begin{aligned}\{\sin[t\log(kN)]-\sin(t\log n)\cos(t\log k)\}^2&=[\sin(t\log n+t\log k)-\sin(t\log n)\cos(t\log k)]^2\\&=\cos^2(t\log n)\sin^2(t\log k),\end{aligned}$$

letting $n\to\infty$ we get

$$\lambda^2[1-\cos(t\log k)]^2=\left(1-\lambda^2\right)\sin^2(t\log k),$$

which implies

$$\frac{\lambda^2}{1-\lambda^2}=\frac{\sin^2(t\log k)}{[1-\cos(t\log k)]^2}=\frac{1+\cos(t\log k)}{1-\cos(t\log k)}.$$

This is impossible since the right-hand side depends on k while the left-hand side does not. □

Stein 6.4.2 Prove that for $\mathrm{Re}(s)>0$

$$\zeta(s)=\frac{s}{s-1}-s\int_1^\infty\frac{\{x\}}{x^{s+1}}\,\mathrm{d}x$$

where $\{x\}$ is the fractional part of x.

Proof We have

$$\begin{aligned}\mathrm{RHS}&=\frac{s}{s-1}-s\sum_{n=1}^\infty\int_n^{n+1}\frac{x-n}{x^{s+1}}\,\mathrm{d}x\\&=\frac{s}{s-1}-s\int_1^\infty\frac{\mathrm{d}x}{x^s}+s\sum_{n=1}^\infty\int_n^{n+1}\frac{n}{x^{s+1}}\,\mathrm{d}x\\&=\frac{s}{s-1}-\frac{s}{s-1}+\sum_{n=1}^\infty n\left[\frac{1}{n^s}-\frac{1}{(n+1)^s}\right]\\&=\sum_{n=1}^\infty\frac{1}{n^{s-1}}-\sum_{n=1}^\infty\frac{n}{(n+1)^s}\\&=\sum_{n=1}^\infty\frac{1}{n^{s-1}}-\sum_{n=2}^\infty\frac{n-1}{n^s}\\&=1+\sum_{n=2}^\infty\frac{1}{n^s}=\mathrm{LHS}.\end{aligned}$$

□

Stein 6.4.3 If $Q(x)=\{x\}-\frac{1}{2}$, then we can write the expression in the previous problem as

$$\zeta(s)=\frac{s}{s-1}-\frac{1}{2}-s\int_1^\infty\frac{Q(x)}{x^{s+1}}\,\mathrm{d}x.$$

Let us construct $Q_k(x)$ recursively so that

$$\int_0^1 Q_k(x)\,\mathrm{d}x = 0, \quad \frac{\mathrm{d}Q_{k+1}}{\mathrm{d}x} = Q_k(x), \quad Q_0(x) = Q(x) \quad \text{and} \quad Q_k(x+1) = Q_k(x).$$

Then we can write

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\int_1^\infty \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}Q_k(x)\right)x^{-s-1}\,\mathrm{d}x,$$

and a k-fold integration by parts gives the analytic continuation for $\zeta(s)$ when $\mathrm{Re}(s) > -k$.

Proof The two identities are clear from what we have proved in Problem 6.4.2 and the recursive definition of $Q_k(x)$. Assume first $\mathrm{Re}(s) > 0$, integrating by parts gives

$$\begin{aligned}
&\int_1^\infty \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}Q_k(x)\right)x^{-s-1}\,\mathrm{d}x \\
=&\sum_{n=1}^\infty \int_n^{n+1} \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}Q_k(x)\right)x^{-s-1}\,\mathrm{d}x \\
=&\sum_{n=1}^\infty \left\{\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}Q_k(x)\right)x^{-s-1}\Big|_n^{n+1} + (s+1)\int_n^{n+1}\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}Q_k(x)\right)x^{-s-2}\,\mathrm{d}x\right\} \\
=&\sum_{n=1}^\infty Q_1(x)x^{-s-1}\big|_n^{n+1} + (s+1)\int_1^\infty\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}Q_k(x)\right)x^{-s-2}\,\mathrm{d}x \\
=&-Q_1(0) + (s+1)\int_1^\infty\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}Q_k(x)\right)x^{-s-2}\,\mathrm{d}x \\
=&-Q_1(0) + (s+1)\left\{-Q_2(0) + (s+2)\int_1^\infty\left(\frac{\mathrm{d}^{k-2}}{\mathrm{d}x^{k-2}}Q_k(x)\right)x^{-s-3}\,\mathrm{d}x\right\} \\
=&\cdots \\
=&-Q_1(0) - \sum_{m=2}^k Q_m(0)(s+1)\cdots(s+m-1) + (s+1)(s+2)\cdots(s+k)\int_1^\infty Q_k(x)x^{-s-k-1}\,\mathrm{d}x.
\end{aligned}$$

Substituting this into the formula of $\zeta(s)$ we get

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} + s\sum_{m=1}^k Q_m(0)s(s+1)\cdots(s+m-1) - s(s+1)\cdots(s+k)\int_1^\infty Q_k(x)x^{-s-k-1}\,\mathrm{d}x.$$

Since $Q_k(x)$ is bounded on $\mathbb{R}$ by its periodicity, the integral converges for $\mathrm{Re}(s) > -k$, which gives the analytic continuation for $\zeta(s)$ when $\mathrm{Re}(s) > -k$. □

Stein 6.4.4 The functions Q_k in the previous problem are related to the Bernoulli polynomials $B_k(x)$ by the formula

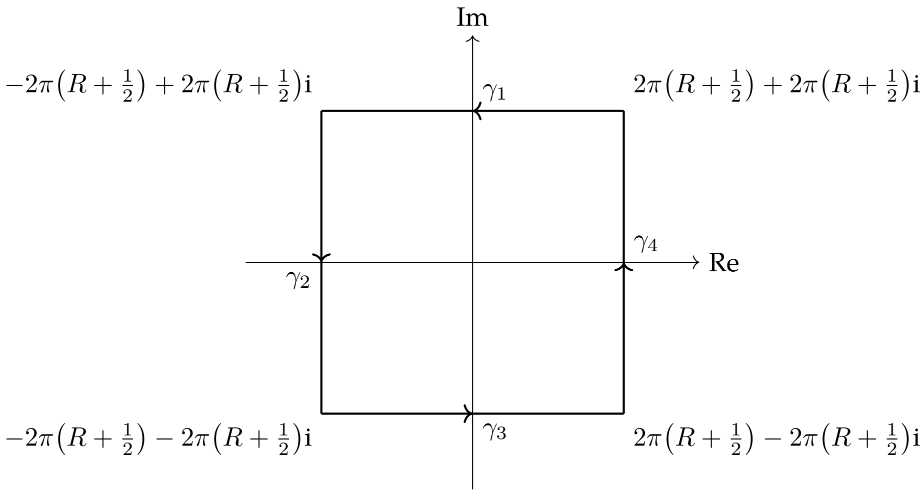
$$Q_k(x) = \frac{B_{k+1}(x)}{(k+1)!} \quad \text{for } 0 \leqslant x \leqslant 1.$$

Also, if k is a positive integer, then

$$2\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{(2k)!}B_{2k},$$

where $B_k = B_k(0)$ are the Bernoulli numbers.

Proof Consider the integral of $f(z)=\dfrac{z^{-2k}}{\mathrm{e}^z-1}$ around the square contour illustrated below,



where R is some large positive integer. Note that $2\pi\mathrm{i}n$ with $n\in\mathbb{Z}$ are all the poles of $f(z)$, and the residues at these poles are

$$\begin{gathered}\operatorname{Res}(f(z),z\pi\mathrm{i}n)=\lim_{z\to2\pi\mathrm{i}n}\frac{(z-2\pi\mathrm{i}n)z^{-2k}}{\mathrm{e}^z-1}=\lim_{z\to2\pi\mathrm{i}n}\frac{z^{-2k}}{\frac{\mathrm{e}^z-1}{z-2\pi\mathrm{i}n}}=(2\pi\mathrm{i}n)^{-2k},\quad n\in\mathbb{Z}\setminus\{0\}\\ \operatorname{Res}(f(z),0)=\operatorname{Res}\left(\frac{z}{\mathrm{e}^z-1}\cdot z^{-2k-1},0\right)=\operatorname{Res}\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}z^{n-2k-1},0\right)=\frac{B_{2k}}{(2k)!}.\end{gathered}$$

Since our contour does not pass through any pole, by the residue formula we have

$$\int_{\gamma}f(z)\,\mathrm{d}z=2\pi\mathrm{i}\left(\sum_{\substack{n\in[-R,R]\cap\mathbb{Z}\\ n\neq0}}(2\pi\mathrm{i}n)^{-2k}+\frac{B_{2k}}{(2k)!}\right),$$

where $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$. Next, observe that for $x\in\mathbb{R}$ and $y=2\pi\left(R+\frac{1}{2}\right)$ we have

$$\left|\mathrm{e}^{x+\mathrm{i}y}-1\right|=\left|-\mathrm{e}^x-1\right|=\mathrm{e}^x+1\geqslant1,$$

then for the integrals along γ_1 we have

$$\left|\int_{\gamma_1}f(z)\,\mathrm{d}z\right|\leqslant\int_{\gamma_1}\frac{|\,\mathrm{d}z|}{|x+\mathrm{i}y|^{2k}|\mathrm{e}^{x+\mathrm{i}y}-1|}\leqslant\frac{4\pi\left(R+\frac{1}{2}\right)}{\left[2\pi\left(R+\frac{1}{2}\right)\right]^{2k}}\xrightarrow{R\to+\infty}0.$$

By the same argument we have

$$\int_{\gamma_3}f(z)\,\mathrm{d}z\xrightarrow{R\to+\infty}0.$$

For the integral along γ_2, note that for $x=2\pi\left(R+\frac{1}{2}\right)$ and $y\in\mathbb{R}$ we have

$$\left|\mathrm{e}^{x+\mathrm{i}y}-1\right|=\left|1-\mathrm{e}^{-2\pi\left(R+\frac{1}{2}\right)+\mathrm{i}y}\right|\geqslant1-\mathrm{e}^{-2\pi\left(R+\frac{1}{2}\right)}>0,$$

hence

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leqslant \int_{\gamma_2} \frac{|\,\mathrm{d}z|}{|x+\mathrm{i}y|^{2k}|\mathrm{e}^{x+\mathrm{i}y}-1|} \leqslant \frac{4\pi\left(R+\frac{1}{2}\right)}{\left[2\pi\left(R+\frac{1}{2}\right)\right]^{2k}\left[1-\mathrm{e}^{-2\pi\left(R+\frac{1}{2}\right)}\right]} \xrightarrow{R\to+\infty} 0.$$

Finally, for the integral along γ_4, note that for $x = 2\pi\left(R+\frac{1}{2}\right)$ and $y \in \mathbb{R}$ we have

$$\left|\mathrm{e}^{x+\mathrm{i}y}-1\right| = \left|\mathrm{e}^{2\pi\left(R+\frac{1}{2}\right)+\mathrm{i}y}-1\right| \geqslant \mathrm{e}^{2\pi\left(R+\frac{1}{2}\right)}-1 > 0,$$

hence

$$\left|\int_{\gamma_4} f(z)\,\mathrm{d}z\right| \leqslant \int_{\gamma_4} \frac{|\,\mathrm{d}z|}{|x+\mathrm{i}y|^{2k}|\mathrm{e}^{x+\mathrm{i}y}-1|} \leqslant \frac{4\pi\left(R+\frac{1}{2}\right)}{\left[2\pi\left(R+\frac{1}{2}\right)\right]^{2k}\left[\mathrm{e}^{2\pi\left(R+\frac{1}{2}\right)}-1\right]} \xrightarrow{R\to+\infty} 0.$$

Therefore, letting $R \to +\infty$ gives

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} (2\pi\mathrm{i}n)^{-2k} + \frac{B_{2k}}{(2k)!} = 0$$

and so

$$2\zeta(2k) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^{2k}} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^k(2\pi)^{2k}}{(2\pi\mathrm{i}n)^{2k}} = (-1)^{k+1}\frac{(2\pi)^{2k}}{(2k)!}B_{2k}.$$

□

Stein 7.3.1 Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the partial sums

$$A_n = a_1 + \cdots + a_n$$

are bounded. Prove that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in this half-plane.

Proof Summation by parts gives

$$\sum_{n=1}^{N} \frac{a_n}{n^s} = \sum_{n=1}^{N} \frac{A_n - A_{n-1}}{n^s} = \frac{A_N}{N^s} - \sum_{n=1}^{N-1} A_n\left[\frac{1}{(n+1)^s} - \frac{1}{n^s}\right].$$

Assume $|A_n| \leqslant M$ for all $n \in \mathbb{N}$, then we have

$$\left|\frac{A_N}{N^s}\right| \leqslant \frac{M}{N^{\mathrm{Re}(s)}} \xrightarrow{N\to\infty} 0$$

uniformly on every compact subset of the half-plane $\mathrm{Re}(s) > 0$. Applying the mean value theorem to z^{-s} one gets

$$\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| \leqslant \frac{|s|}{n^{\mathrm{Re}(s)+1}}.$$

Therefore, on every compact subset K of the half-plane $\mathrm{Re}(s) > 0$ we have

$$\sum_{n=1}^{\infty}\left|A_n\left[\frac{1}{(n+1)^s}-\frac{1}{n^s}\right]\right| \leqslant \sum_{n=1}^{\infty}\frac{M|s|}{n^{\mathrm{Re}(s)+1}} \leqslant MS\sum_{n=1}^{\infty}\frac{1}{n^{\delta+1}}$$

where

$$S=\max_{s\in K}|s|<+\infty \quad \text{and} \quad \delta=\min_{s\in K}\mathrm{Re}(s)>0.$$

These two estimates gives the uniform convergence of the series on every compact subset of the half-plane $\mathrm{Re}(s) > 0$, which implies the holomorphicity of the function defined by this series. □

Stein 7.3.2 The following links the multiplication of Dirichlet series with the divisibility properties of their coefficients.

(1) Show that if $\{a_m\}$ and $\{b_k\}$ are two bounded sequences of complex numbers, then

$$\left(\sum_{m=1}^{\infty}\frac{a_m}{m^s}\right)\left(\sum_{k=1}^{\infty}\frac{b_k}{k^s}\right)=\sum_{n=1}^{\infty}\frac{c_n}{n^s} \quad \text{where } c_n=\sum_{mk=n}a_m b_k.$$

The above series converge absolutely when $\mathrm{Re}(s) > 1$.

(2) Prove as a consequence that one has

$$[\zeta(s)]^2=\sum_{n=1}^{\infty}\frac{d(n)}{n^s} \quad \text{and} \quad \zeta(s)\zeta(s-a)=\sum_{n=1}^{\infty}\frac{\sigma_a(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$ and Res$(s-a) > 1$, respectively. Here $d(n)$ equals the number of divisors of n, and $\sigma_a(n)$ is the sum of the a-th powers of the divisors of n. In particular, one has $\sigma_0(n) = d(n)$.

Proof (1) The convolution identity is obtained by noticing that $m^{-s}k^{-s} = n^{-s}$ iff $mk = n$. Assume $\{a_m\}$ and $\{b_k\}$ are bounded by A and B respectively, then $|c_n| \leqslant ABd(n)$. A classical result of the arithmetic functions $d(n)$ states that

$$\limsup_{n\to\infty}\frac{\log d(n)\log\log n}{\log n}=\log 2,$$

hence $d(n) \leqslant C\log n$ for some constant $C > 0$ and

$$\sum_{n=1}^{\infty}\left|\frac{c_n}{n^s}\right| \leqslant ABC\sum_{n=1}^{\infty}\frac{\log n}{n^{\mathrm{Re}(s)}}<+\infty \quad \text{when } \mathrm{Re}(s)>1.$$

(2) Taking $a_m = b_k \equiv 1$ we get the first identity. For the second identity, note that

$$\zeta(s-a)=\sum_{k=1}^{\infty}\frac{1}{k^{s-a}}=\sum_{k=1}^{\infty}\frac{k^a}{k^s},$$

hence by (1) we have

$$\zeta(s)\zeta(s-a)=\sum_{n=1}^{\infty}\frac{c_n}{n^s} \quad \text{where } c_n=\sum_{k|n}k^a=\sigma_a(n).$$

□

Stein 7.3.3 In line with the previous exercise, we consider the Dirichlet series for $1/\zeta$.

(1) Prove that for $\text{Re}(s) > 1$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n = p_1, \cdots p_k, \text{ and the } p_j \text{ are distinct primes}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mu(nm) = \mu(n)\mu(m)$ whenever n and m are relatively prime.

(2) Show that

$$\sum_{k|n} \mu(k) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof Our proof of (1) is based on the formula in (2).

(1) By the Dirichlet convolution formula in Exercise 7.3.2 (1) we have

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \left(\sum_{m=1}^{\infty} \frac{1}{m^s} \right) \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \right) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

where

$$c_n = \sum_{k|n} \mu(k) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

hence

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \quad \text{for } \text{Re}(s) > 1$$

as we want.

(2) The case $n = 1$ is clear. Now assume $n > 1$ and write $n = p_1^{r_1} \cdots p_m^{r_m}$ where $p_1, \cdots, p_m$ are distinct primes and $r_1, \cdots, r_m$ are positive integers. Since $\mu(n)$ is a multiplicative function, we have

$$\begin{aligned} \sum_{k|n} \mu(k) &= \sum_{0 \leqslant s_i \leqslant r_i} \mu(p_1^{s_1} \cdots p_m^{s_m}) = \mu(1) + \sum_{s_i = 0,1} \mu(p_1^{s_1} \cdots p_m^{s_m}) \\ &= 1 + \sum_{k=1}^{m} \binom{m}{k} (-1)^k = (1-1)^m = 0. \end{aligned}$$

□

Remark One can also prove (1) directly by using the Euler product formula for $\zeta(s)$ to write

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s} \right),$$

and the result is clear from the definition of the Möbius function $\mu(n)$.

Stein 7.3.4 Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $a_n = a_m$ if $n \equiv m \pmod q$ for some positive integer q. Define the Dirichlet L-series associated to $\{a_n\}$ by

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{for } \mathrm{Re}(s) > 1.$$

Also, with $a_0 = a_q$, let

$$Q(x) = \sum_{m=0}^{q-1} a_{q-m} \mathrm{e}^{mx}.$$

Show, as in Exercise 6.3.15 and 6.3.16, that

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Q(x) x^{s-1}}{\mathrm{e}^{qx} - 1} \,\mathrm{d}x, \quad \text{for } \mathrm{Re}(s) > 1.$$

Prove as a result that $L(s)$ is continuable into the complex plane, with the only possible singularity a pole at $s = 1$. In fact, $L(s)$ is regular at $s = 1$ if and only if $\sum_{m=0}^{q-1} a_m = 0$. Note the connection with the Dirichlet $L(s, \chi)$ series, taken up in Book I, Chapter 8, and that as a consequence, $L(s, \chi)$ is regular at $s = 1$ if and only if χ is a non-trivial character.

Proof As in Exercise 6.3.15, for positive integer q and $x > 0$, we have

$$\frac{1}{\mathrm{e}^{qx} - 1} = \sum_{n=1}^{\infty} \mathrm{e}^{-nqx}.$$

Substituting this into the integral and using Fubini's theorem we get

$$\begin{aligned}
\int_0^{\infty} \frac{Q(x) x^{s-1}}{\mathrm{e}^{qx} - 1} \,\mathrm{d}x &= \int_0^{\infty} x^{s-1} \left(\sum_{m=0}^{q-1} a_{q-m} \mathrm{e}^{mx} \right) \left(\sum_{n=1}^{\infty} \mathrm{e}^{-nqx} \right) \mathrm{d}x \\
&= \sum_{m=0}^{q-1} \sum_{n=1}^{\infty} a_{q-m} \int_0^{\infty} x^{s-1} \mathrm{e}^{(m-nq)x} \,\mathrm{d}x \\
&= \sum_{k=1}^{\infty} a_k \int_0^{\infty} x^{s-1} \mathrm{e}^{-kx} \,\mathrm{d}x \\
&= \sum_{k=1}^{\infty} \frac{a_k}{k^s} \int_0^{\infty} (kx)^{s-1} \mathrm{e}^{-kx} \,\mathrm{d}(kx) \\
&= \Gamma(s) L(s).
\end{aligned}$$

For each $m \in \{0, 1, \cdots, q-1\}$, as what we have done in Exercise 6.3.16, let us consider the integral

$$\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{\mathrm{e}^{mx} x^{s-1}}{\mathrm{e}^{qx} - 1} \,\mathrm{d}x = \frac{1}{\Gamma(s)} \int_0^{\frac{1}{q}} \frac{\mathrm{e}^{mx} x^{s-1}}{\mathrm{e}^{qx} - 1} \,\mathrm{d}x + \frac{1}{\Gamma(s)} \int_{\frac{1}{q}}^{\infty} \frac{\mathrm{e}^{mx} x^{s-1}}{\mathrm{e}^{qx} - 1} \,\mathrm{d}x.$$

The second integral defines an entire function because of exponential decay near infinity, while

$$\int_0^{\frac{1}{q}} \frac{\mathrm{e}^{mx} x^{s-1}}{\mathrm{e}^{qx} - 1} \,\mathrm{d}x = \int_0^1 \frac{\mathrm{e}^{\frac{mx}{q}} x^{s-1}}{q^s (\mathrm{e}^x - 1)} \,\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{q^s}\int_0^1 e^{\frac{mx}{q}} x^{s-2}\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n\,dx \\
&= \frac{1}{q^s}\sum_{n=0}^{\infty}\frac{B_n}{n!}\int_0^1 x^{n+s-2}e^{\frac{mx}{q}}\,dx \\
&= \frac{1}{q^s}\sum_{n=0}^{\infty}\frac{B_n}{n!}\int_0^1 x^{n+s-2}\sum_{k=0}^{\infty}\frac{m^k x^k}{k!q^k}\,dx \\
&= \frac{1}{q^s}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_n}{n!}\cdot\frac{m^k}{k!q^k}\int_0^1 x^{n+k+s-2}\,dx \\
&= \frac{1}{q^s}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_n}{n!}\cdot\frac{m^k}{k!q^k}\cdot\frac{1}{n+k+s-1}
\end{aligned}
$$

by Fubini's theorem. For $R > 0$ and $s \in \overline{\mathbb{B}(0,R)}$ we split the sum into two parts:

$$
\frac{1}{q^s}\sum_{n+k<R+2}\frac{B_n}{n!}\cdot\frac{m^k}{k!q^k}\cdot\frac{1}{n+k+s-1}+\frac{1}{q^s}\sum_{n+k\geqslant R+2}\frac{B_n}{n!}\cdot\frac{m^k}{k!q^k}\cdot\frac{1}{n+k+s-1}.
$$

The first sum is the sum of a finite number of meromorphic functions, with simple poles at $1, 0, -1, -2, \cdots$, and since $\Gamma(s)$ has simple poles at $0, -1, -2, \cdots$, it becomes a holomorphic function on $\overline{\mathbb{B}(0,R)}\setminus\{1\}$ after being divided by $\Gamma(s)$. For the second sum, notice that

$$
|n+k+s-1| \geqslant |n+k-1| - |s| \geqslant n+k-1-R \geqslant R+2-1-R = 1
$$

when $|s| \leqslant R$ and $n+k \geqslant R+2$. Therefore,

$$
\sum_{n+k\geqslant R+2}\left|\frac{B_n}{n!}\cdot\frac{m^k}{k!q^k}\cdot\frac{1}{n+k+s-1}\right| \leqslant \sum_{n+k\geqslant R+2}\frac{B_n}{n!}\cdot\frac{m^k}{k!q^k} \leqslant \sum_{n=0}^{\infty}\frac{B_n}{n!}\left(\sum_{k=0}^{\infty}\frac{m^k}{k!q^k}\right) = \frac{e^{\frac{m}{q}}}{e-1}.
$$

With these two facts, letting $R \to +\infty$ we conclude that $L(s)$ is continuable into the complex plane, with the only possible singularity a pole at $s = 1$. We refer the last statements to Problem 7.4.4. □

Stein 7.3.5 Consider the following function

$$
\tilde{\zeta}(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}.
$$

(1) Prove that the series defining $\tilde{\zeta}(s)$ converges for $\operatorname{Re}(s) > 0$ and defines a holomorphic function in that half-plane.

(2) Show that for $s > 1$ one has $\tilde{\zeta}(s) = \left(1 - 2^{1-s}\right)\zeta(s)$.

(3) Conclude, since $\tilde{\zeta}$ is given as an alternating series, that ζ has no zeros on the segment $0 < s < 1$. Extend this last assertion to $s = 0$ by using the functional equation.

Proof (1) Since the partial sums $\sum_{n=1}^{N}(-1)^n$ are bounded, Exercise 7.3.1 applies.

(2) On $s > 1$, since $\zeta(s)$ and $\tilde{\zeta}(s)$ are absolutely convergent (as series), we have

$$\zeta(s) - \tilde{\zeta}(s) = \sum_{n=1}^{\infty} \left[\frac{1}{n^s} - \frac{(-1)^{n+1}}{n^s} \right] = \sum_{n=1}^{\infty} \frac{2}{(2n)^s} = 2^{1-s}\zeta(s).$$

(3) Note that at $s = 1$, the simple pole of $\zeta(s)$ cancels with the zero of $1 - 2^{1-s}$, so both sides of the identity in (2) are holomorphic functions on $\mathrm{Re}(s) > 0$ that agree on $\mathrm{Re}(s) > 1$. Thus this identity holds on the whole half-plane $\mathrm{Re}(s) > 0$. Focusing on $0 < s < 1$, we have

$$\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s} > 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence the alternating series $\tilde{\zeta}(s) > 0$ when $0 < s < 1$, and $\zeta(s) \neq 0$ on the segment $0 < s < 1$ by the identity in (2). Finally, the functional equation $\xi(s) = \xi(1-s)$ or equivalently,

$$\pi^{-\frac{s}{2}} \Gamma\big(\tfrac{s}{2}\big) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\big(\tfrac{1-s}{2}\big) \zeta(1-s),$$

in Theorem 2.3 of Chapter 6, shows that $\zeta(0) \neq 0$ since the simple pole of $\zeta(1-s)$ at $s = 0$ cancels with the simple zero of $\dfrac{1}{\Gamma\big(\frac{s}{2}\big)}$. This concludes that $\zeta(s) \neq 0$ on the segment $0 < s < 1$. □

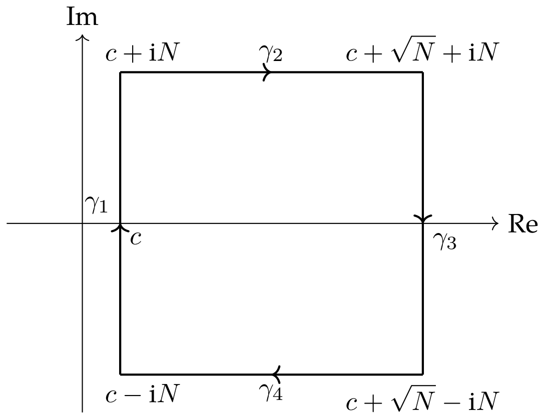
Stein 7.3.6 Show that for every $c > 0$

$$\lim_{N \to \infty} \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}N}^{c+\mathrm{i}N} a^s \frac{\mathrm{d}s}{s} = \begin{cases} 1, & \text{if } a > 1, \\ \frac{1}{2}, & \text{if } a = 1, \\ 0, & \text{if } 0 \leqslant a < 1. \end{cases}$$

This integral is taken over the vertical segment from $c - \mathrm{i}N$ to $c + \mathrm{i}N$.

Proof Let $f(s) = \dfrac{a^s}{s}$ be the integrand.

(1) For $0 \leqslant a < 1$, choose the rectangular contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as illustrated below.



Since $f(s)$ is holomorphic on $\mathbb{C} \setminus \{0\}$, we have $\int\limits_{\gamma} f(s)\,\mathrm{d}s = 0$. For the integral along γ_2, one has

$$\left|\int\limits_{\gamma_2} f(s)\,\mathrm{d}s\right| \leqslant \sqrt{N} \cdot \frac{a^c}{N} \xrightarrow{N\to\infty} 0.$$

By the same argument we have

$$\left|\int\limits_{\gamma_4} f(s)\,\mathrm{d}s\right| \leqslant \sqrt{N} \cdot \frac{a^c}{N} \xrightarrow{N\to\infty} 0.$$

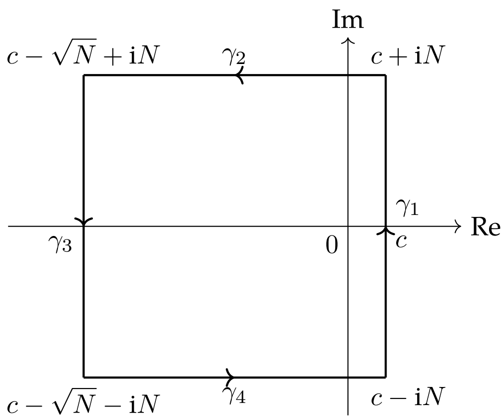
For the integral along γ_3, we have

$$\left|\int\limits_{\gamma_3} f(s)\,\mathrm{d}s\right| \leqslant 2N \cdot \frac{a^{c+\sqrt{N}}}{c+\sqrt{N}} \xrightarrow[0\leqslant a<1]{N\to\infty} 0.$$

Therefore, letting $N \to \infty$ gives

$$\lim_{N\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}N}^{c+\mathrm{i}N} a^s \frac{\mathrm{d}s}{s} = 0 \quad \text{when } 0 \leqslant a < 1.$$

(2) For $a > 1$, choose the rectangular contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as illustrated below.



The residue of $f(s)$ at $s = 0$ is 1, hence by the residue formula we have

$$\int\limits_{\gamma} f(s)\,\mathrm{d}s = 2\pi\mathrm{i}\,\mathrm{Res}(f, 0) = 2\pi\mathrm{i}.$$

For the integral along γ_2, one has

$$\left|\int\limits_{\gamma_2} f(s)\,\mathrm{d}s\right| \leqslant \sqrt{N} \cdot \frac{a^c}{N} \xrightarrow{N\to\infty} 0.$$

By the same argument we have

$$\left|\int_{\gamma_4} f(s)\,\mathrm{d}s\right| \leqslant \sqrt{N}\cdot\frac{a^c}{N} \xrightarrow{N\to\infty} 0.$$

For the integral along γ_3, we have

$$\left|\int_{\gamma_3} f(s)\,\mathrm{d}s\right| \leqslant 2N\cdot\frac{a^{c-\sqrt{N}}}{\sqrt{N}-c} \xrightarrow[a>1]{N\to\infty} 0.$$

Therefore, letting $N\to\infty$ gives

$$\lim_{N\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}N}^{c+\mathrm{i}N} a^s\frac{\mathrm{d}s}{s} = 1 \quad \text{when } a>1.$$

(3) For $a=1$, we compute directly to get

$$\lim_{N\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}N}^{c+\mathrm{i}N}\frac{\mathrm{d}s}{s} = \frac{1}{2\pi\mathrm{i}}\lim_{N\to\infty}\mathrm{i}[\arg(c+\mathrm{i}N)-\arg(c-\mathrm{i}N)] = \frac{\pi}{2\pi} = \frac{1}{2}.$$

Here we choose the principal branch of the logarithm in the slit plane $\mathbb{C}\setminus(-\infty,0]$. □

Stein 7.3.7 Show that the function

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right)\zeta(s)$$

is real when s is real, or when $\mathrm{Re}(s)=\frac{1}{2}$.

Proof The case when s is real is clear, since the two holomorphic functions $\zeta(s)$ and $\overline{\zeta(\bar{s})}$ agrees on $\mathrm{Re}(s)>1$, and hence must be identical for all $s\in\mathbb{R}\setminus\{1\}$. This implies that $\zeta(s)$ is real when s is real, and so does $\xi(s)$ by its definition. For $\mathrm{Re}(s)=\frac{1}{2}$, we observe that

- $\Gamma(\bar{s})=\overline{\Gamma(s)}$, which is clear from the integral representation of the gamma function.
- $\zeta\left(\frac{1}{2}-\mathrm{i}t\right)=\overline{\zeta\left(\frac{1}{2}+\mathrm{i}t\right)}$, which can be seen by the formula $\tilde{\zeta}(s)=\left(1-2^{1-s}\right)\zeta(s)$ in Exercise 7.3.5 (2).

With these two facts and the functional equation $\xi(s)=\xi(1-s)$, one has

$$\begin{aligned}
\overline{\xi\left(\tfrac{1}{2}+\mathrm{i}t\right)} &= \overline{\pi^{-\frac{\frac{1}{2}+\mathrm{i}t}{2}}}\cdot\overline{\Gamma\left(\frac{\frac{1}{2}+\mathrm{i}t}{2}\right)}\cdot\overline{\xi\left(\tfrac{1}{2}+\mathrm{i}t\right)} \\
&= \pi^{-\frac{\frac{1}{2}-\mathrm{i}t}{2}}\Gamma\left(\frac{\frac{1}{2}-\mathrm{i}t}{2}\right)\xi\left(\tfrac{1}{2}-\mathrm{i}t\right) \\
&= \pi^{-\frac{1-\left(\frac{1}{2}+\mathrm{i}t\right)}{2}}\Gamma\left(\frac{1-\left(\frac{1}{2}+\mathrm{i}t\right)}{2}\right)\xi\left(1-\left(\tfrac{1}{2}+\mathrm{i}t\right)\right) \\
&= \xi\left(1-\left(\tfrac{1}{2}+\mathrm{i}t\right)\right) = \xi\left(\tfrac{1}{2}+\mathrm{i}t\right).
\end{aligned}$$

Therefore we conclude that $\xi(s)$ is real when $\mathrm{Re}(s)=\frac{1}{2}$. □

Stein 7.3.8 The function ζ has infinitely many zeros in the critical strip. This can be seen as follows.

(1) Let

$$F(s)=\xi\left(\tfrac{1}{2}+s\right), \quad \text{where} \quad \xi(s)=\pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right)\zeta(s).$$

Show that $F(s)$ is an even function of s, and as a result, there exists G so that $G(s^2) = F(s)$.

(2) Show that the function $(s-1)\zeta(s)$ is an entire function of growth order 1, that is

$$|(s-1)\zeta(s)| \leqslant A_\varepsilon \mathrm{e}^{a_\varepsilon |s|^{1+\varepsilon}}.$$

As a consequence $G(s)$ is of growth order $\frac{1}{2}$.

(3) Deduce from the above that ζ has infinitely many zeros in the critical strip.

Proof (1) The functional equation $\xi(s) = \xi(1-s)$ implies that $\xi\left(\frac{1}{2}+s\right) = \xi\left(\frac{1}{2}-s\right)$.

(2) By the proof of Theorem 2.3 in Chapter 6 we have

$$\xi(s) = \frac{1}{s-1} + \frac{1}{s} + \int_1^\infty \left(u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right)\psi(u)\,\mathrm{d}u,$$

where

$$\psi(u) = \frac{\vartheta(u)-1}{2} = \sum_{n=1}^\infty \mathrm{e}^{-\pi n^2 u}.$$

For $s = \sigma + \mathrm{i}t$, take $N \in \mathbb{N}$ such that $\dfrac{|\sigma|+1}{2} \leqslant N \leqslant |\sigma|+2$, then

$$\begin{aligned}
\left|\int_1^\infty \left(u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right)\psi(u)\,\mathrm{d}u\right| &\leqslant \int_1^\infty \left|u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right|\psi(u)\,\mathrm{d}u \\
&\leqslant \int_1^\infty \left(u^{-\frac{\sigma}{2}-\frac{1}{2}} + u^{\frac{\sigma}{2}-1}\right)\psi(u)\,\mathrm{d}u \\
\left(\because -\tfrac{\sigma}{2}-\tfrac{1}{2} \leqslant N-1 \text{ and } \tfrac{\sigma}{2}-1 \leqslant N-1\right) \quad &\leqslant 2\int_1^\infty u^{N-1}\sum_{n=1}^\infty \mathrm{e}^{-\pi n^2 u}\,\mathrm{d}u \\
&\leqslant 2\sum_{n=1}^\infty \int_0^\infty u^{N-1}\mathrm{e}^{-\pi n^2 u}\,\mathrm{d}u \\
&\overset{v=\pi n^2 u}{=\!=} 2\sum_{n=1}^\infty \frac{1}{(\pi n^2)^N}\int_0^\infty v^{N-1}\mathrm{e}^{-v}\,\mathrm{d}v \\
&\leqslant \sum_{n=1}^\infty \frac{2(N-1)!}{\pi n^2} \\
&= \frac{\pi}{3}(N-1)! \\
&\leqslant \frac{\pi}{3}(N-1)^{N-1} \\
&= \frac{\pi}{3}\mathrm{e}^{(N-1)\log(N-1)} \\
&\leqslant \frac{\pi}{3}\mathrm{e}^{(|\sigma|+1)\log(|\sigma|+1)}.
\end{aligned}$$

This shows that the growth order of the integral is at most 1. Next, by

$$(s-1)\zeta(s) = \frac{(s-1)\pi^{\frac{s}{2}}\xi(s)}{\Gamma\left(\frac{s}{2}\right)},$$

we see that the simple poles 0 and 1 of $\xi(s)$ are canceled by the simple pole of $\Gamma\left(\frac{s}{2}\right)$ and the simple zero of $s-1$ respectively. Since both $\pi^{\frac{s}{2}}$ and $1/\Gamma\left(\frac{s}{2}\right)$ are of growth order 1 (the latter by Theorem 1.6

in Chapter 6), we conclude that $(s-1)\zeta(s)$ is an entire function of growth order 1 and consequently $G(s)$ is of growth order $\frac{1}{2}$.

(3) By (2) and Exercise 5.6.14 we see that $G(s)$, and so $F(s)$ and $\xi(s)$, have infinitely many zeros. The $\pi^{-\frac{s}{2}}$ and $\Gamma\left(\frac{s}{2}\right)$ factors in the defining equation of ξ are nonvanishing when $\mathrm{Re}(s) > 1$, and $\zeta(s)$ has no zeros when $\mathrm{Re}(s) > 1$ by Theorem 1.1 in Chapter 7, so $\xi(s)$ is nonzero in this half-plane. With this fact and the functional equation $\xi(s) = \xi(1-s)$ we see that $\xi(s)$ is also nonvanishing when $\mathrm{Re}(s) < 0$. Therefore the nontrivial zeros of $\zeta(s)$ are the same thing as all zeros of $\xi(s)$, which implies that $\zeta(s)$ has infinitely many zeros in the critical strip. □

Stein 7.3.9 Refine the estimates in Proposition 2.7 in Chapter 6 and Proposition 1.6 to show that

(1) $|\zeta(1+\mathrm{i}t)| \leqslant A \log|t|$,

(2) $|\zeta'(1+\mathrm{i}t)| \leqslant A(\log|t|)^2$,

(3) $\dfrac{1}{|\zeta(1+\mathrm{i}t)|} \leqslant A(\log|t|)^a$,

when $|t| \geqslant 2$ (with $a = 7$).

Proof (1) See Problem 6.4.1 (2).

(2) See Problem 6.4.1 (3).

(3) One can check that the estimates above are still valid for $\mathrm{Re}(s) > 1$. By Corollary 1.5 in Chapter 7 we have

$$|\zeta(\sigma+\mathrm{i}t)| \geqslant |\zeta(\sigma)|^{-\frac{3}{4}}|\zeta(\sigma+2\mathrm{i}t)|^{-\frac{1}{4}} \geqslant A_1(\sigma-1)^{\frac{3}{4}}(\log|t|)^{-\frac{1}{4}}$$

for $\sigma > 1$ and $t \in \mathbb{R}$. If $\sigma - 1 \geqslant B(\log|t|)^b$ for some appropriate constant B and b (whose value we choose later), then

$$|\zeta(\sigma+\mathrm{i}t)| \geqslant A_1 B^{\frac{3}{4}}(\log|t|)^{\frac{3b-1}{4}}.$$

If, however, $\sigma - 1 < B(\log|t|)^b$, then we first select $\sigma' > \sigma$ with $\sigma' - 1 = B(\log|t|)^b$. The mean value theorem, together with the estimate of ζ', gives

$$|\zeta(\sigma'+\mathrm{i}t) - \zeta(\sigma+\mathrm{i}t)| \leqslant (\sigma'-\sigma)A_2(\log|t|)^2 \leqslant (\sigma'-1)A_2(\log|t|)^2.$$

Then

$$\begin{aligned}|\zeta(\sigma+\mathrm{i}t)| &\geqslant |\zeta(\sigma'+\mathrm{i}t)| - |\zeta(\sigma'+\mathrm{i}t) - \zeta(\sigma+\mathrm{i}t)| \\ &\geqslant A_1(\sigma'-1)^{\frac{3}{4}}(\log|t|)^{-\frac{1}{4}} - (\sigma'-1)A_2(\log|t|)^2 \\ &= A_1 B^{\frac{3}{4}}(\log|t|)^{\frac{3b-1}{4}} - A_2(\log|t|)^{b+2}.\end{aligned}$$

Now let $\frac{3b-1}{4} = b+2$, then $b = -9$ and we get

$$\left(A_1 B^{\frac{3}{4}} - A_2\right)(\log|t|)^{-7}.$$

Next choose $B = \left(\frac{A_2+1}{A_1}\right)^{\frac{4}{3}}$, which gives precisely $A_1 B^{\frac{3}{4}} = A_2 + 1$, to get

$$|\zeta(\sigma+\mathrm{i}t)| \geqslant (\log|t|)^{-7}.$$

To conclude, we have shown in two separate cases the desired result. □

Stein 7.3.10 In the theory of primes, a better approximation to $\pi(x)$ (instead of $x/\log x$) turns out to be $\mathrm{Li}(x)$ defined by

$$\mathrm{Li}(x)=\int_2^x \frac{\mathrm{d}t}{\log t}.$$

(1) Prove that

$$\mathrm{Li}(x)=\frac{x}{\log x}+O\left(\frac{x}{(\log x)^2}\right)\quad \text{as } x\to\infty,$$

and that as a consequence

$$\pi(x)\sim \mathrm{Li}(x)\quad \text{as } x\to\infty.$$

(2) Refine the previous analysis ny showing that for every integer $N>0$ one has the following asymptotic expansion

$$\mathrm{Li}(x)=\frac{x}{\log x}+\frac{x}{(\log x)^2}+2\frac{x}{(\log x)^3}+\cdots+(N-1)!\frac{x}{(\log x)^N}+O\left(\frac{x}{(\log x)^{N+1}}\right)$$

as $x\to\infty$.

Proof (1) Integration by parts gives

$$\mathrm{Li}(x)=\int_2^x \frac{\mathrm{d}t}{\log t}=\frac{x}{\log x}-\frac{2}{\log 2}+\int_2^x \frac{\mathrm{d}t}{(\log t)^2}.$$

Therefore it suffices to show that

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2}=O\left(\frac{x}{(\log x)^2}\right),$$

which can be obtained by the estimate

$$\int_2^{\sqrt{x}} \frac{\mathrm{d}t}{(\log t)^2}+\int_{\sqrt{x}}^x \frac{\mathrm{d}t}{(\log t)^2}\leqslant C\sqrt{x}+\frac{x-\sqrt{x}}{(\log\sqrt{x})^2}.$$

(2) Apply again integration by parts to the second integral in (1),

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2}=\frac{x}{(\log x)^2}-\frac{2}{(\log 2)^2}+2\int_2^x \frac{\mathrm{d}t}{(\log t)^3}.$$

Then repeat this process to decompose the new integral until one finally has

$$\begin{aligned}\mathrm{Li}(x)=&\frac{x}{\log x}+\frac{x}{(\log x)^2}+2\frac{x}{(\log x)^3}+\cdots+(N-1)!\frac{x}{(\log x)^N}\\ &-\frac{2}{\log 2}-\frac{2}{(\log 2)^2}-2\frac{2}{\log 2}^{3}-\cdots-(N-1)!\frac{2}{(\log 2)^N}\\ &+N!\int_2^x \frac{\mathrm{d}t}{(\log t)^{N+1}}.\end{aligned}$$

As in (1), we write

$$\int_2^{\sqrt{x}} \frac{\mathrm{d}t}{(\log t)^{N+1}}+\int_{\sqrt{x}}^x \frac{\mathrm{d}t}{(\log t)^{N+1}}\leqslant C'\sqrt{x}+\frac{x-\sqrt{x}}{(\log\sqrt{x})^{N+1}}$$

to show that the last term is $O\left(\frac{x}{(\log x)^{N+1}}\right)$ as $x \to \infty$. □

Stein 7.3.11 Let

$$\varphi(x) = \sum_{p \leqslant x} \log p$$

where the sum is taken over all primes $\leqslant x$. Prove that the following are equivalent as $x \to \infty$:

(1) $\varphi(x) \sim x$,

(2) $\pi(x) \sim \frac{x}{\log x}$,

(3) $\psi(x) \sim x$,

(4) $\psi_1(x) \sim \frac{x^2}{2}$.

Proof "(4) ⇒ (3)" and "(3) ⇒ (2)" are proved in Proposition 2.2 and 2.1 respectively.

(2) ⇒ (1) Summation by parts gives

$$\varphi(x) = \sum_{n=1}^{\lfloor x \rfloor} \log(n)[\pi(n) - \pi(n-1)] = \log(\lfloor x \rfloor + 1)\pi(x) - \sum_{n=1}^{\lfloor x \rfloor} \pi(n)[\log(n+1) - \log(n)].$$

Since $n \log\left(1 + \frac{1}{n}\right) < 1$, we have

$$\begin{aligned} \left| \sum_{n=1}^{\lfloor x \rfloor} \pi(n)[\log(n+1) - \log(n)] \right| &\leqslant \left| \sum_{n \leqslant x} \pi(n) \log\left(1 + \tfrac{1}{n}\right) \right| \leqslant \sum_{n \leqslant x} \frac{\pi(n)}{n} \\ &\leqslant \sum_{2 \leqslant n \leqslant \sqrt{x}} \frac{\pi(n)}{n} + \sum_{\sqrt{x} < n \leqslant x} \frac{C}{\log n} \leqslant \sqrt{x} + \frac{Cx}{\log x}. \end{aligned}$$

Therefore $\varphi(x) \sim x$ as $x \to \infty$.

(1) ⇒ (3) Assume (1), then for any fixed $m \in \mathbb{N}$ we have

$$\sum_{p^m \leqslant x} \log p = \sum_{p \leqslant x^{\frac{1}{m}}} \log p = \varphi\left(x^{\frac{1}{m}}\right) \sim x^{\frac{1}{m}}.$$

Letting m take value in each positive integer leads to

$$\psi(x) = \sum_{p^m \leqslant x} \log p \sim x + x^{\frac{1}{2}} + x^{\frac{1}{3}} + \cdots,$$

which implies (3).

(3) ⇒ (4) Assume (3), then given any $\varepsilon > 0$, there exists $x_0 > 0$ such that $|\psi(x) - x| \leqslant \varepsilon x$ for all $x \geqslant x_0$. Then

$$\left| \psi_1(x) - \frac{x^2}{2} + \frac{1}{2} \right| = \left| \int_1^x [\psi(u) - u] \, \mathrm{d}u \right| \leqslant \int_1^{x_0} |\psi(u) - u| \, \mathrm{d}u + \varepsilon \int_{x_0}^x u \, \mathrm{d}u \to \frac{\varepsilon}{2}$$

as $x \to \infty$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$. □

Stein 7.3.12 If p_n denotes the n-th prime, the prime number theorem implies that $p_n \sim n \log n$ as $n \to \infty$.

(1) Show that $\pi(x) \sim \frac{x}{\log x}$ implies that

$$\log \pi(x) + \log \log x \sim \log x.$$

(2) As a consequence, prove that $\log \pi(x) \sim \log x$, and take $x = p_n$ to conclude the proof.

Proof (1) $\dfrac{\pi(x) \log x}{x} \to 1$ as $x \to \infty$ implies $\dfrac{\log(\pi(x) \log x)}{\log x} \to 1$ as $x \to \infty$.

(2) Since

$$\frac{\log \log x}{\log x} \to 0 \quad \text{as } x \to \infty,$$

from (1) we obtain $\log \pi(x) \sim \log x$ as $x \to \infty$. Taking $x = p_n$ in $\pi(x) \sim \dfrac{x}{\log x}$ gives

$$n = \pi(p_n) \sim \frac{p_n}{\log p_n} \sim \frac{p_n}{\log \pi(p_n)} = \frac{p_n}{\log n},$$

or equivalently, $p_n \sim n \log n$, as $n \to \infty$. □

Stein 7.4.1 Let $F(s) = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where $|a_n| \leqslant M$ for all n.

(1) Then

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |F(\sigma + \mathrm{i}t)|^2 \, \mathrm{d}t = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} \quad \text{if } \sigma > 1.$$

How is this reminiscent of the Parseval-Plancherel theorem? See e.g. Chapter 3 in Book I.

(2) Show as a consequence the uniqueness of the Dirichlet series: If $F(s) = \displaystyle\sum_{n=1}^{\infty} a_n n^{-s}$, where the coefficients are assumed to satisfy $|a_n| \leqslant cn^k$ for some k, and $F(s) \equiv 0$, then $a_n = 0$ for all n.

Proof (1) By Fubini's theorem we have

$$\begin{aligned}\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |F(\sigma + \mathrm{i}t)|^2 \, \mathrm{d}t &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma + \mathrm{i}t}} \right) \left(\sum_{m=1}^{\infty} \frac{\overline{a_m}}{m^{\sigma - \mathrm{i}t}} \right) \mathrm{d}t \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n \overline{a_m}}{(nm)^{\sigma}} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{m}{n} \right)^{\mathrm{i}t} \mathrm{d}t.\end{aligned}$$

Since

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{m}{n} \right)^{\mathrm{i}t} \mathrm{d}t = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases}$$

the left-hand side of the above reduces to $\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}$.

(2) By (1) we have $\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} = 0$, hence $a_n \equiv 0$ for all $n \in \mathbb{N}$. Note that the assumption $|a_n| \leqslant cn^k$ guarantees the convergence of the defining series of $F(s)$ when $\mathrm{Re}(s)$ is sufficiently large. □

Stein 7.4.2 One of the "explicit formulas" in the theory of primes is as follows: if ψ_1 is the integrated Tchebychev function considered in Section 2, then

$$\psi_1(x) = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - E(x)$$

where the sum is taken over all zeros ρ of the zeta function in the critical strip. The error term is given by

$$E(x) = c_1 x + c_0 + \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)},$$

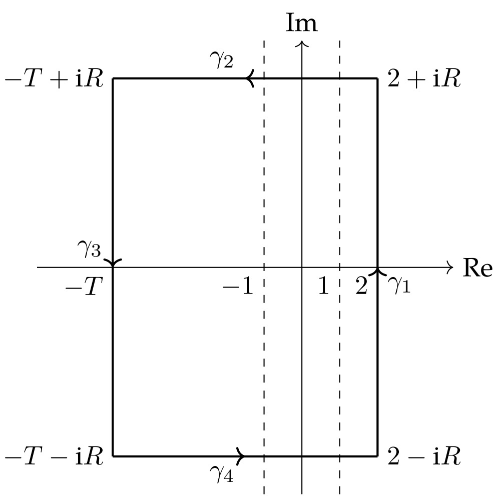
where

$$c_1 = \frac{\zeta'(0)}{\zeta(0)} \quad \text{and} \quad c_0 = -\frac{\zeta'(-1)}{\zeta(-1)}.$$

Proof By Proposition 2.3 in Chapter 7 we have

$$\psi_1(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \mathrm{d}s \quad \text{for all } c > 1.$$

Now fix $c = 2$ and consider the integral of $f(s) = \dfrac{x^{s+1}}{s(s+1)} \left(-\dfrac{\zeta'(s)}{\zeta(s)} \right)$ along the rectangular contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as illustrated below.



It is necessary to choose R with a little care, so that the horizontal sides of the rectangle shall avoid, as far as possible, the zeros of $\zeta(s)$ in the critical strip. Similarly, here T is chosen to be a large odd integer, so that the left vertical side passes halfway between two of the trival zeros of $\zeta(s)$.

We first calculate the residues of $f(s)$ at $1, 0, -1$ and all zeros of ζ:

$$\mathrm{Res}(f, 1) = -\frac{x^2}{2} \operatorname{ord}(\zeta, 1) = \frac{x^2}{2},$$

$$\mathrm{Res}(f, 0) = \lim_{s \to 0} \frac{x^{s+1}}{s+1} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) = -c_1 x \quad \text{where } c_1 = \frac{\zeta'(0)}{\zeta(0)},$$

$$\mathrm{Res}(f, -1) = \lim_{s \to -1} \frac{x^{s+1}}{s} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) = -c_0 \quad \text{where } c_0 = -\frac{\zeta'(-1)}{\zeta(-1)},$$

$$\mathrm{Res}(f, -2k) = -\frac{x^{-2k+1}}{-2k(-2k+1)} \operatorname{ord}(\zeta, -2k) = -\frac{x^{1-2k}}{2k(2k-1)} \quad \text{for } k = 1, 2, 3, \cdots,$$

$$\mathrm{Res}(f,\rho) = -\frac{x^{\rho+1}}{\rho(\rho+1)}\,\mathrm{ord}(\zeta,\rho) \quad \text{for any nontrivial zero } \rho \text{ of } \zeta.$$

Note that in the formula we are going to prove the nontrivial zeros of ζ are to be counted with multiplicities, i.e., each ρ appears in the summation as many times as its order, since we actually don't know whether they are simple or not.

In Exercise 7.3.8 we have shown that $(s-1)\zeta(s)$ is an entire function of growth order 1, thus by Theorem 2.1 in Chapter 5 we have $\sum_\rho \frac{1}{|\rho|^{1+\varepsilon}} < \infty$ for every $\varepsilon > 0$. Hence

$$\sum_\rho \left|\frac{x^{\rho+1}}{\rho(\rho+1)}\right| \leqslant \sum_\rho \frac{x^2}{|\rho|^2} < \infty.$$

Also, it is obvious that $E(x) = O(x)$ as $x \to \infty$. So we are allowed to apply the residue theorem and let R and T tend to infinity to find

$$\psi_1(x) + \frac{1}{2\pi\mathrm{i}} \lim_{R,T\to+\infty} \int_{\gamma_2\cup\gamma_3\cup\gamma_4} f(s)\,\mathrm{d}s = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - E(x),$$

and it remains to show that the integral of $f(s)$ along $\gamma_2\cup\gamma_3\cup\gamma_4$ vanishes as R and T tend to infinity. To achieve this, we need an estimate for $|\zeta'/\zeta|$, and we shall prove that

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = \begin{cases} O(\log|2s|), & \text{if } \mathrm{Re}(s) \leqslant -1 \text{ and all circles of radius } \frac{1}{2} \text{ around the trival zeros are excluded,} \\ O\left(\log^2 R\right), & \text{if } -1 < \mathrm{Re}(s) \leqslant 2 \text{ and } \mathrm{Im}(s) = R. \end{cases}$$

With this in hand, it is clear that the integral of $f(s)$ along $\gamma_2\cup\gamma_3\cup\gamma_4$ vanishes as R and T tend to infinity, thus completing the proof.

Case 1: Re$(s) \leqslant -1$ with "circles" excluded First recall two functional relations satisfied by $\Gamma(s)$:

- ◇ $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$.
- ◇ $\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2s}\Gamma(2s)$, which has been proved in Exercise 6.3.3.

Combined, one has

$$\Gamma\left(\tfrac{1-s}{2}\right) = \Gamma\left(1-\tfrac{1+s}{2}\right) = \frac{\pi}{\sin\left(\pi\frac{1+s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)} = \frac{\pi}{\cos\frac{\pi s}{2}\Gamma\left(\frac{1+s}{2}\right)} = \frac{\pi}{\cos\frac{\pi s}{2}} \cdot \frac{\Gamma\left(\frac{s}{2}\right)}{2^{1-s}\sqrt{\pi}\Gamma(s)}$$
$$= \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)}{2^{1-s}\cos\frac{\pi s}{2}\Gamma(s)},$$

thus giving

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \pi^{-\frac{1}{2}}2^{1-s}\cos\frac{\pi s}{2}\Gamma(s).$$

If this is used in the functional equation of $\zeta(s)$, we get

$$\zeta(1-s) = \frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)} = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s).$$

Taking the logarithmic derivative of both sides gives

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} = -\frac{\pi}{2}\tan\frac{\pi s}{2} + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)} - \log 2\pi.$$

Since we are interested in the left-hand side under $\mathrm{Re}(1-s) \leqslant -1$, the right-hand side can be considered only for $\mathrm{Re}(s) \geqslant 2$. The first term is bounded if s is not close to any odd integer, or more specifically, $|s-(2m+1)| \geqslant \frac{1}{2}$ for all $m \in \mathbb{N}$. Note that this is equivalent to

$$|(1-s)-(-2m)| \geqslant \frac{1}{2},$$

which is precisely satisfied by our assumption that all circles of radius $\frac{1}{2}$ around the trival zeros are excluded. The third term is bounded since

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = \left|\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right| \leqslant \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2} = -\frac{\zeta'(2)}{\zeta(2)} \quad \text{for } \mathrm{Re}(s) \geqslant 2. \tag{7.4.2–1}$$

Finally, the digamma function $\Gamma'(s)/\Gamma(s)$ is $O(\log|s|)$, and so is $O(\log 2|1-s|)$. The asymptotic behavior we use here can be deduced from Exercise 6.3.13, where we have shown that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \frac{\mathrm{d}}{\mathrm{d}s}\log\Gamma(s) = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+s}\right).$$

By Euler-Maclaurin summation formula on $(x+s)^{-1}$ we have

$$\sum_{n=0}^{N}\frac{1}{n+s} = \log(N+s) - \log s + \frac{1}{2s} + \frac{1}{2(s+N)} + O\big(|s|^{-2}\big),$$

then

$$\sum_{n=1}^{N}\frac{1}{n+s} = \log(N+s) - \log s - \frac{1}{2s} + \frac{1}{2(s+N)} + O\big(|s|^{-2}\big).$$

Hence

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O\big(|s|^{-2}\big). \tag{7.4.2–2}$$

Case 2: $-1 < \mathrm{Re}(s) \leqslant 2$ **and** $\mathrm{Im}(s) = R$ We refer to two results which we shall prove later:

① For large R (not coinciding with the ordinate of a zero) and $-1 \leqslant \mathrm{Re}(s) \leqslant 2$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\mathrm{Im}(\rho)-R|<1}\frac{1}{s-\rho} + O(\log R). \tag{7.4.2–3}$$

② For any large R, the number of zeros ρ of ζ with $|\mathrm{Im}(\rho)-R| < 1$ is $O(\log R)$.

As a consequence of ②, among the ordinates of these zeros there must be a gap of length at least $C(\log R)^{-1}$ for some constant $C > 0$ independent of R. Hence by varying T by a bounded amount (say 1) we can ensure that

$$|\mathrm{Im}(\rho) - R| \geqslant \frac{C'}{\log R}$$

for all zeros ρ of ζ. Now we apply reslut ① with the present choice of R to find

$$|s-\rho| \geqslant |\mathrm{Im}(\rho)-R| \geqslant \frac{C'}{\log R}$$

and the number of terms is also $O(\log R)$. So on the new horizontal lines of integration we have

$$\frac{\zeta'(s)}{\zeta(s)} = O\left(\log^2 R\right) \quad \text{for } -1 \leqslant \mathrm{Re}(s) \leqslant 2.$$

Now we prove the two results ① and ② mentioned above. Define

$$\tilde{\xi}(s) = \frac{1}{2}s(s-1)\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}+1\right)\zeta(s), \tag{7.4.2–4}$$

then by the deduction in Exercise 7.3.8 we see $\tilde{\xi}(s)$ is an entire function of order 1. Hadamard's factorization theorem shows that

$$\tilde{\xi}(s) = \mathrm{e}^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)\mathrm{e}^{\frac{s}{\rho}},$$

where the product is taken over all nontrivial zeros of ζ. Logarithmic differentiation of this gives

$$\frac{\tilde{\xi}'(s)}{\tilde{\xi}(s)} = B + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Since by our definition

$$\frac{\tilde{\xi}'(s)}{\tilde{\xi}(s)} = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} + \frac{\zeta'(s)}{\zeta(s)},$$

we find

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - B - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} - \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \tag{7.4.2–5}$$

By the asymptotic behavior (7.4.2–2) of the digamma function we see the Γ term above is less than $A\log t$ if $t \geqslant 2$ and $1 \leqslant \sigma \leqslant 2$ for $s = \sigma + \mathrm{i}t$. Hence, in this region,

$$-\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) < A\log t - \sum_{\rho}\mathrm{Re}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

In this inequality we take $s = 2+\mathrm{i}R$, and since $|\zeta'/\zeta|$ is bounded for such s as shown in (7.4.2–1), we obtain

$$\sum_{\rho}\mathrm{Re}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) < A\log R.$$

Note that $\mathrm{Re}\left(\frac{1}{\rho}\right) > 0$ for each ρ, and

$$\mathrm{Re}\left(\frac{1}{s-\rho}\right) = \mathrm{Re}\left(\frac{1}{2+\mathrm{i}R-\rho}\right) = \frac{2-\mathrm{Re}(\rho)}{[2-\mathrm{Re}(\rho)]^2+[R-\mathrm{Im}(\rho)]^2} \geqslant \frac{1}{4+[R-\mathrm{Im}(\rho)]^2},$$

we get

$$\sum_{\rho}\frac{1}{1+[R-\mathrm{Im}(\rho)]^2}=O(\log R).$$

As a consequence, we see that

$$\frac{1}{2}\#\{\rho:|\mathrm{Im}(\rho)-R|<1\}\leqslant\sum_{|\mathrm{Im}(\rho)-R|<1}\frac{1}{1+[R-\mathrm{Im}(\rho)]^2}\leqslant\sum_{\rho}\frac{1}{1+[R-\mathrm{Im}(\rho)]^2}=O(\log R),$$

which implies result ②. Also note as a byproduct that

$$\frac{1}{2}\sum_{|\mathrm{Im}(\rho)-R|\geqslant 1}\frac{1}{|\mathrm{Im}(\rho)-R|^2}\leqslant\sum_{|\mathrm{Im}(\rho)-R|\geqslant 1}\frac{1}{1+|R-\mathrm{Im}(\rho)|^2}\leqslant\sum_{\rho}\frac{1}{1+[R-\mathrm{Im}(\rho)]^2}=O(\log R),$$

hence we find

$$\sum_{|\mathrm{Im}(\rho)-R|\geqslant 1}\frac{1}{|\mathrm{Im}(\rho)-R|^2}=O(\log R). \tag{7.4.2–6}$$

By formula (7.4.2–5), applied at $s=\sigma+\mathrm{i}R$ (here $-1<\sigma\leqslant 2$) and $2+\mathrm{i}R$ and subtracted,

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)}&=\frac{\zeta'(2+\mathrm{i}R)}{\zeta(2+\mathrm{i}R)}-\frac{1}{s-1}+\frac{1}{1+\mathrm{i}R}+\frac{1}{2}\frac{\Gamma'\left(2+\frac{\mathrm{i}R}{2}\right)}{\Gamma\left(2+\frac{\mathrm{i}R}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)}+\sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2+\mathrm{i}R-\rho}\right)\\&=O(\log R)+\sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2+\mathrm{i}R-\rho}\right),\end{aligned}$$

where we have used (7.4.2–1) and (7.4.2–2) to estimate the ζ and Γ terms. Now we focus on the sum. For the terms with $|\mathrm{Im}(\rho)-R|\geqslant 1$, we have

$$\left|\frac{1}{s-\rho}-\frac{1}{2+\mathrm{i}R-\rho}\right|=\frac{2-\sigma}{|s-\rho||2+\mathrm{i}R-\rho|}\leqslant\frac{3}{|\mathrm{Im}(\rho)-R|^2},$$

and their contribution to the sum is $O(\log R)$ by (7.4.2–6). As for the terms with $|\mathrm{Im}(\rho)-R|<1$, we have $|2+\mathrm{i}R-\rho|\geqslant|(2+\mathrm{i}R)-(1+\mathrm{i}R)|=1$, and the number of terms is $O(\log R)$ by result ② above. Therefore we have proved result ①. □

Stein 7.4.3 Using the previous problem one can show that

$$\pi(x)-\mathrm{Li}(x)=O\big(x^{\alpha+\varepsilon}\big)\quad\text{as } x\to\infty$$

for every $\varepsilon>0$, where α is fixed and $\frac{1}{2}\leqslant\alpha<1$ if and only if $\zeta(s)$ has no zeros in the strip $\alpha<\mathrm{Re}(s)<1$. The case $\alpha=\frac{1}{2}$ corresponds to the Riemann hypothesis.

Proof Using the explicit formula for $\psi_1(x)$ in Problem 7.4.2 we have

$$\psi(x)=x-\sum_{\rho}\frac{x^{\rho}}{\rho}+O(1)\quad\text{as } x\to\infty. \tag{7.4.3–1}$$

Here the termwise differentiation of the series is justified by Fubini's theorem. Fix $\alpha\in\left[\frac{1}{2},1\right)$, we shall prove the following statements are equivalent:

(1) $\pi(x)-\mathrm{Li}(x)=O\big(x^{\alpha+\varepsilon}\big)$ as $x\to\infty$ for every $\varepsilon>0$.

(2) $\psi(x) = x + O\big(x^{\alpha+\varepsilon}\big)$ as $x \to \infty$ for every $\varepsilon > 0$.

(3) $\zeta(s)$ has no zeros in the strip $\alpha < \mathrm{Re}(s) < 1$.

(1) ⇒ (2) Consider the first Tchebychev function

$$\theta(x) = \sum_{p \leqslant x} \log p,$$

where the sum extends over all primes p that are less than or equal to x. Now write $\theta(x)$ as a Riemann-Stieltjes integral and integrate by parts,

$$\theta(x) = \int_{[2,x]} \log t \,\mathrm{d}\pi(t) = \pi(x)\log x - \pi(2^-)\log 2 - \int_2^x \pi(t)\,\mathrm{d}\log t = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,\mathrm{d}t.$$

If we assume that $\pi(x) = \mathrm{Li}(x) + Q(x)$, where $Q(x) = O\big(x^{\alpha+\varepsilon}\big)$ as $x \to \infty$, then

$$\begin{aligned}\int_2^x \frac{\pi(t)}{t}\,\mathrm{d}t &= \int_2^x \frac{\mathrm{Li}(t)}{t}\,\mathrm{d}t + \int_2^x \frac{Q(t)}{t}\,\mathrm{d}t \geqslant \int_2^x \frac{1}{t}\left(\int_2^t \frac{\mathrm{d}u}{\log u}\right)\mathrm{d}t - \int_2^x C_1 t^{\alpha+\varepsilon-1}\,\mathrm{d}t \\ &\geqslant \int_2^x \frac{1}{\log u}\left(\int_u^x \frac{\mathrm{d}t}{t}\right)\mathrm{d}u - \frac{C_1}{\alpha}x^{\alpha+\varepsilon} \geqslant \log x \cdot \mathrm{Li}(x) - (x-2) - 2C_1 x^{\alpha+\varepsilon}.\end{aligned}$$

Hence

$$\begin{aligned}\theta(x) &\leqslant [\pi(x) - \mathrm{Li}(x)]\log x + (x-2) + 2C_1 x^{\alpha+\varepsilon} \\ &\leqslant C_2 x^{\alpha+\varepsilon}\log x + (x-2) + 2C_1 x^{\alpha+\varepsilon} \\ &\leqslant x + C_3 x^{\alpha+2\varepsilon},\end{aligned}$$

and by replacing 2ε with ε we find $\theta(x) = x + O\big(x^{\alpha+\varepsilon}\big)$ as $x \to \infty$. Observe that

$$\psi(x) = \sum_{n=1}^{\infty} \theta\big(x^{\frac{1}{n}}\big) = \theta(x) + \theta\big(x^{\frac{1}{2}}\big) + \cdots + \theta\big(x^{1/\lfloor \log_2 x \rfloor}\big), \tag{7.4.3–2}$$

therefore

$$\psi(x) = x + O\big(x^{\alpha+\varepsilon}\big) \iff \theta(x) = x + O\big(x^{\alpha+\varepsilon}\big),$$

which gives the desired result.

(2) ⇒ (1) To deduce the asymptotic formula for $\pi(x)$, we first pass to the function

$$\pi_1(x) = \sum_{n \leqslant x} \frac{\Lambda(n)}{\log n}.$$

This is expressed in terms of the function $\psi(x)$ by

$$\begin{aligned}\pi_1(x) &= \sum_{n \leqslant x} \Lambda(n)\left(\int_n^x \frac{\mathrm{d}t}{t\log^2 t} + \frac{1}{\log x}\right) = \sum_{n \leqslant x} \Lambda(n)\int_n^x \frac{\mathrm{d}t}{t\log^2 t} + \frac{1}{\log x}\sum_{n \leqslant x}\Lambda(n) \\ &= \int_2^x \sum_{n \leqslant t}\Lambda(n)\frac{\mathrm{d}t}{t\log^2 t} + \frac{\psi(x)}{\log x} = \int_2^x \frac{\psi(t)\,\mathrm{d}t}{t\log^2 t} + \frac{\psi(x)}{\log x}.\end{aligned}$$

The effect of replacing $\psi(t)$ by t is to give

$$\int_2^x \frac{\mathrm{d}t}{\log^2 t} + \frac{x}{\log x} = \mathrm{Li}(x) + \frac{2}{\log 2}.$$

on integrating by parts. Thus it remains to consider the estimate for the error term $O(x^{\alpha+\varepsilon})$, which is

$$\ll \int_2^x Ct^{\alpha+\varepsilon-1}\,\mathrm{d}t + Cx^{\alpha+\varepsilon} < C\left(1+\frac{1}{\alpha}\right)x^{\alpha+\varepsilon} \leqslant 3Cx^{\alpha+\varepsilon} \quad \text{as } x \to \infty.$$

Finally, since

$$\pi_1(x) = \sum_{p^m \leqslant x} \frac{\log p}{m \log p} = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots,$$

and $\pi(x^{\frac{1}{2}}) \leqslant x^{\frac{1}{2}}, \pi(x^{\frac{1}{3}}) \leqslant x^{\frac{1}{3}}, \cdots$, the difference between $\pi_1(x)$ and $\pi(x)$ is $O(x^{\frac{1}{2}})$. Thus

$$\pi(x) - \mathrm{Li}(x) = O(x^{\alpha+\varepsilon}) \quad \text{as } x \to \infty.$$

(2) ⇒ (3) For $\mathrm{Re}(s) > 1$, Fubini's theorem gives

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty} \Lambda(n) \int_n^{\infty} sx^{-s-1}\,\mathrm{d}x = s\sum_{n=1}^{\infty} \int_1^{\infty} \Lambda(n) x^{-s-1} \mathbb{1}_{\{x \geqslant n\}}\,\mathrm{d}x \\ &= s\int_1^{\infty} \sum_{n=1}^{\infty} \Lambda(n) x^{-s-1} \mathbb{1}_{\{n \leqslant x\}}\,\mathrm{d}x = s\int_1^{\infty} \sum_{1 \leqslant n \leqslant x} \Lambda(n) x^{-s-1}\,\mathrm{d}x = s\int_1^{\infty} \psi(x) x^{-s-1}\,\mathrm{d}x. \end{aligned}$$

If we assume that $\psi(x) = x + R(x)$, where $R(x) = O(x^{\alpha+\varepsilon})$, then

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} + s\int_1^{\infty} R(x) x^{-s-1}\,\mathrm{d}x.$$

The assumption that $R(x) = O(x^{\alpha+\varepsilon})$ implies that the integral represents a holomorphic function in the half-plane $\mathrm{Re}(s) > \alpha + \varepsilon$. And since ε is arbitrary, we have shown that $\zeta(s)$ has no zeros in the strip $\alpha < \mathrm{Re}(s) < 1$.

(3) ⇒ (2) If we assume that $\zeta(s)$ has no zeros in the strip $\alpha < \mathrm{Re}(s) < 1$, then in the asymptotic formula (7.4.3–1) we have $|x^\rho| \leqslant x^\alpha$, and it remains to estimate the sum $\sum_\rho \frac{1}{|\rho|}$. Since $|\rho| > |\mathrm{Im}(\rho)|$, we shall deduce an estimate for the sum

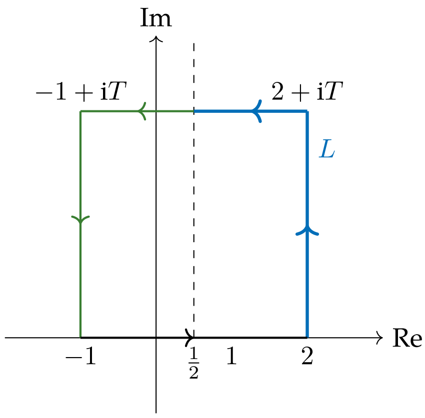
$$\sum_{0 < \mathrm{Im}(\rho) < T} \frac{1}{\mathrm{Im}(\rho)} \tag{7.4.3–3}$$

when T is large. Let $N(T)$ denote the number of nontrivial zeros ρ of ζ with $0 < \mathrm{Im}(\rho) < T$. Recall the function $\tilde{\xi}(s)$ defined in (7.4.2–4). Assuming for simplicity that T (which we suppose to be large) does not coincide with the ordinate of a zero, by the argument principle we have

$$2\pi N(T) = \Delta_R \operatorname{Arg} \tilde{\xi}(s),$$

where R is the rectangle in the s plane with vertices at

$$2, \quad 2+\mathrm{i}T, \quad -1+\mathrm{i}T, \quad -1.$$



There is no change in $\operatorname{Arg}\tilde{\xi}(s)$ as s moves along the base of this rectangle, since $\tilde{\xi}(s)$ is then real and nonvanishing. Further, the change as s moves from $\frac{1}{2}+\mathrm{i}T$ to $-1+\mathrm{i}T$ and then to -1 is equal to the change as s moves from 2 to $2+\mathrm{i}T$ and then to $\frac{1}{2}+\mathrm{i}T$, since

$$\tilde{\xi}(\sigma+\mathrm{i}t)=\tilde{\xi}(1-\sigma-\mathrm{i}t)=\overline{\tilde{\xi}(1-\sigma+\mathrm{i}t)}.$$

Hence

$$\pi N(T)=\Delta_L \operatorname{Arg}\tilde{\xi}(s),$$

where L denotes the line from 2 to $2+\mathrm{i}T$ and then to $\frac{1}{2}+\mathrm{i}T$. To find out $\Delta_L \operatorname{Arg}\tilde{\xi}(s)$, we consider each term in the defining equation (7.4.2–4):

$$\Delta_L \operatorname{Arg}(s-1)=\arg\left(-\tfrac{1}{2}+\mathrm{i}T\right)=\frac{\pi}{2}+O\left(T^{-1}\right),$$
$$\Delta_L \operatorname{Arg}\left(\pi^{-\frac{s}{2}}\right)=\Delta_L \operatorname{Arg}\left(\mathrm{e}^{-\frac{\log\pi}{2}s}\right)=-\frac{\log\pi}{2}T.$$

As for the Γ term, we have by Stirling's formula (Theorem 2.3 in Appendix A)

$$\log\Gamma(s)=\left(s-\tfrac{1}{2}\right)\log s-s+\frac{1}{2}\log 2\pi+O\left(|s|^{-1}\right)$$

as $|s|\to\infty$, in the angle $-\pi+\delta<\arg s<\pi-\delta$, for any fixed $\delta>0$. Hence

$$\begin{aligned}\Delta_L \operatorname{Arg}\Gamma\left(\tfrac{s}{2}+1\right)&=\operatorname{Im}\left\{\log\Gamma\left(\tfrac{\frac{1}{2}+\mathrm{i}T}{2}+1\right)\right\}=\operatorname{Im}\left\{\log\left(\tfrac{5}{4}+\tfrac{\mathrm{i}T}{2}\right)\right\}\\&=\operatorname{Im}\left\{\left(\tfrac{3}{4}+\tfrac{\mathrm{i}T}{2}\right)\log\left(\tfrac{5}{4}+\tfrac{\mathrm{i}T}{2}\right)-\tfrac{5}{4}-\tfrac{\mathrm{i}T}{2}+\tfrac{1}{2}\log 2\pi+O\left(T^{-1}\right)\right\}\\&=\frac{T}{2}\log\frac{T}{2}-\frac{T}{2}+\frac{3\pi}{8}+O\left(T^{-1}\right),\end{aligned}$$

and

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+\frac{1}{\pi}\Delta_L \operatorname{Arg}\zeta(s)+O\left(T^{-1}\right).$$

Note that

$$\begin{aligned}\Delta_L \operatorname{Arg}\zeta(s) &= \arg\{\zeta(\tfrac{1}{2}+\mathrm{i}T)\} = \operatorname{Im}\{\log\zeta(\tfrac{1}{2}+\mathrm{i}T)\} = \operatorname{Im}\left\{\int_L \frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s)\,\mathrm{d}s\right\} \\ &= \int_L \operatorname{Im}\left(\frac{\zeta'(s)}{\zeta(s)}\right)\mathrm{d}s = O(1) - \int_{\frac{1}{2}+\mathrm{i}T}^{2+\mathrm{i}T}\operatorname{Im}\left(\frac{\zeta'(s)}{\zeta(s)}\right)\mathrm{d}s,\end{aligned}$$

where the $O(1)$ term comes from the variation along $\operatorname{Re}(s)=2$. Recall formula (7.4.2–3), and note that the integral of the imaginary part of the summands is bounded by π:

$$\begin{aligned}\left|\int_{\frac{1}{2}+\mathrm{i}T}^{2+\mathrm{i}T}\operatorname{Im}\left(\frac{1}{s-\rho}\right)\mathrm{d}s\right| &= \left|\operatorname{Im}\{\log(2+\mathrm{i}T-\rho)-\log(\tfrac{1}{2}+\mathrm{i}T-\rho)\}\right| \\ &= \left|\arg(2+\mathrm{i}T-\rho)-\arg(\tfrac{1}{2}+\mathrm{i}T-\rho)\right| \leqslant \pi,\end{aligned}$$

and the number of terms in the sum in (7.4.2–3) is $O(\log T)$ by result ② in the proof of Problem 7.4.2. Therefore, we find $\Delta_L \operatorname{Arg}\zeta(s) = O(\log T)$ and so

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{7.4.3–4}$$

Now the sum (7.4.3–3) can be written as a Riemann-Stieltjes integral and integration by parts gives

$$\sum_{0<\operatorname{Im}(\rho)<T}\frac{1}{\operatorname{Im}(\rho)} = \int_0^T t^{-1}\,\mathrm{d}N(t) = \frac{1}{T}N(T) + \int_0^T t^{-2}N(t)\,\mathrm{d}t.$$

Since $N(t) = O(t\log t)$ by (7.4.3–4), the sum above is $O\left(\log^2 T\right)$. Hence by taking $T = x + O(1)$ we deduce from (7.4.3–1) that

$$\psi(x) - x = O\left(x^\alpha\log^2 T\right) + O(1) = O\left(x^\alpha\log^2 x\right) = O\left(x^{\alpha+\varepsilon}\right)$$

for every $\varepsilon > 0$. □

Stein 7.4.4 One can combine ideas from the prime number theorem with the proof of Dirichlet's theorem about primes in arithmetic progressions (given in Book I) to prove the following. Let q and ℓ be relatively prime integers. We consider the primes belonging to the arithmetic progression $\{qk+\ell\}_{k=1}^{\infty}$, and let $\pi_{q,\ell}(x)$ denote the number of such primes $\leqslant x$. Then one has

$$\pi_{q,\ell}(x) \sim \frac{x}{\varphi(q)\log x} \quad \text{as } x\to\infty,$$

where $\varphi(q)$ denotes the number of positive integers less than q and relatively prime to q.

Proof Define the quantities

$$\begin{aligned}\theta_{q,\ell}(x) &= \sum_{\substack{1\leqslant p\leqslant x\\ p\equiv\ell \pmod q}}\log p,\\ \psi_{q,\ell}(x) &= \sum_{\substack{p^m\leqslant x\\ p^m\equiv\ell \pmod q}}\log p = \sum_{\substack{1\leqslant n\leqslant x\\ p\equiv\ell \pmod q}}\Lambda(n),\end{aligned}$$

and the series $\{a_n\}$ by

$$a_n = \begin{cases} \Lambda(n), & \text{if } n \equiv \ell \pmod q, \\ 0, & \text{otherwise.} \end{cases} \tag{7.4.4–1}$$

Then the Dirichlet L-series (see Exercise 7.3.4) associated to $\{a_n\}$ is given by

$$L(s) = \sum_{\substack{n=1 \\ n\equiv \ell \pmod q}}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Let $G(q)$ be the group of characters modulo q. By Fubini's theorem we have

$$\int_1^\infty \frac{\psi_{q,\ell}(x)}{x^{s+1}}\,\mathrm{d}x = \int_1^\infty \sum_{\substack{1\leqslant n\leqslant x \\ n\equiv \ell \pmod q}} \Lambda(n)\frac{\mathrm{d}x}{x^{s+1}} = \sum_{\substack{n=1 \\ n\equiv \ell \pmod q}}^{\infty} \Lambda(n)\int_n^\infty \frac{\mathrm{d}x}{x^{s+1}} = L(s).$$

Recall the following orthogonality relation for characters:

$$\frac{1}{|\widehat{G}|}\sum_{\chi\in\widehat{G}} \chi(a)\overline{\chi(b)} = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

By this relation we can rewrite $L(s)$ as

$$L(s) = \frac{1}{\varphi(q)}\sum_{\chi\in G(q)} \overline{\chi(\ell)}\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^s}. \tag{7.4.4–2}$$

For the inner sum we observe that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^s} &= \sum_{p^m}\frac{\chi(p^m)\log p}{p^{ms}} \\
&= \sum_p \log p \sum_{m=1}^{\infty}\left(\frac{\chi(p)}{p^s}\right)^m \\
&= \sum_p \frac{\chi(p)p^{-s}\log p}{1-\chi(p)p^{-s}} \\
&= \sum_p \frac{\mathrm{d}}{\mathrm{d}s}\log(1-\chi(p)p^{-s}) \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p \frac{1}{1-\chi(p)p^{-s}}.
\end{aligned}$$

Note that the infinite product above is similar to the Euler product for $\zeta(s)$ as shown in Section 7.1. We define the Dirichlet L-function by

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}. \tag{7.4.4–3}$$

Since $\chi \in G(q)$ is a strongly multiplicative arithmetic function, we have

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

by the fundamental theorem of arithmetic. Combining the above into formula (7.4.4–2) shows

$$L(s) = -\frac{1}{\varphi(q)} \sum_{\chi \in G(q)} \overline{\chi(\ell)} \cdot \frac{L'(s,\chi)}{L(s,\chi)}. \tag{7.4.4–4}$$

When $\chi = \chi_0$ is the principal character defined by

$$\chi_0(a) = \begin{cases} 1, & \text{if } (a,q) = 1, \\ 0, & \text{otherwise}, \end{cases}$$

for all $a \in \mathbb{Z}$, applying (7.4.4–3) one has

$$L(s,\chi_0) = \prod_{p \nmid q} \frac{1}{1-p^{-s}} = \zeta(s) \prod_{p \mid q} \frac{1}{1-p^{-s}}. \tag{7.4.4–5}$$

Since the product over $p \mid q$ is finite, it follows that $L(s,\chi_0)$ is meromorphic in the half-plane $\operatorname{Re}(s) > 0$ with a simple pole at $s = 1$. If χ is not the principal character, by the orthogonality relation

$$\frac{1}{|G|} \sum_{a \in G} \chi(a) = \begin{cases} 1, & \text{if } \chi = 1, \\ 0, & \text{if } \chi \neq 1, \end{cases}$$

we have

$$\sum_{n=1}^{q} \chi(n+a) = 0$$

for any $a \in \mathbb{Z}$. Therefore, the partial sums $\sum_{n=1}^{N} \chi(n)$ (for $N \in \mathbb{N}$) are bounded by $\varphi(q)$, and by Exercise 7.3.1 we see that $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ converges absolutely for $\operatorname{Re}(s) > 0$. Hence $L(s,\chi)$ is holomorphic in the half-plane $\operatorname{Re}(s) > 0$ when $\chi \neq \chi_0$. In fact, the above two results are direct consequences of Exercise 7.3.4.

Logarithmic differentiation of (7.4.4–5) gives

$$-\frac{L'(s,\chi_0)}{L(s,\chi_0)} = -\frac{\zeta'(s)}{\zeta(s)} - \sum_{p \mid q} \frac{\log p}{p^s - 1}.$$

Substituting this into (7.4.4–4) we see the residue of $L(s)$ at $s = 1$ is

$$\operatorname{Res}(L,1) = -\frac{1}{\varphi(q)} \operatorname{ord}(L(s,\chi),1) = \frac{1}{\varphi(q)}.$$

To prove the prime number theorem for arithmetic progressions, we need the following lemmas:

(1) $\theta_{q,\ell}(x) = O(x)$ as $x \to \infty$.

(2) $\psi_{q,\ell}(x) = \theta_{q,\ell}(x) + O(\sqrt{x})$ as $x \to \infty$.

(3) $\psi_{q,\ell}(x) = O(x)$ as $x \to \infty$.

Proof of (1) Recall the function $\theta(x)$ defined in the proof of Problem 7.4.3. Since

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \geqslant \binom{2n}{n} \geqslant \prod_{n<p\leqslant 2n} p,$$

take $n = 2^{k-1}$, then

$$\prod_{2^{k-1}<p\leqslant 2^k} p \leqslant 2^{2^k} \quad \text{and} \quad \sum_{2^{k-1}<p\leqslant 2^k} \log p \leqslant 2^k \log 2.$$

Hence

$$\begin{aligned}\theta_{q,\ell}(x) \leqslant \theta(x) \leqslant \sum_{p\leqslant 2^{1+\lfloor \log_2 x\rfloor}} \log p = \sum_{k=1}^{1+\lfloor \log_2 x\rfloor} \sum_{2^{k-1}<p\leqslant 2^k} \log p \leqslant \sum_{k=1}^{1+\lfloor \log_2 x\rfloor} 2^k \log 2 \\ \leqslant 2^{2+\lfloor \log_2 x\rfloor} \log 2 \leqslant (4\log 2)x.\end{aligned} \tag{7.4.4–6}$$

Proof of (2) Note that

$$\psi_{q,\ell}(x) - \theta_{q,\ell}(x) = \sum_{m\geqslant 2} \sum_{\substack{p^m\leqslant x \\ p^m\equiv \ell \pmod q}} \log p \leqslant \sum_{m\geqslant 2} \sum_{p^m\leqslant x} \log p = \psi(x) - \theta(x),$$

and formula (7.4.3–2) shows that

$$\begin{aligned}\psi(x) - \theta(x) &= \theta\big(x^{\frac{1}{2}}\big) + \theta\big(x^{\frac{1}{3}}\big) + \cdots + \theta\big(x^{1/\lfloor \log_2 x\rfloor}\big) \\ &\leqslant \theta\big(x^{\frac{1}{2}}\big) + \left(\frac{\log x}{\log 2} - 2\right)\theta\big(x^{\frac{1}{3}}\big) = O\big(\sqrt{x} + \sqrt[3]{x}\cdot \log x\big) \\ &= O\big(\sqrt{x}\big) \quad \text{as } x \to \infty.\end{aligned}$$

Proof of (3) Combine (1) and (2).

Now we are ready to prove the desired result. We shall use the following **Tauberian theorem for Dirichlet series**. Let $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series and set $A(x) = \sum_{n\leqslant x} a_n$. Suppose $L(s)$ satisfies the following conditions:

(1) $a_n \geqslant 0$ for all n.

(2) There exist $C > 0$ and $\sigma > 0$ such that $|A(x)| \leqslant Cx^{\sigma}$ for all $x \geqslant 1$.

(3) $L(s)$ converges for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma$, where σ comes from (2).

(4) There exists an open subset $U \subset \mathbb{C}$ containing $\{s \in \mathbb{C} : \mathrm{Re}(s) \geqslant \sigma\}$ such that $L(s)$ can be continued analytically to $U \setminus \{\sigma\}$ and for which $\lim_{s\to\sigma}(s-\sigma)L(s) = \alpha$.

Then

$$\lim_{x\to\infty}\frac{A(x)}{x^\sigma}=\frac{\alpha}{\sigma}.$$

Lemma (3), together with the analytic continuation of $L(s)$ mentioned above, implies that $\{a_n\}$ (defined by (7.4.4–1)) and the corresponding $L(s)$ satisfy the conditions of the Tauberian theorem, with $\sigma=1$ and $\alpha=\dfrac{1}{\varphi(q)}$. Hence

$$\frac{\psi_{q,\ell}(x)}{x}\to\frac{1}{\varphi(q)}\quad\text{as } x\to\infty,$$

and then by Lemma (2),

$$\frac{\theta_{q,\ell}(x)}{x}=\frac{\psi_{q,\ell}(x)-O(\sqrt{x})}{x}\to\frac{1}{\varphi(q)}\quad\text{as } x\to\infty.$$

Write $\pi_{q,\ell}(x)$ as a Riemann-Stieltjes integral and integrate by parts:

$$\pi_{q,\ell}(x)=\sum_{\substack{1\leqslant p\leqslant x\\ p\equiv\ell\ (\mathrm{mod}\ q)}}\log p\cdot\frac{1}{\log p}=\int_2^x\frac{\mathrm{d}\theta_{q,\ell}(t)}{\log t}=\frac{\theta_{q,\ell}(x)}{\log x}+\int_2^x\frac{\theta_{q,\ell}(t)}{t\log^2 t}\,\mathrm{d}t.$$

Since $\theta_{q,\ell}(t)\leqslant(4\log 2)t$ by (7.4.4–6), we have

$$0\leqslant\int_2^x\frac{\theta_{q,\ell}(t)}{t\log^2 t}\,\mathrm{d}t\leqslant 4\log 2\int_2^x\frac{\mathrm{d}t}{\log^2 t}=O\left(\frac{x}{\log^2 x}\right)\quad\text{as } x\to\infty.$$

Therefore,

$$\frac{\pi_{q,\ell}(x)\log x}{x}=\frac{\theta_{q,\ell}(x)}{x}+O\left(\frac{1}{\log x}\right)\to\frac{1}{\varphi(q)}\quad\text{as } x\to\infty.$$

□

Stein 9.3.1 Suppose that a meromorphic function f has two periods ω_1 and ω_2, with $\omega_2/\omega_1\in\mathbb{R}$.

(1) Suppose ω_2/ω_1 is rational, say equal to p/q, where p and q are relatively prime integers. Prove that as a result the periodicity assumption is equivalent to the assumption that f is periodic with the simple period $\omega_0=\frac{1}{q}\omega_1$.

(2) If ω_2/ω_1 is irrational, then f is constant. To prove this, use the fact that $\{m-n\tau\}$ is dense in $\mathbb{R}$ whenever τ is irrational and m,n range over the integers.

Proof (1) Since p and q are relatively prime, there exist integers m and n such that $mp+nq=1$. Then

$$n\omega_2=\frac{np}{q}\omega_1=\frac{1-mq}{q}\omega_1=\frac{1}{q}\omega_1-m\omega_1,$$

hence

$$f\left(z+\tfrac{1}{q}\omega_1\right)=f\left(z+\tfrac{1}{q}\omega_1-m\omega_1\right)=f(z+n\omega_2)=f(z)\quad\text{for all } z\in\mathbb{C}.$$

This shows that $\frac{1}{q}\omega_1$ is a period of f. And by $\left|\frac{1}{q}\omega_1\right|=\frac{1}{q}|\omega_1|=\frac{1}{p}|\omega_2|$ we see that f is periodic with a simple period. The converse is trivial.

(2) The continuity of f, together with the given fact, implies that f is constant. □

Stein 9.3.2 Suppose that $a_1, \cdots, a_r$ and $b_1, \cdots, b_r$ are the zeros and poles, respectively, in the fundamental parallelogram of an elliptic function f. Show that

$$a_1 + \cdots + a_r - b_1 - \cdots - b_r = n\omega_1 + m\omega_2$$

for some integers n and m.

Proof After translating the parallelogram by a small amount if necessary, we may assume that f has no zeros or poles on the boundary of the fundamental parallelogram P. Since $a_1, \cdots, a_r$ and $b_1, \cdots, b_r$ are all simple poles of the function $\dfrac{f'(z)}{f(z)}$ and nonzero by our assumption, we have

$$\operatorname{Res}\left(z\frac{f'(z)}{f(z)}, c\right) = \lim_{z \to c}(z-c)z\frac{f'(z)}{f(z)} = c \operatorname{ord}(f), \quad \text{if } c \text{ is a zero or pole of } f.$$

Hence by the residue theorem we have

$$\begin{aligned}
&2\pi \mathrm{i}(a_1 + \cdots + a_r - b_1 - \cdots - b_r) \\
=&\int_0^{\omega_1} z\frac{f'(z)}{f(z)}\,\mathrm{d}z + \int_{\omega_1}^{\omega_1+\omega_2} z\frac{f'(z)}{f(z)}\,\mathrm{d}z + \int_{\omega_1+\omega_2}^{\omega_2} z\frac{f'(z)}{f(z)}\,\mathrm{d}z + \int_{\omega_2}^{0} z\frac{f'(z)}{f(z)}\,\mathrm{d}z \\
=&\int_0^{\omega_1} z\frac{f'(z)}{f(z)}\,\mathrm{d}z + \int_0^{\omega_2} (z+\omega_1)\frac{f'(z)}{f(z)}\,\mathrm{d}z + \int_{\omega_1}^{0} (z+\omega_2)\frac{f'(z)}{f(z)}\,\mathrm{d}z + \int_{\omega_2}^{0} z\frac{f'(z)}{f(z)}\,\mathrm{d}z \\
=&\omega_2\int_{\omega_1}^{0} \frac{f'(z)}{f(z)}\,\mathrm{d}z + \omega_1\int_0^{\omega_2} \frac{f'(z)}{f(z)}\,\mathrm{d}z \\
=&\omega_2 \operatorname{Log} f(z)\Big|_{\omega_1}^{0} + \omega_1 \operatorname{Log} f(z)\Big|_{0}^{\omega_2} \\
=&\mathrm{i}\omega_2 \operatorname{Arg} f(z)\Big|_{\omega_1}^{0} + \mathrm{i}\omega_1 \operatorname{Arg} f(z)\Big|_{0}^{\omega_2} \\
=&2\pi\mathrm{i}(m\omega_2 + n\omega_1) \quad \text{for some integers } n \text{ and } m.
\end{aligned}$$

□

Stein 9.3.3 In contrast with the result in Lemma 1.5, prove that the series

$$\sum_{n+m\tau \in \Lambda^*} \frac{1}{|n+m\tau|^2} \quad \text{where } \tau \in \mathbb{H}$$

does not converge. In fact, show that

$$\sum_{1 \leqslant n^2+m^2 \leqslant R^2} \frac{1}{n^2+m^2} = 2\pi \log R + O(1) \quad \text{as } R \to \infty.$$

Proof Since

$$\iint\limits_{1 \leqslant x^2+y^2 \leqslant R^2} \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2} = \int_1^R \int_0^{2\pi} \frac{1}{r}\,\mathrm{d}\theta\,\mathrm{d}r = 2\pi \log R$$

and $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 \leqslant x^2 + y^2$ in the first quadrant, we have

$$\left| \sum_{\substack{1 \leqslant n^2+m^2 \leqslant R^2 \\ n,m \geqslant 0}} \frac{1}{n^2+m^2} - \frac{\pi}{2}\log R \right|$$

$$
\begin{aligned}
&\leqslant \left| \iint\limits_{\substack{1\leqslant x^2+y^2\leqslant (R+\sqrt{2})^2 \\ x,y\geqslant 0}} \frac{\mathrm{d}x\,\mathrm{d}y}{\lfloor x\rfloor^2+\lfloor y\rfloor^2} - \iint\limits_{\substack{1\leqslant x^2+y^2\leqslant R^2 \\ x,y\geqslant 0}} \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2} \right| \\
&\leqslant \left| \iint\limits_{\substack{1\leqslant x^2+y^2\leqslant R^2 \\ x,y\geqslant 0}} \left(\frac{1}{\lfloor x\rfloor^2+\lfloor y\rfloor^2} - \frac{1}{x^2+y^2} \right) \mathrm{d}x\,\mathrm{d}y \right| + \iint\limits_{\substack{R^2< x^2+y^2\leqslant (R+\sqrt{2})^2 \\ x,y\geqslant 0}} \frac{\mathrm{d}x\,\mathrm{d}y}{\lfloor x\rfloor^2+\lfloor y\rfloor^2} \\
&\leqslant \iint\limits_{\substack{1\leqslant x^2+y^2\leqslant R^2 \\ x,y\geqslant 0}} \frac{\left(x^2-\lfloor x\rfloor^2\right)+\left(y^2-\lfloor y\rfloor^2\right)}{\left(\lfloor x\rfloor^2+\lfloor y\rfloor^2\right)^2} \mathrm{d}x\,\mathrm{d}y + \frac{\pi\left[\left(R+\sqrt{2}\right)^2-R^2\right]}{R^2-2} \\
&\leqslant \iint\limits_{\substack{1\leqslant x^2+y^2\leqslant R^2 \\ x,y\geqslant 0}} \frac{(x+\lfloor x\rfloor)+(y+\lfloor y\rfloor)}{\left(\lfloor x\rfloor^2+\lfloor y\rfloor^2\right)^2} \mathrm{d}x\,\mathrm{d}y + \frac{\pi\left(2\sqrt{2}R+2\right)}{R^2-2} \\
&= O(1) \quad \text{as } R\to\infty.
\end{aligned}
$$

With the fact that $\sum_{n\neq 0}\frac{1}{n^2}=\frac{\pi^2}{3}$ on each axis, we conclude that

$$
\sum_{1\leqslant n^2+m^2\leqslant R^2} \frac{1}{n^2+m^2} = 2\pi\log R + O(1) \quad \text{as } R\to\infty.
$$

Finally, observe that $|n+m\tau|^2 \leqslant C\left(n^2+m^2\right)$, where $C=\max\left\{1,[\mathrm{Im}(\tau)]^2,\mathrm{Re}(\tau)\right\}$, hence the given series does not converge. □

Stein 9.3.4 By rearranging the series

$$
\frac{1}{z^2}+\sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right],
$$

show directly, without differentiation, that $\wp(z+\omega)=\wp(z)$ whenever $\omega\in\Lambda$.

Proof Define

$$
\wp^R(z)=\frac{1}{z^2}+\sum_{0<|\omega|<R}\left[\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right].
$$

Fix $\rho>1$. For $|z|\leqslant\rho$ and $|\omega|\geqslant 2\rho$ we have

$$
\left|\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right| = \left|\frac{z^2+2z\omega}{\omega^2(z+\omega)^2}\right| \leqslant \frac{\rho(\rho+2|\omega|)}{|\omega|^2(|\omega|-\rho)^2} \leqslant \frac{\frac{5}{2}\rho|\omega|}{|\omega|^2\left(|\omega|-\frac{|\omega|}{2}\right)^2} = \frac{10\rho}{|\omega|^3}.
$$

Then

$$
\left|\wp(z)-\wp^R(z)\right| \leqslant \sum_{|\omega|\geqslant R}\left|\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right| \leqslant \sum_{|\omega|\geqslant R}\frac{C_1}{|\omega|^3}.
$$

Since

$$
|n+m\tau|^2 \geqslant C_2(|n|+|m|)^2 \geqslant C_2\left(n^2+m^2\right)
$$

$$|n+m\tau|^2 \leqslant C_3(|n|+|m|)^2 \leqslant 4C_3\max\{n^2,m^2\} \leqslant 4C_3\big(n^2+m^2\big)$$

for $n, m \in \mathbb{Z}$, the last series can be estimated by

$$\begin{aligned}
\sum_{|\omega|\geqslant R}\frac{C_1}{|\omega|^3} &= \sum_{|n+m\tau|^2\geqslant R^2}\frac{C_1}{(|n+m\tau|^2)^{\frac{3}{2}}} \leqslant \sum_{n^2+m^2\geqslant\frac{R^2}{4C_3}}\frac{C_1}{C_2^{\frac{3}{2}}(n^2+m^2)^{\frac{3}{2}}}\\
&\leqslant C_1C_2^{-\frac{3}{2}}\int\limits_{x^2+y^2\geqslant\left(\frac{R}{2\sqrt{C_3}}-\sqrt{2}\right)^2}\frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^{\frac{3}{2}}} = 2\pi C_1C_2^{-\frac{3}{2}}\int_{\frac{R}{2\sqrt{C_3}}-\sqrt{2}}^{\infty}\frac{\mathrm{d}r}{r^2}\\
&= \frac{2\pi C_1C_2^{-\frac{3}{2}}}{\frac{R}{2\sqrt{C_3}}-\sqrt{2}} = O\left(\frac{1}{R}\right) \quad \text{as } R\to\infty.
\end{aligned}$$

Therefore, we have

$$\wp(z) = \wp^R(z) + O\left(\frac{1}{R}\right) \quad \text{as } R\to\infty.$$

Next, observe that

$$\begin{aligned}
\wp^R(z+1)-\wp^R(z) &= \frac{1}{(z+1)^2}-\frac{1}{z^2}+\sum_{0<|\omega|<R}\left[\frac{1}{(z+1+\omega)^2}-\frac{1}{(z+\omega)^2}\right]\\
&= \sum_{0\leqslant|\omega|<R}\left[\frac{1}{(z+1+\omega)^2}-\frac{1}{(z+\omega)^2}\right]\\
&= \sum_{0\leqslant|\omega-1|<R}\frac{1}{(z+\omega)^2}-\sum_{0\leqslant|\omega|<R}\frac{1}{(z+\omega)^2}\\
&= \sum_{|\omega-1|<R\leqslant|\omega|}\frac{1}{(z+\omega)^2}-\sum_{|\omega|<R\leqslant|\omega-1|}\frac{1}{(z+\omega)^2},
\end{aligned}$$

and then

$$\left|\wp^R(z+1)-\wp^R(z)\right| \leqslant \sum_{|\omega-1|<R\leqslant|\omega|}\frac{1}{|z+\omega|^2}+\sum_{|\omega|<R\leqslant|\omega-1|}\frac{1}{|z+\omega|^2} \leqslant \sum_{R-1\leqslant|\omega|\leqslant R+1}\frac{1}{|z+\omega|^2}.$$

Since for $|z|\leqslant\rho$ and $|\omega|\geqslant 2\rho$ we have

$$\frac{1}{|z+\omega|^2} \leqslant \frac{1}{(|\omega|-\rho)^2} \leqslant \frac{1}{\left(|\omega|-\frac{|\omega|}{2}\right)^2} \leqslant \frac{4}{|\omega|^2},$$

then

$$\begin{aligned}
\left|\wp^R(z+1)-\wp^R(z)\right| &\leqslant \sum_{R-1\leqslant|\omega|\leqslant R+1}\frac{C_4}{|\omega|^2} \leqslant \sum_{\frac{(R-1)^2}{4C_3}\leqslant n^2+m^2\leqslant\frac{(R+1)^2}{C_2}}\frac{C_4C_2^{-1}}{n^2+m^2}\\
&\leqslant \frac{4C_2^{-1}C_3C_4}{(R-1)^2}\cdot 2\pi\cdot\frac{R+1}{\sqrt{C_2}}\cdot C_5 = O\left(\frac{1}{R}\right) \quad \text{as } R\to\infty.
\end{aligned}$$

The same argument shows that

$$\wp^R(z+\tau) = \wp^R(z) + O\left(\frac{1}{R}\right) \quad \text{as } R\to\infty.$$

Hence

$$|\wp(z+1)-\wp(z)| \leqslant \left|\wp(z+1)-\wp^R(z+1)\right| + \left|\wp^R(z+1)-\wp^R(z)\right| + \left|\wp^R(z)-\wp(z)\right| = O\left(\frac{1}{R}\right),$$

and by letting $R\to\infty$ we obtain $\wp(z+1)=\wp(z)$. The same argument shows that $\wp(z+\tau)=\wp(z)$. Since ρ is arbitrary, we conclude that $\wp(z+\omega)=\wp(z)$ for all $\omega\in\Lambda$. □

Stein 9.3.5 Let $\sigma(z)$ be the canonical product

$$\sigma(z) = z\prod_{j=1}^{\infty} E_2\left(\tfrac{z}{\tau_j}\right)$$

where τ_j is an enumeration of the periods $\{n+m\tau\}$ with $(n,m)\neq(0,0)$, and $E_2(z)=(1-z)\mathrm{e}^{z+\frac{z^2}{2}}$.

(1) Show that $\sigma(z)$ is an entire function of order 2 that has simple zeros at all the periods $n+m\tau$, and vanishes nowhere else.

(2) Show that

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{(n,m)\neq(0,0)}\left[\frac{1}{z-n-m\tau}+\frac{1}{n+m\tau}+\frac{z}{(n+m\tau)^2}\right],$$

and that this series converges whenever z is not a lattice point.

(3) Let $L(z) = -\dfrac{\sigma'(z)}{\sigma(z)}$. Then

$$L'(z) = \frac{[\sigma'(z)]^2-\sigma(z)\sigma''(z)}{[\sigma(z)]^2} = \wp(z).$$

Proof (1) Given z, the set of $\omega\in\Lambda$ for which $|\omega|<2|z|$ is finite. If ω is not in that set, we have

$$\begin{aligned}\left|\log E_2\left(\tfrac{z}{\omega}\right)\right| &= \left|\log\left(1-\frac{z}{\omega}\right)+\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2\right| = \left|\frac{1}{3}\left(\frac{z}{\omega}\right)^3+\frac{1}{4}\left(\frac{z}{\omega}\right)^4+\cdots\right|\\ &\leqslant \frac{1}{3}\left|\frac{z}{\omega}\right|^{2+\varepsilon}\left(\left|\frac{z}{\omega}\right|^{1-\varepsilon}+\left|\frac{z}{\omega}\right|^{2-\varepsilon}+\cdots\right) \leqslant \frac{1}{3}\left|\frac{z}{\omega}\right|^{2+\varepsilon}\left(\left|\frac{z}{\omega}\right|^{\frac{1}{2}}+\left|\frac{z}{\omega}\right|^{\frac{3}{2}}+\cdots\right)\\ &\leqslant \frac{1}{3}\left|\frac{z}{\omega}\right|^{2+\varepsilon}\left[\left(\frac{1}{2}\right)^{\frac{1}{2}}+\left(\frac{1}{2}\right)^{\frac{3}{2}}+\cdots\right] = \frac{\sqrt{2}}{3}\left|\frac{z}{\omega}\right|^{2+\varepsilon}\end{aligned}$$

for any $\varepsilon\in\left(0,\frac{1}{2}\right)$. So the infinite product converges absolutely and uniformly on bounded subsets of $\mathbb{C}$ by Lemma 1.5 in Chapter 9. Accordingly, it defines an entire function and it is clear that its zeros are at the points of Λ and are simple. Moreover, since ε can be chosen arbitrarily close to 0, we conclude that $\sigma(z)$ is of order 2.

(2) Logarithmic differentiation of $\sigma(z)$ gives

$$\begin{aligned}\frac{\sigma'(z)}{\sigma(z)} &= \frac{1}{z}+\sum_{j=1}^{\infty}\left[\frac{1}{z-\tau_j}+\frac{1}{\tau_j}+\frac{z}{\tau_j^2}\right]\\ &= \frac{1}{z}+\sum_{(n,m)\neq(0,0)}\left[\frac{1}{z-n-m\tau}+\frac{1}{n+m\tau}+\frac{z}{(n+m\tau)^2}\right].\end{aligned}$$

For $|z| \leqslant \rho$ and $|\omega| \geqslant 2\rho$ we have

$$\left|\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right| = \frac{|z|^2}{|\omega|^2|z-\omega|} \leqslant \frac{\rho^2}{|\omega|^2(|\omega|-|z|)} \leqslant \frac{2\rho^2}{|\omega|^3},$$

hence the last series converges whenever z is not a lattice point.

(3) Termwise differentiation of the series in (2) gives

$$L'(z) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)} \left[\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2}\right] = \wp(z).$$

□

Stein 9.3.6 Prove that $\wp''$ is a quadratic polynomial in $\wp$.

Proof We have

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

by Corollary 2.3 in Chapter 9. Differentiating both sides gives

$$2\wp'\wp'' = 12\wp^2\wp' - g_2\wp'$$

and then

$$\wp'' = 6\wp^2 - \frac{g_2}{2}.$$

□

Stein 9.3.7 Setting $\tau = \frac{1}{2}$ in the expression

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^2} = \frac{\pi^2}{\sin^2(\pi\tau)},$$

deduce that

$$\sum_{m\geqslant 1,\, m \text{ odd}} \frac{1}{m^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{m\geqslant 1} \frac{1}{m^2} = \frac{\pi^2}{6} = \zeta(2).$$

Similarly, using $\sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^4}$ deduce that

$$\sum_{m\geqslant 1,\, m \text{ odd}} \frac{1}{m^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{m\geqslant 1} \frac{1}{m^4} = \frac{\pi^4}{90} = \zeta(4).$$

These results were already obtained using Fourier series in the exercises at the end of Chapters 2 and 3 in Book I.

Proof Substituting $\tau = -\frac{1}{2}$ shows that

$$\pi^2 = \sum_{m=-\infty}^{\infty} \frac{1}{\left(m+\frac{1}{2}\right)^2} = \sum_{m=-\infty}^{\infty} \frac{4}{(2m+1)^2} = 8 \sum_{m\geqslant 1,\, m \text{ odd}} \frac{1}{m^2}.$$

For the second identity, observe that

$$\frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{m^2}=\sum_{m\geqslant 2,\, m \text{ even}}\frac{1}{m^2}=\sum_{m=1}^{\infty}\frac{1}{m^2}-\sum_{m\geqslant 1,\, m \text{ odd}}\frac{1}{m^2},$$

hence

$$\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{4}{3}\sum_{m\geqslant 1,\, m \text{ odd}}\frac{1}{m^2}=\frac{\pi^2}{6}.$$

Similarly, by differentiating the given expression twice we get

$$\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^4}=\frac{\pi^4}{3}\cdot\frac{\sin^2(\pi\tau)+3\pi\cos^2(\pi\tau)}{\sin^4(\pi\tau)}$$

and then

$$\sum_{m=-\infty}^{\infty}\frac{1}{\left(m+\frac{1}{2}\right)^4}=\sum_{m=-\infty}^{\infty}\frac{16}{(2m+1)^4}=\frac{\pi^4}{3}.$$

Hence

$$\sum_{m\geqslant 1,\, m \text{ odd}}\frac{1}{m^4}=\frac{\pi^4}{96},$$

and by

$$\frac{1}{16}\sum_{m=1}^{\infty}\frac{1}{m^4}=\sum_{m\geqslant 2,\, m \text{ even}}\frac{1}{m^4}=\sum_{m=1}^{\infty}\frac{1}{m^4}-\sum_{m\geqslant 1,\, m \text{ odd}}\frac{1}{m^4}$$

we find

$$\sum_{m=1}^{\infty}\frac{1}{m^4}=\frac{\pi^4}{90}.$$

□

Stein 9.3.8 Let

$$E_4(\tau)=\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^4}$$

be the Eisenstein series of order 4.

(1) Show that $E_4(\tau)\to\frac{\pi^4}{45}$ as $\operatorname{Im}(\tau)\to\infty$.

(2) More precisely,

$$\left|E_4(\tau)-\frac{\pi^4}{45}\right|\leqslant c\mathrm{e}^{-2\pi t}\quad \text{if } \tau=x+\mathrm{i}t \text{ and } t\geqslant 1.$$

(3) Deduce that

$$\left|E_4(\tau)-\tau^{-4}\frac{\pi^4}{45}\right|\leqslant ct^{-4}\mathrm{e}^{-\frac{2\pi}{t}}\quad \text{if } \tau=\mathrm{i}t \text{ and } 0<t\leqslant 1.$$

Proof (1) This is a consequence of (2).

(2) By Theorem 2.5 in Chapter 9 we have

$$E_4(\tau)=2\zeta(4)+\frac{(2\pi)^4}{3}\sum_{r=1}^{\infty}\sigma_3(r)\mathrm{e}^{2\pi\mathrm{i}\tau r}\quad \text{for } \operatorname{Im}(\tau)>0.$$

We know that $\zeta(4) = \frac{\pi^4}{90}$ by Exercise 9.3.7, hence for $\tau = x + \mathrm{i}t$ and $t \geqslant 1$ we have

$$\begin{aligned}\left|E_4(\tau) - \frac{\pi^4}{45}\right| &= \left|\frac{(2\pi)^4}{3}\sum_{r=1}^{\infty}\sigma_3(r)\mathrm{e}^{2\pi\mathrm{i}\tau r}\right| \\ &\leqslant \frac{(2\pi)^4}{3}\sum_{r=1}^{\infty}\sigma_3(r)\mathrm{e}^{-2\pi rt} \\ &\leqslant \frac{(2\pi)^4}{3}\sum_{r=1}^{\infty}r^4\mathrm{e}^{-2\pi rt} \\ &= \frac{(2\pi)^4}{3}\mathrm{e}^{-2\pi t}\sum_{r=1}^{\infty}r^4\mathrm{e}^{-2\pi(r-1)t} \\ &\leqslant c\mathrm{e}^{-2\pi t}\end{aligned}$$

because of exponential decay of the summands.

(3) By Theorem 2.1 (iii) in Chapter 9 we have

$$E_4(\tau) = \tau^{-4}E_4\left(-\tfrac{1}{\tau}\right).$$

Hence for $\tau = \mathrm{i}t$ and $0 < t \leqslant 1$ we have

$$\left|E_4(\tau) - \tau^{-4}\frac{\pi^4}{45}\right| = t^{-4}\left|E_4\left(\tfrac{\mathrm{i}}{t}\right) - \frac{\pi^4}{45}\right| \leqslant ct^{-4}\mathrm{e}^{-\frac{2\pi}{t}}$$

by (2). □

Stein 9.4.1 Besides the approach in Section 1.2, there are several alternate ways of dealing with the sum $\displaystyle\sum_{\omega\in\Lambda}\frac{1}{(z+\omega)^2}$, where $\omega = n + m\tau$. For example, one may sum either (1) circularly, (2) first in n then in m, (3) or first in m then in n.

(1) Prove that if $z \notin \Lambda$, then

$$\lim_{R\to\infty}\sum_{n^2+m^2\leqslant R^2}\frac{1}{(z+n+m\tau)^2} = S_1(z)$$

exists and $S_1(z) = \wp(z) + c_1$.

(2) Similarly,

$$\sum_m\left(\sum_n\frac{1}{(z+n+m\tau)^2}\right) = S_2(z)$$

exists and $S_2(z) = \wp(z) + c_2$, where $c_2 = F(\tau)$, and F is the forbidden Eisenstein series.

(3) Also

$$\sum_n\left(\sum_m\frac{1}{(z+n+m\tau)^2}\right) = S_3(z)$$

exists with $S_3(z) = \wp(z) + c_3$, and $c_3 = \widetilde{F}(\tau)$, the reverse of F.

Proof (1) It suffices to show that the limit

$$\lim_{R\to\infty}\sum_{1\leqslant n^2+m^2\leqslant R^2}\frac{1}{(n+m\tau)^2}$$

exists. We first observe that

$$\sum_{m\neq 0}\left[\sum_{n\in\mathbb{Z}}\left(\frac{1}{n+m\tau}-\frac{1}{n+1+m\tau}\right)\right]=0.$$

To see this, note that for any $m \neq 0$, the inner sum converges absolutely since the summands are $O\left(\frac{1}{n^2}\right)$ as $n \to \infty$, hence it can be evaluated by

$$\lim_{N\to\infty}\sum_{n=-N}^{N-1}\left(\frac{1}{n+m\tau}-\frac{1}{n+1+m\tau}\right)=\lim_{N\to\infty}\left(\frac{1}{-N+m\tau}-\frac{1}{N+m\tau}\right)=0.$$

Thus, we can rewrite the forbidden Eisenstein series $F(\tau)$ as

$$\begin{aligned}
F(\tau)&=\sum_{m}\left(\sum_{n}\frac{1}{(n+m\tau)^2}\right)=\sum_{n\neq 0}\frac{1}{n^2}+\sum_{m\neq 0}\left(\sum_{n\in\mathbb{Z}}\frac{1}{(n+m\tau)^2}\right)\\
&=\sum_{n\neq 0}\frac{1}{n^2}+\sum_{m\neq 0}\left(\sum_{n\in\mathbb{Z}}\frac{1}{(n+m\tau)^2}\right)-\sum_{m\neq 0}\left[\sum_{n\in\mathbb{Z}}\left(\frac{1}{n+m\tau}-\frac{1}{n+1+m\tau}\right)\right]\\
&=\sum_{n\neq 0}\frac{1}{n^2}+\sum_{m\neq 0}\left(\sum_{n\in\mathbb{Z}}\frac{1}{(n+m\tau)^2(n+1+m\tau)}\right).
\end{aligned}$$

The last series converges absolutely by comparison with $\displaystyle\sum_m\sum_n\frac{1}{(n+m\tau)^3}$. Since

$$\lim_{R\to\infty}\sum_{1\leqslant n^2+m^2\leqslant R^2}\frac{1}{(n+m\tau)^2}=\sum_{n\neq 0}\frac{1}{n^2}+\lim_{R\to\infty}\sum_{\substack{-R\leqslant m\leqslant R\\ m\neq 0}}\sum_{|n|\leqslant\sqrt{R^2-m^2}}\frac{1}{(n+m\tau)^2},$$

we have

$$\begin{aligned}
&\lim_{R\to\infty}\sum_{1\leqslant n^2+m^2\leqslant R^2}\frac{1}{(n+m\tau)^2}-\sum_{n\neq 0}\frac{1}{n^2}-\lim_{R\to\infty}\sum_{\substack{-R\leqslant m\leqslant R\\ m\neq 0}}\sum_{|n|\leqslant\sqrt{R^2-m^2}}\left(\frac{1}{n+m\tau}-\frac{1}{n+1+m\tau}\right)\\
&=\lim_{R\to\infty}\sum_{\substack{-R\leqslant m\leqslant R\\ m\neq 0}}\sum_{|n|\leqslant\sqrt{R^2-m^2}}\left(\frac{1}{(n+m\tau)^2}-\frac{1}{n+m\tau}-\frac{1}{n+1+m\tau}\right)\\
&=\lim_{R\to\infty}\sum_{\substack{-R\leqslant m\leqslant R\\ m\neq 0}}\sum_{|n|\leqslant\sqrt{R^2-m^2}}\frac{1}{(n+m\tau)^2(n+1+m\tau)}\\
&=\sum_{m\neq 0}\left(\sum_{n\in\mathbb{Z}}\frac{1}{(n+m\tau)^2(n+1+m\tau)}\right),
\end{aligned}$$

where in the last equality we have appealed to absolute convergence to justify rearranging the series. From the above we see that

$$\lim_{R\to\infty}\sum_{1\leqslant n^2+m^2\leqslant R^2}\frac{1}{(n+m\tau)^2}=F(\tau)+\lim_{R\to\infty}\sum_{1\leqslant n^2+m^2\leqslant R^2}\left(\frac{1}{n+m\tau}-\frac{1}{n+1+m\tau}\right).$$

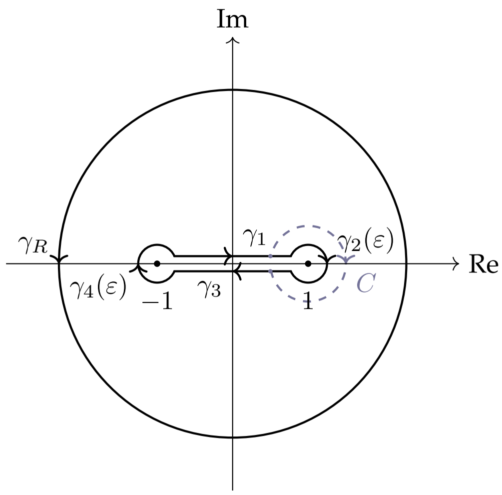
Now we shall focus on the last series and note that for $f(x)=\sqrt{1-x^2}$, we have

$$
\begin{aligned}
&\lim_{R\to\infty}\sum_{1\leqslant n^2+m^2\leqslant R^2}\left(\frac{1}{n+m\tau}-\frac{1}{n+1+m\tau}\right)\\
=&\lim_{R\to\infty}\sum_{\substack{-R\leqslant m\leqslant R\\ m\neq 0}}\sum_{-\sqrt{R^2-m^2}\leqslant n\leqslant\sqrt{R^2-m^2}}\left(\frac{1}{n+m\tau}-\frac{1}{n+1+m\tau}\right)\\
=&\lim_{R\to\infty}\sum_{\substack{-R\leqslant m\leqslant R\\ m\neq 0}}\left(\frac{1}{m\tau-\lfloor Rf(\frac{m}{R})\rfloor}-\frac{1}{m\tau+\lfloor Rf(\frac{m}{R})\rfloor}+\frac{1}{m\tau+\lfloor Rf(\frac{m}{R})\rfloor}-\frac{1}{m\tau+\lfloor Rf(\frac{m}{R})\rfloor+1}\right)\\
=&2\lim_{R\to\infty}\sum_{\substack{-R\leqslant m\leqslant R\\ m\neq 0}}\left(\frac{2\lfloor Rf(\frac{m}{R})\rfloor}{m^2\tau^2-\lfloor Rf(\frac{m}{R})\rfloor^2}+\frac{1}{(m\tau+\lfloor Rf(\frac{m}{R})\rfloor)(m\tau+\lfloor Rf(\frac{m}{R})\rfloor+1)}\right)\\
=&2\lim_{R\to\infty}\sum_{\substack{-R\leqslant m\leqslant R\\ m\neq 0}}\frac{2\lfloor Rf(\frac{m}{R})\rfloor}{m^2\tau^2-\lfloor Rf(\frac{m}{R})\rfloor^2}\\
=&4\lim_{R\to\infty}\frac{1}{R}\sum_{1\leqslant m\leqslant R}\frac{R^{-1}\lfloor Rf(\frac{m}{R})\rfloor}{R^{-2}m^2\tau^2-R^{-2}\lfloor Rf(\frac{m}{R})\rfloor^2}.
\end{aligned}
$$

Since $R^{-1}\lfloor Rf(\frac{m}{R})\rfloor\sim f(\frac{m}{R})$ and $R^{-2}\lfloor Rf(\frac{m}{R})\rfloor^2\sim f^2(\frac{m}{R})$ as $R\to\infty$, the above limit is the same as the limit of the Riemann sum

$$
4\lim_{R\to\infty}\frac{1}{R}\sum_{1\leqslant m\leqslant R}\frac{f(\frac{m}{R})}{\tau^2(\frac{m}{R})^2-f^2(\frac{m}{R})}=4\int_0^1\frac{f(x)}{\tau^2x^2-f^2(x)}\,\mathrm{d}x=2\int_{-1}^1\frac{\sqrt{1-x^2}}{(1+\tau^2)x^2-1}\,\mathrm{d}x.
$$

To evaluate the last integral, we define $F(z)=\dfrac{\sqrt{1-z^2}}{(1+\tau^2)z^2-1}$ and consider its integral along the contour $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4\cup\gamma_R$ as shown below. Note that we can choose a single-valued analytic branch for $F(z)$ in $\mathbb{C}\setminus[-1,1]$.



It is obvious that the integrals along γ_2 and γ_4 vanish as $\varepsilon\to 0^+$. For $x_0\in(-1,1)$, if we denote x_0^{U} the point close to x_0 in the upper half-plane and x_0^{L} the point close to x_0 in the lower half-plane, then

$$\begin{aligned}\operatorname{Arg} F(z)\big|_{z=x_0^{\mathrm{L}}} &= \operatorname{Arg} F(z)\big|_{z=x_0^{\mathrm{U}}} + \Delta_C \operatorname{Arg} F(z) = 0 + \frac{1}{2}[\Delta_C \operatorname{Arg}(z+1) + \Delta_C \operatorname{Arg}(z-1)] \\ &= \frac{1}{2}(0-2\pi) = -\pi,\end{aligned}$$

and then

$$\lim_{\varepsilon\to 0^+} \int_{\gamma_3} F(z)\,\mathrm{d}z = \int_1^{-1} \mathrm{e}^{-\pi\mathrm{i}} F(x)\,\mathrm{d}x = \int_{-1}^{1} F(x)\,\mathrm{d}x.$$

Next we consider the integral along γ_R (the circle with radius R). Since

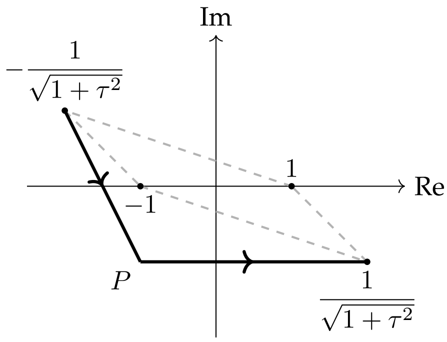
$$\operatorname{Arg}\big(1-z^2\big)\big|_{z=R\mathrm{e}^{\mathrm{i}\theta}} \xrightarrow{R\to\infty} \operatorname{Arg}\big(-R^2\mathrm{e}^{2\mathrm{i}\theta}\big) = 2\theta - \pi,$$

the integral of $F(z)$ along γ_R as $R\to\infty$ becomes

$$\int_0^{2\pi} \frac{R\mathrm{e}^{\mathrm{i}\frac{2\theta-\pi}{2}}}{(1+\tau^2)R^2\mathrm{e}^{2\mathrm{i}\theta}-1} R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \to \int_0^{2\pi} \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\left(2\theta-\frac{\pi}{2}\right)}}{(1+\tau^2)\mathrm{e}^{2\mathrm{i}\theta}}\,\mathrm{d}\theta = \frac{2\pi}{1+\tau^2}.$$

Therefore, by the residue theorem we have

$$2\int_{-1}^{1} \frac{\sqrt{1-x^2}}{(1+\tau^2)x^2-1}\,\mathrm{d}x + \frac{2\pi}{1+\tau^2} = 2\pi\mathrm{i}\left[\operatorname{Res}\left(F, \frac{1}{\sqrt{1+\tau^2}}\right) + \operatorname{Res}\left(F, -\frac{1}{\sqrt{1+\tau^2}}\right)\right]. \qquad (9.4.1\text{–}1)$$



$$\Delta_P \operatorname{Arg}\big(1-z^2\big) = \Delta_P \operatorname{Arg}(1+z) + \Delta_P \operatorname{Arg}(1-z) = 2\pi$$

These two residues at the simple poles of $F(z)$ add up to

$$\begin{aligned}&\lim_{z\to\frac{1}{\sqrt{1+\tau^2}}} \frac{\sqrt{1-z^2}}{(1+\tau^2)\left(z+\frac{1}{\sqrt{1+\tau^2}}\right)} + \lim_{z\to-\frac{1}{\sqrt{1+\tau^2}}} \frac{\sqrt{1-z^2}}{(1+\tau^2)\left(z-\frac{1}{\sqrt{1+\tau^2}}\right)} \\ =&\frac{1}{2\sqrt{1+\tau^2}}\sqrt{1-z^2}\Bigg|_{-\frac{1}{\sqrt{1+\tau^2}}}^{\frac{1}{\sqrt{1+\tau^2}}} \\ =&\frac{1}{2\sqrt{1+\tau^2}}\left|\frac{\tau^2}{1+\tau^2}\right|^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}}{2}\operatorname{Arg}\frac{\tau^2}{1+\tau^2}}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\Delta_P \operatorname{Arg}\left(1-z^2\right)} - 1\right) \\ =&-\frac{1}{\sqrt{1+\tau^2}}\left|\frac{\tau^2}{1+\tau^2}\right|^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}}{2}\operatorname{Arg}\frac{\tau^2}{1+\tau^2}}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\sqrt{1+\tau^2}}\left(\frac{\tau^2}{1+\tau^2}\right)^{\frac{1}{2}} \\
&= -\frac{\tau}{1+\tau^2}.
\end{aligned}
$$

Substituting this into (9.4.1–1) gives

$$
2\int_{-1}^{1}\frac{\sqrt{1-x^2}}{(1+\tau^2)x^2-1}\,\mathrm{d}x = 2\pi\mathrm{i}\left(-\frac{\tau}{1+\tau^2}\right) - \frac{2\pi}{1+\tau^2} = \frac{-2\pi\mathrm{i}}{\tau+\mathrm{i}}.
$$

Hence we conclude that

$$
\lim_{R\to\infty}\sum_{1\leqslant n^2+m^2\leqslant R^2}\frac{1}{(n+m\tau)^2} = F(\tau) - \frac{2\pi\mathrm{i}}{\tau+\mathrm{i}}.
$$

(2) $S_2(z) - \wp(z) = \sum_m\left(\sum_n \frac{1}{(m+n\tau)^2}\right) = F(\tau).$

(3) $S_3(z) - \wp(z) = \sum_n\left(\sum_m \frac{1}{(m+n\tau)^2}\right) = \widetilde{F}(\tau).$ □

Stein 9.4.2 Show that

$$
\wp(z) = c + \pi^2\sum_{m=-\infty}^{\infty}\frac{1}{\sin^2[(z+m\tau)\pi]}
$$

where c is an appropriate constant. In fact, by part (2) of the previous problem $c = -F(\tau)$.

Proof By Problem 9.4.1 and Exercise 4.4.7 we have

$$
\begin{aligned}
\wp(z) = S_2(z) - F(\tau) &= \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{1}{(z+n+m\tau)^2}\right) - F(\tau) \\
&= \sum_{m=-\infty}^{\infty}\frac{\pi^2}{\sin^2[(z+m\tau)\pi]} - F(\tau).
\end{aligned}
$$

□

Stein 9.4.3 Suppose Ω is a simply connected domain that excludes the three roots of the polynomial $4z^3 - g_2 z - g_3$. For $\omega_0 \in \Omega$ and ω_0 fixed, define the function I on Ω by

$$
I(\omega) = \int_{\omega_0}^{\omega}\frac{\mathrm{d}z}{\sqrt{4z^3 - g_2 z - g_3}}, \quad \omega \in \Omega.
$$

Then the function I has an inverse given by $\wp(z+\alpha)$ for some constant α; that is,

$$
I(\wp(z+\alpha)) = z
$$

for appropriate α.

Proof By Corollary 2.3 in Chapter 9 we have

$$
(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{9.4.3–1}
$$

where $g_2 = 60E_4$ and $g_3 = 140E_6$. Now consider the complex cubic curve C defined by the equation

$$y^2 = 4x^3 - g_2 x - g_3, \quad x, y \in \mathbb{C}.$$

By (9.4.3–1), any point $(x, y) \in C$ can be written as $(\wp(u), \wp'(u))$ for some u. Recall that $\wp(z) - \wp(u)$ has a single zero of order 2 if u is a half-period, and two distinct zeros at u and $-u$ otherwise, and that $\wp'$ is an odd function with zeros at the half-periods. Hence in both cases there is a unique u in the fundamental parallelogram that corresponds to the given (x, y). Note that

$$I(\omega) = \int_{\omega_0}^{\omega} \frac{\mathrm{d}x}{\sqrt{4z^3 - g_2 z - g_3}} = \int_{\wp^{-1}(\omega_0)}^{\wp^{-1}(\omega)} \frac{\wp'(u)\,\mathrm{d}u}{\sqrt{4\wp^3(u) - g_2\wp(u) - g_3}} = \int_{\wp^{-1}(\omega_0)}^{\wp^{-1}(\omega)} \frac{\wp'(u)\,\mathrm{d}u}{\sqrt{[\wp'(u)]^2}},$$

hence a suitable substitution $x = \wp(u)$ such that $\sqrt{[\wp'(u)]^2} = \wp'(u)$ gives

$$I(\wp(u)) = u - \alpha$$

for some constant α. Therefore, $I(\wp(z + \alpha)) = z$. □

Stein 9.4.4 Suppose τ is purely imaginary, say $\tau = it$ with $t > 0$. Consider the division of the complex plane into congruent rectangles obtained by considering the lines $x = \frac{n}{2}, y = \frac{tm}{2}$ as n and m range over the integers. (An example is the rectangle whose vertices are $0, \frac{1}{2}, \frac{1}{2} + \frac{\tau}{2}$, and $\frac{\tau}{2}$.)

(1) Show that $\wp$ is real-valued on all these lines, and hence on the boundaries of all these rectangles.

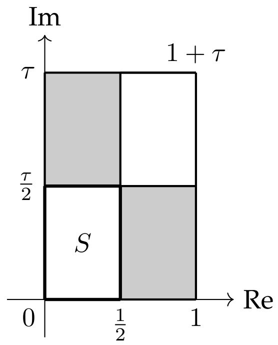
(2) Prove that $\wp$ maps the interior of each rectangle conformally to the upper (or lower) half-plane.

Proof (1) Since Λ is invariant under both negation and complex conjugation, we have

$$\wp(\bar{z}) = \wp(-\bar{z}) = \overline{\wp(z)}.$$

Combining this with the doubly-periodicity of $\wp$ gives the desired result.

(2) Let $S = \left(0, \frac{1}{2}\right) \times \left(0, \frac{\tau}{2}\right)$.



Note that $\wp(z) \sim z^{-2}$ for small z in S, hence

$$\lim_{x \to 0^+} \wp(x) = +\infty \quad \text{and} \quad \lim_{y \to 0^+} \wp(iy) = -\infty.$$

And since $\wp(\partial S) \subset \overline{\mathbb{R}}$ by (1), we obtain $\wp(\partial S) = \overline{\mathbb{R}}$. Now we claim that $\wp(z) \notin \mathbb{R}$ for any $z \in S$. In fact, if $a \in S$ satisfies $\wp(a) \in \mathbb{R}$, then $\wp(-a) \in \mathbb{R}$ and $\wp\left(-a + \frac{1+\tau}{2}\right) \in \mathbb{R}$. Hence $\wp(z) = \wp(a)$

has at least 3 roots in the fundamental parallelogram $[0,1)\times[0,\tau)$, which contradicts with the fact that $\wp$ is an elliptic function of order 2. Since S is a component of the complement of $\wp^{-1}(\overline{\mathbb{R}})$ and contains no poles of $\wp$, it is mapped conformally to $\mathbb{H}$ or $-\mathbb{H}$. In fact, the unshaded rectangles are mapped to $-\mathbb{H}$ since $\operatorname{Im}\wp(z)\sim\operatorname{Im}z^{-2}$ for small z in S, and by $\wp(\bar z)=\overline{\wp(z)}$ we see that the shaded rectangles are mapped to $\mathbb{H}$. Finally, for any $\xi\in\mathbb{H}$, the equation $\wp(z)=\xi$ has a unique solution in S, for $\wp(z)-\xi$ is an elliptic function of order 2 and all those unshaded rectangles have the same image under $\wp$ by $\wp(z)=\wp(-z)$. □

Stein 10.4.1 Prove that

$$\frac{[\Theta'(z\mid\tau)]^2-\Theta(z\mid\tau)\Theta''(z\mid\tau)}{[\Theta(z\mid\tau)]^2}=\wp_\tau\left(z-\tfrac{1}{2}-\tfrac{\tau}{2}\right)+c_\tau,$$

where c_τ can be expressed in terms of the first three derivatives of $\Theta(z\mid\tau)$, with respect to z, at $z=\frac{1}{2}+\frac{\tau}{2}$. Compare this formula with the result in Exercise 9.3.5.

Proof By Corollary 1.5 in Chapter 10, we know that the left-hand side, denoted by $L(z)$, is an elliptic function of order 2 with periods 1 and τ, and with a double pole at $z=z_0:=\frac{1}{2}+\frac{\tau}{2}$. Also note that the coefficient of the double pole $(z-z_0)^{-2}$ is 1, for z_0 is a simple pole of $\Theta(z\mid\tau)$ by Proposition 1.2 (iv) in Chapter 10. Hence $L(z)-\wp_\tau(z-z_0)$ is an entire elliptic function, thus being a constant. This establishes the desired equality. To get c_τ, we take the derivative of both sides with respect to z, and then square both sides. By Theorem 1.7 in Chapter 9 we get

$$\begin{aligned}[L'(z)]^2=[\wp_\tau'(z-z_0)]^2&=4[\wp_\tau(z-z_0)-e_1][\wp_\tau(z-z_0)-e_2][\wp_\tau(z-z_0)-e_3]\\&=4[L(z)-c_\tau-e_1][L(z)-c_\tau-e_2][L(z)-c_\tau-c_3],\end{aligned}$$

where

$$e_1=\wp_\tau\left(\tfrac{1}{2}\right),\quad e_2=\wp_\tau\left(\tfrac{\tau}{2}\right),\quad e_3=\wp_\tau\left(\tfrac{1+\tau}{2}\right).$$

Setting $z=z_0$, all $\Theta(z\mid\tau)$ vanish, thus giving an equation of c_τ and the first three derivatives of $\Theta(z\mid\tau)$ at $z=z_0$, which completes the proof. □

Stein 10.4.2 Consider the Fibonacci numbers $\{F_n\}_{n=0}^\infty$, defined by the two initial values $F_0=0, F_1=1$ and the recursion relation

$$F_n=F_{n-1}+F_{n-2}\quad\text{for } n\geqslant 2.$$

(1) Consider the generating function $F(x)=\displaystyle\sum_{n=0}^\infty F_nx^n$ associated to $\{F_n\}$, and prove that

$$F(x)=x^2F(x)+xF(x)+x$$

for all x in a neighborhood of 0.

(2) Show that the polynomial $q(x)=1-x-x^2$ can be factored as

$$q(x)=(1-\alpha x)(1-\beta x),$$

where α and β are the roots of the polynomial $p(x)=x^2-x-1$.

(3) Expand the expression for F in partial fractions and obtain

$$F(x) = \frac{x}{1 - x - x^2} = \frac{x}{(1 - \alpha x)(1 - \beta x)} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x},$$

where $A = \frac{1}{\alpha - \beta}$ and $B = \frac{1}{\beta - \alpha}$.

(4) Conclude that $F_n = A\alpha^n + B\beta^n$ for $n \geqslant 0$. The two roots of p are actually

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2},$$

so that $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$.

The number $\frac{1}{\alpha} = \frac{\sqrt{5} - 1}{2}$, which is known as the golden mean, satisfies the following property: given a line segment $[AC]$ of unit length (Figure 1), there exists a unique point B on this segment so that the following proportion holds

$$\frac{AC}{AB} = \frac{AB}{BC}.$$

If $\ell = AB$, this reduces to the equation $\ell^2 + \ell - 1 = 0$, whose only positive solution is the golden mean. This ratio arises also in the construction of the regular pentagon. It has played a role in architecture and art, going back to the time of ancient Greece.

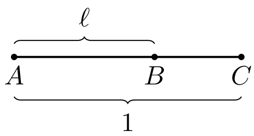


Figure 1: Appearance of the golden mean

Proof (1) Basic induction shows that $F_n \leqslant 2^n$, hence the defining series of $F(x)$ converges for $|x| \leqslant \frac{1}{2}$, and then

$$x^2 F(x) + xF(x) = \sum_{n=2}^{\infty} F_{n-2} x^n + \sum_{n=1}^{\infty} F_{n-1} x^n = \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n = \sum_{n=2}^{\infty} F_n x^n = F(x) - x.$$

(2) Just note that $\alpha\beta = -1$ and $\alpha + \beta = 1$.

(3) Already done.

(4) By (3) we have for all x in a neighborhood of 0

$$F(x) = A \sum_{n=0}^{\infty} (\alpha x)^n + B \sum_{n=0}^{\infty} (\beta x)^n = \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n) x^n.$$

□

Stein 10.4.3 More generally, consider the difference equation given by the initial values u_0 and u_1, and the recurrence relation $u_n = au_{n-1} + bu_{n-2}$ for $n \geqslant 2$. Define the generating function associated to $\{u_n\}_{n=0}^{\infty}$ by $U(x) = \sum_{n=0}^{\infty} u_n x^n$. The recurrence relation implies that $U(x)\bigl(1 - ax - bx^2\bigr) = u_0 + (u_1 - au_0)x$ in a neighborhood of the origin. If α and β denote the roots of the polynomial $p(x) = x^2 - ax - b$, then we may write

$$U(x) = \frac{u_0 + (u_1 - au_0)x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = A\sum_{n=0}^{\infty}\alpha^n x^n + B\sum_{n=0}^{\infty}\beta^n x^n,$$

where it is an easy matter to solve for A and B. Finally, this gives $u_n = A\alpha^n + B\beta^n$. Note that this approach yields a solution to our problem if the roots of p are distinct, namely $\alpha \neq \beta$. A variant of the formula holds if $\alpha = \beta$.

Proof For $\alpha = \beta$ we have

$$\begin{aligned} U(x) &= \frac{u_0 + (u_1 - au_0)x}{(1-\alpha x)^2} = [u_0 + (u_1 - au_0)x]\sum_{n=0}^{\infty}(n+1)\alpha^n x^n \\ &= \sum_{n=0}^{\infty}\alpha^{n-1}[(n+1)u_0\alpha + n(u_1 - au_0)]x^n. \end{aligned}$$

□

Stein 10.4.4 Using the generating formula for $p(n)$, prove the recurrence formula

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) - \cdots \\ &= \sum_{k\neq 0}(-1)^{k+1}p\Big(n - \tfrac{k(3k+1)}{2}\Big), \end{aligned}$$

where the right-hand side is the finite sum taken over those $k \in \mathbb{Z}$, $k \neq 0$, with $\frac{k(3k+1)}{2} \leqslant n$. Use this formula to calculate $p(5), p(6), p(7), p(8), p(9)$, and $p(10)$; check that $p(10) = 42$.

Proof By Theorem 2.1 and Proposition 2.2 in Chapter 10 we have

$$\left(\sum_{n=0}^{\infty}p(n)x^n\right)\left(\sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k+1)}{2}}\right) = \prod_{n=1}^{\infty}\frac{1}{1-x^n}\sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k+1)}{2}} = 1.$$

Therefore, the desired formula follows by comparing the coefficients of x^n on both sides. With this formula and the first five values of $p(n)$, we calculate

$$\begin{aligned} p(5) &= -p(0) + p(3) + p(4) = -1 + 3 + 5 = 7, \\ p(6) &= -p(1) + p(4) + p(5) = -1 + 5 + 7 = 11, \\ p(7) &= -p(0) - p(2) + p(5) + p(6) = -1 - 2 + 7 + 11 = 15, \\ p(8) &= -p(1) - p(3) + p(6) + p(7) = -1 - 3 + 11 + 15 = 22, \\ p(9) &= -p(2) - p(4) + p(7) + p(8) = -2 - 5 + 15 + 22 = 30, \\ p(10) &= -p(3) - p(5) + p(8) + p(9) = -3 - 7 + 22 + 30 = 42. \end{aligned}$$

□

The next two exercises give elementary results related to the assymptotics of the partition function. More refined statements can be found in Appendix A.

Stein 10.4.5 Let

$$F(x) = \sum_{n=0}^{\infty}p(n)x^n = \prod_{n=1}^{\infty}\frac{1}{1-x^n}$$

be the generating function for the partitions. Show that

$$\log F(x) \sim \frac{\pi^2}{6(1-x)} \quad \text{as } x \to 1, \text{ with } 0 < x < 1.$$

Proof Taking the logarithm of the product formula for $F(x)$ gives

$$\log F(x) = \sum_{n=1}^{\infty} \log \frac{1}{1-x^n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} x^{nm} = \sum_{m=1}^{\infty} \frac{1}{m} \frac{x^m}{1-x^m}.$$

By the intermediate value theorem we have

$$\frac{1^m - x^m}{1-x} = m\xi^{m-1} \in \left[mx^{m-1}, m\right] \quad \text{for } 0 < x < 1 \text{ and some } \xi \in (x, 1),$$

hence

$$mx^{m-1}(1-x) \leqslant 1 - x^m \leqslant m(1-x).$$

Then

$$\sum_{m=1}^{\infty} \frac{1}{m^2} \frac{x^m}{1-x} \leqslant \log F(x) \leqslant \sum_{m=1}^{\infty} \frac{1}{m^2} \frac{x}{1-x},$$

thus giving

$$\log F(x) \sim \frac{\pi^2}{6(1-x)} \quad \text{as } x \to 1.$$

□

Stein 10.4.6 Show as a consequence of Exercise 10.4.5 that

$$\mathrm{e}^{c_1 n^{\frac{1}{2}}} \leqslant p(n) \leqslant \mathrm{e}^{c_2 n^{\frac{1}{2}}}$$

for two positive constants c_1 and c_2.

Proof Use Exercise 10.4.5 and take $x = \mathrm{e}^{-y}$ to get

$$\log F\left(\mathrm{e}^{-y}\right) \sim \frac{\pi^2}{6(1-\mathrm{e}^{-y})} \sim \frac{\pi^2}{6y} \quad \text{as } y \to 0.$$

Then

$$F\left(\mathrm{e}^{-y}\right) = \sum_{n=0}^{\infty} p(n)\mathrm{e}^{-ny} \leqslant C_1 \mathrm{e}^{\frac{C_2}{y}} \tag{10.4.6–1}$$

and

$$F\left(\mathrm{e}^{-y}\right) = \sum_{n=0}^{\infty} p(n)\mathrm{e}^{-ny} \geqslant C_3 \mathrm{e}^{\frac{C_4}{y}} \tag{10.4.6–2}$$

for some positive constants C_1, C_2, C_3, and C_4. Using (10.4.6–1) we get

$$p(n)\mathrm{e}^{-ny} \leqslant C_1 \mathrm{e}^{\frac{C_2}{y}},$$

and taking $y = n^{-\frac{1}{2}}$ yields

$$p(n) \leqslant C_1 \mathrm{e}^{(1+C_2)n^{\frac{1}{2}}} \leqslant \mathrm{e}^{c_2 n^{\frac{1}{2}}}.$$

For the opposite direction, (10.4.6–2) gives

$$\sum_{n=0}^{m} p(n)\mathrm{e}^{-ny} \geqslant C_3\mathrm{e}^{\frac{C_4}{y}} - \sum_{n=m+1}^{\infty} p(n)\mathrm{e}^{-ny} \geqslant C_3\mathrm{e}^{\frac{C_4}{y}} - \sum_{n=m+1}^{\infty} \mathrm{e}^{c_2 n^{\frac{1}{2}}}\mathrm{e}^{-ny}.$$

Taking $y = Am^{-\frac{1}{2}}$ and using the fact that the sequence $p(m)$ is increasing,

$$(m+1)p(m) \geqslant \sum_{n=0}^{m} p(n)\mathrm{e}^{-n\frac{A}{\sqrt{m}}} \geqslant C_3\mathrm{e}^{\frac{C_4\sqrt{m}}{A}} - \sum_{n=m+1}^{\infty} \mathrm{e}^{c_2 n^{\frac{1}{2}}}\mathrm{e}^{-n\frac{A}{\sqrt{m}}},$$

and then by choosing A to be sufficiently large we get

$$p(m) \geqslant \mathrm{e}^{c_1 n^{\frac{1}{2}}}$$

for some positive constant c_1. □

Stein 10.4.7 Use the product formula for Θ to prove:

(1) The "triangular number" identity

$$\prod_{n=0}^{\infty}(1+x^n)\left(1-x^{2n+2}\right) = \sum_{n=-\infty}^{\infty} x^{\frac{n(n+1)}{2}},$$

which holds for $|x| < 1$.

(2) The "septagonal number" identity

$$\prod_{n=0}^{\infty}\left(1-x^{5n+1}\right)\left(1-x^{5n+4}\right)\left(1-x^{5n+5}\right) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+3)}{2}},$$

which holds for $|x| < 1$.

Proof The product formula for Θ is

$$\Theta(z \mid \tau) := \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi \mathrm{i} n^2\tau}\mathrm{e}^{2\pi \mathrm{i} nz} = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+q^{2n-1}\mathrm{e}^{2\pi \mathrm{i}z}\right)\left(1+q^{2n-1}\mathrm{e}^{-2\pi \mathrm{i}z}\right),$$

where $q = \mathrm{e}^{\pi \mathrm{i}\tau}$.

(1) Take $z = \frac{\tau}{2}$, then

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi \mathrm{i}\left(n^2+n\right)\tau} = \sum_{n=-\infty}^{\infty} q^{n^2+n} = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+q^{2n}\right)\left(1+q^{2n-2}\right).$$

Replace q^2 by x to get

$$\sum_{n=-\infty}^{\infty} x^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty}(1-x^n)(1+x^n)\left(1+x^{n-1}\right) = \prod_{n=0}^{\infty}(1+x^n)\left(1-x^{2n+2}\right).$$

(2) Take $z = \frac{3\tau}{10} + \frac{1}{2}$, then

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi \mathrm{i} n + \pi \mathrm{i} \tau \left(n^2 + \frac{3n}{5}\right)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2 + \frac{3n}{5}} = \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\left(1 - q^{2n - \frac{2}{5}}\right)\left(1 - q^{2n - \frac{8}{5}}\right).$$

Replace $q^{\frac{2}{5}}$ by x to get

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+3)}{2}} = \prod_{n=1}^{\infty} \left(1 - x^{5n}\right)\left(1 - x^{5n-1}\right)\left(1 - x^{5n-4}\right)$$
$$= \prod_{n=0}^{\infty} \left(1 - x^{5n+1}\right)\left(1 - x^{5n+4}\right)\left(1 - x^{5n+5}\right). \qquad \square$$

Stein 10.4.8 Consider Pythagorean triples (a, b, c) with $a^2 + b^2 = c^2$, and with $a, b, c \in \mathbb{Z}$. Suppose moreover that a and b have no common factors.

(1) Show that either a or b must be odd, and the other even.

(2) Show in this case (assuming a is odd and b even) that there are integers m, n so that $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$.

(3) Conversely, show that whenever c is a sum of two-squares, then there exist integers a and b such that $a^2 + b^2 = c^2$.

Proof (1) If both a and b are even, then c is even, and then a and b have a common factor 2. If both a and b are odd, then $4 \nmid c^2$, which is impossible since c must be even.

(2) From (1) we have $2 \nmid a + b$, and then $2 \nmid a$ and $2 \nmid c$ since b is even. Therefore, we can write

$$\left(\frac{b}{2}\right)^2 = \frac{c+a}{2} \cdot \frac{c-a}{2}. \qquad (10.4.8\text{–}1)$$

Also note that

$$\gcd\left(\frac{c+a}{2}, \frac{c-a}{2}\right) \mid a, \quad \gcd\left(\frac{c+a}{2}, \frac{c-a}{2}\right) \mid c,$$

this fact, together with our assumption that $\gcd(a, c) = 1$, gives

$$\gcd\left(\frac{c+a}{2}, \frac{c-a}{2}\right) = 1. \qquad (10.4.8\text{–}2)$$

Without loss of generality, assume $a, b, c \geqslant 0$. With (10.4.8–1) and (10.4.8–2) we conclude that

$$\frac{c+a}{2} = m^2 \quad \text{and} \quad \frac{c-a}{2} = n^2$$

for some positive integers m and n with $m > n$. This leads to the desired result.

(3) If $c = m^2 + n^2$ for some integers m and n, then $a = m^2 - n^2$ and $b = 2mn$ satisfy $a^2 + b^2 = c^2$. $\square$

Stein 10.4.9 Use the formula for $r_2(n)$ to prove the following:

(1) If $n = p$, where p is a prime of the form $4k + 1$, then $r_2(n) = 8$. This implies that n can be written in a unique way as $n = n_1^2 + n_2^2$, except for the signs and reordering of n_1 and n_2.

(2) If $n = q^a$, where q is a prime of the form $4k+3$ and a is a positive integer, then $r_2(n) > 0$ if and only if a is even.

(3) In general, n can be represented as the sum of two squares if and only if all the primes of the form $4k+3$ that arise in the prime decomposition of n occur with even exponents.

Proof The formula for $r_2(n)$ is

$$r_2(n) = 4[d_1(n) - d_3(n)]$$

by Theorem 3.1 in Chapter 10.

(1) $r_2(p) = 4[d_1(p) - d_3(p)] = 4 \times (2 - 0) = 8$.

(2) Observe that

$$d_1(q^a) = \#\left\{q^0, q^2, \cdots, q^{2\lfloor \frac{a}{2} \rfloor}\right\} = \left\lfloor \frac{a}{2} \right\rfloor + 1,$$
$$d_3(q^a) = \#\left\{q^1, q^3, \cdots, q^{2\lfloor \frac{a-1}{2} \rfloor + 1}\right\} = \left\lfloor \frac{a-1}{2} \right\rfloor + 1.$$

Hence

$$r_2(q^a) > 0 \iff d_1(q^a) > d_2(q^a) \iff \left\lfloor \frac{a}{2} \right\rfloor > \left\lfloor \frac{a-1}{2} \right\rfloor \iff 2 \mid a.$$

(3) Note that for positive integers a and b that are coprime, we have

$$d_1(ab) = d_1(a)d_1(b) + d_3(a)d_3(b) \quad \text{and} \quad d_3(ab) = d_1(a)d_3(b) + d_3(a)d_1(b).$$

Hence the function $d_1(n) - d_3(n)$ is a multiplicative function:

$$d_1(ab) - d_3(ab) = [d_1(a) - d_3(a)][d_1(b) - d_3(b)].$$

Since

$$d_1(2^t) - d_3(2^t) = 1 \quad \text{for all } t \geqslant 1$$

and

$$d_1(p^b) - d_3(p^b) = d_1(p^b) = b + 1 > 0 \quad \text{for all } b \geqslant 1 \tag{10.4.9–1}$$

where p is a prime of the form $4k+1$, we conclude that $r_2(n) > 0$ if and only if

$$d_1(q^a) - d_3(q^a) > 0$$

for any prime factor q of n of the form $4k+3$ and a the exponent of q in the prime decomposition of n. Therefore, with (2) we see that this is equivalent to $2 \mid a$, which completes the proof. □

Stein 10.4.10 Observe the following irregularities of the functions $r_2(n)$ and $r_4(n)$ as n becomes large:

(1) $r_2(n) = 0$ for infinitely many n, while $\limsup\limits_{n\to\infty} r_2(n) = \infty$.

(2) $r_4(n) = 24$ for infinitely many n, while $\limsup\limits_{n\to\infty} \dfrac{r_4(n)}{n} = \infty$.

Proof (1) With Exercise 10.4.9, we know that $r_2(q) = 0$ whenever q is a prime of the form $4k+3$. Then recall that there are infinitely many primes of the form $4k+3$, which is a direct consequence of Problem 7.4.4. With (10.4.9–1) we have $r_2(p^b) = 4(b+1)$ for any prime p of the form $4k+1$ and $b \geqslant 1$, which tends to infinity as $b \to \infty$.

(2) Note that $r_4(2^k) = 8\sigma_1^*(2^k) = 8 \times (1+2) = 24$ for $k \geqslant 1$, by Theorem 3.6 in Chapter 10, where $\sigma_1^*(n)$ equals the sum of divisors of n that are not divisible by 4. Let $a_n = (2n-1)!!$, then

$$\frac{r_4(a_n)}{a_n} = \frac{8\sigma^*(a_n)}{a_n} \geqslant 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} = \sum_{k=1}^{n} \frac{1}{2k-1},$$

which tends to infinity as $n \to \infty$. □

Stein 10.4.11 Recall from Problem 2 in Chapter 2, that

$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}, \quad |z| < 1$$

where $d(n)$ denotes the number of divisors of n.

More generally, show that

$$\sum_{n=1}^{\infty} \sigma_\ell(n) z^n = \sum_{n=1}^{\infty} \frac{n^\ell z^n}{1-z^n}, \quad |z| < 1$$

where $\sigma_\ell(n)$ is the sum of the ℓ-th powers of divisors of n.

Proof The fisrt identity is a special case of the second one, since $\sigma_0(n) = d(n)$. For $|z| < 1$,

$$\sum_{n=1}^{\infty} \frac{n^\ell z^n}{1-z^n} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n^\ell z^{n(1+m)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n^\ell z^{nm} = \sum_{n=1}^{\infty} \sigma_\ell(n) z^n.$$

□

Stein 10.4.12 Here we give another identity involving θ^4, which is equivalent to the four-squares theorem.

(1) Show that for $|q| < 1$

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}.$$

(2) Show as a result that

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \sum_{n=1}^{\infty} \frac{4nq^{4n}}{1-q^{4n}} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - 4\sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2} = \sum_{n=1}^{\infty} \sigma_1^*(n) q^n$$

where $\sigma_1^*(n)$ is the sum of the divisors of n that are not divisible by 4.

(3) Show that the four-squares theorem is equivalent to the identity

$$\theta(\tau)^4 = 1 + 8\sum_{n=1}^{\infty} \frac{q^n}{[1+(-1)^n q^n]^2}, \quad q = e^{\pi i \tau}.$$

Proof (1) Using $\dfrac{1}{(1-x)^2} = \sum_{m=1}^{\infty} m x^{m-1}$ for $|x| < 1$ we get

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty} q^n \sum_{m=1}^{\infty} m(q^n)^{m-1} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m q^{nm} = \sum_{m=1}^{\infty} \frac{mq^m}{1-q^m}.$$

(2) Using $\sigma^*(n) = \sigma_1(n) - 4\sigma_1\left(\frac{n}{4}\right)$ we have

$$\sum_{n=1}^{\infty} \sigma_1^*(n) q^n = \sum_{m=1}^{\infty} m \sum_{k=1}^{\infty} q^{km} - \sum_{m=1}^{\infty} 4m \sum_{k=1}^{\infty} q^{4km} = \sum_{m=1}^{\infty} \frac{mq^m}{1-q^m} - \sum_{m=1}^{\infty} \frac{4mq^{4m}}{1-q^{4m}}.$$

(3) Since

$$\theta(\tau)^4 = \left(\sum_{n=-\infty}^{\infty} q^{n_1^2}\right)\left(\sum_{n=-\infty}^{\infty} q^{n_2^2}\right)\left(\sum_{n=-\infty}^{\infty} q^{n_3^2}\right)\left(\sum_{n=-\infty}^{\infty} q^{n_4^2}\right) = \sum_{n=0}^{\infty} r_4(n) q^n,$$

the four-squares theorem, namely $r_4(n) = 8\sigma_1^*(n)$ for all $n \geqslant 1$, is equivalent to

$$\theta(\tau)^4 = 1 + 8\sum_{n=1}^{\infty} \sigma_1^*(n) q^n.$$

By (2) this can be reduced to showing

$$\theta(\tau)^4 = 1 + 8\left(\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - 4\sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2}\right).$$

Then from

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - 4\sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2} &= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - \sum_{n=1}^{\infty}\left(\frac{q^{2n}}{(1-q^{2n})^2} - \frac{q^{2n}}{(1+q^{2n})^2}\right) \\ &= \sum_{n=1}^{\infty} \frac{q^n}{[1+(-1)^n q^n]^2}\end{aligned}$$

we prove the desired equivalence. □

Stein 10.5.1 Suppose n is of the form $n = 4^a(8k+7)$, where a and k are positive integers. Show that n cannot be written as the sum of three-squares. The converse, that every n that is not of that form can be written as the sum of three-squares, is a difficult theorem of Legendre and Gauss.

Proof If $4^a(8k+7) = p^2 + q^2 + r^2$, then p, q and r must be all even, and then by dividing by 4

$$4^{a-1}(8k+7) = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + \left(\frac{r}{2}\right)^2.$$

Repeating this finally reduces the problem to the case $n = 8k+7$. Note that every square is congruent to 0, 1, or 4 modulo 8, therefore the sum of three squares cannot be congruent to 7 modulo 8, which completes the proof. □

Stein 10.5.2 Let $\mathrm{SL}_2(\mathbb{Z})$ denote the set of 2×2 matrices with integer entries and determinant 1, that is,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \text{ and } ad-bc = 1 \right\}.$$

This group acts on the upper half-plane by the fractional linear transformation $g(\tau) = \frac{a\tau+b}{c\tau+d}$. Together with this action comes the so-called fundamental domain $\mathcal{F}_1$ in the complex plane defined by

$$\mathcal{F}_1 = \left\{ \tau \in \mathbb{C} : |\tau| \geqslant 1,\ |\mathrm{Re}(\tau)| \leqslant \tfrac{1}{2} \text{ and } \mathrm{Im}(\tau) \geqslant 0 \right\}.$$

It is illustrated in Figure 2.

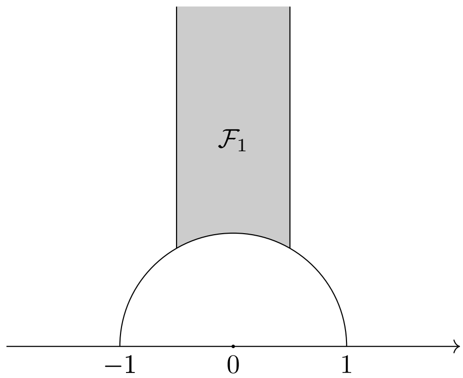


Figure 2: The fundamental domain $\mathcal{F}_1$

Consider the two elements in $\mathrm{SL}_2(\mathbb{Z})$ defined by $S(\tau) = -\frac{1}{\tau}$ and $T_1(\tau) = \tau + 1$. These correspond (for example) to the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

respectively. Let $\mathfrak{g}$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T_1.

(1) Show that for every $\tau \in \mathbb{H}$ there exists $g \in \mathfrak{g}$ such that $g(\tau) \in \mathcal{F}_1$.

(2) We say that two points τ and τ' are congruent if there exists $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $g(\tau) = \tau'$. Prove that if $\tau, \tau' \in \mathcal{F}_1$ are congruent, then either $\mathrm{Re}(\tau) = \pm\frac{1}{2}$ and $\tau' = \tau \mp 1$ or $|\tau| = 1$ and $\tau' = -\frac{1}{\tau}$.

(3) Prove that S and T_1 generate the modular group in the sense that every fractional linear transformation corresponding to $g \in \mathrm{SL}_2(\mathbb{Z})$ is a composition of finitely many S's and T_1's, and their inverses. Strictly speaking, the matrices associated to S and T_1 generate the projective special linear group $\mathrm{PSL}_2(\mathbb{Z})$, which equals $\mathrm{SL}_2(\mathbb{Z})$ modulo $\pm I$.

Proof (1) For $\tau \in \mathbb{H}$ and $g(\tau) = \frac{a\tau+b}{c\tau+d} \in \mathfrak{g}$ we have

$$\mathrm{Im}(g(\tau)) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}. \tag{10.5.2–1}$$

Since c and d are integers, we may choose a $g_0 \in \mathfrak{g}$ such that $\mathrm{Im}(g_0(\tau))$ is maximal. Since the translations T_1 and their inverses do not change imaginary parts, we may apply finitely many of them to see that there exists $g_1 \in G$ with $|\mathrm{Re}(g_1(\tau))| \leqslant \frac{1}{2}$ and $\mathrm{Im}(g_1(\tau))$ is maximal. It now suffices to prove that $|g_1(\tau)| \geqslant 1$ to conclude that $g_1(\tau) \in \mathcal{F}_1$. If this were not true, that is, $|g_1(\tau)| < 1$, then

$$\mathrm{Im}(Sg_1(\tau)) = \mathrm{Im}\left(-\frac{1}{g_1(\tau)}\right) = -\frac{\mathrm{Im}\left(\overline{g_1(\tau)}\right)}{|g_1(\tau)|^2} > \mathrm{Im}(g_1(\tau)),$$

and this contradicts the maximality of $\mathrm{Im}(g_1(\tau))$.

(2) Say $\tau' = g(\tau)$ for $\tau, \tau' \in \mathbb{H}$ and $g(\tau) = \frac{a\tau+b}{c\tau+d} \in \mathrm{SL}_2(\mathbb{Z})$. We may assume that $\mathrm{Im}(\tau') \geqslant \mathrm{Im}(\tau)$, for otherwise we can relabel τ and τ'. Hence by (10.5.2–1) we have $|c\tau + d| \leqslant 1$, and since $\tau \in \mathcal{F}_1$, this implies that $|c| \leqslant 1$.

- If $c = 0$, then $ad = 1$ and $g(\tau) = \tau \pm b$. Since $\tau, \tau' \in \mathcal{F}_1$, we get $g(\tau) = \tau \pm 1$, and then $\mathrm{Re}(\tau) = \pm\frac{1}{2}$ and $\tau' = \tau \mp 1$.
- If $c = \pm 1$, then $|c\tau + d| = |\tau \pm d| \leqslant 1$.
  - If $d = 0$, then $b = \mp 1$ and $g(\tau) = -\frac{1}{\tau} \pm a$. Hence τ must be at one of the cusps.
  - If $d \neq 0$, then $|\tau + 1| \leqslant 1$ or $|\tau - 1| \leqslant 1$, and again τ must be at one of the cusps.

(3) For any $g \in \mathrm{SL}_2(\mathbb{Z})$, since $2\mathrm{i} \in \mathcal{F}_1 \subset \mathbb{H}$, by (10.5.2–1) we have $g(2\mathrm{i}) \in \mathbb{H}$. Then by (1), there exists $h \in \mathfrak{g}$ such that $h(g(2\mathrm{i})) \in \mathcal{F}_1$. Now both $2\mathrm{i}$ and $h(g(2\mathrm{i}))$ are in the interior of $\mathcal{F}_1$.

- If $h(g(2\mathrm{i})) \neq 2\mathrm{i}$, by (2) there exists $\ell \in \mathfrak{g}$ such that $\ell \circ h \circ g(2\mathrm{i}) = 2\mathrm{i}$. Note that if

  $$\frac{a(2\mathrm{i}) + b}{c(2\mathrm{i}) + d} = 2\mathrm{i} \quad \text{for } a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1$$

  then $a = d = \pm 1$ and $b = c = 0$, which means that $\ell \circ h \circ g$ is the identity. Hence $g = h^{-1} \circ \ell^{-1}$.
- If $h(g(2\mathrm{i})) = 2\mathrm{i}$, then by the same argument $g = h^{-1}$. □

Stein 10.5.3 In this problem, consider the group G of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries, determinant 1, and such that a and d have the same parity, b and c have the same parity, and c and d have opposite parity. This group also acts on the upper half-plane by fractional linear transformations. To the group G corresponds the fundamental domain $\mathcal{F}$ defined by $|\tau| \geqslant 1$, $|\mathrm{Re}(\tau)| \leqslant 1$, and $\mathrm{Im}(\tau) \geqslant 0$ (see Figure 3). Also, let

$$S(\tau) = -\frac{1}{\tau} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T_2(\tau) = \tau + 2 \leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Prove that every fractional linear transformation corresponding to $g \in G$ is a composition of finitely many S, T_2 and their inverses, in analogy with the previous problem.

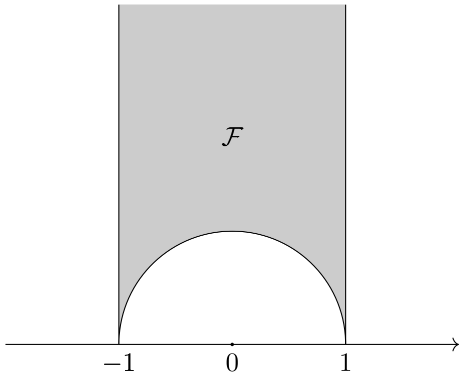


Figure 3: The fundamental domain $\mathcal{F}$

Proof (1) By Lemma 3.5 in Chapter 10, every point in the upper half-plane can be mapped into $\mathcal{F}$ using repeatedly S, T_2 and their inverses.

(2) Suppose $\tau' = g(\tau)$ for $\tau, \tau' \in \mathcal{F}$, $g(\tau) = \frac{a\tau+b}{c\tau+d} \in G$ and $\mathrm{Im}(\tau') \geqslant \mathrm{Im}(\tau)$. Then by (10.5.2–1) we have $|c\tau + d| \leqslant 1$, and since $\tau \in \mathcal{F}$, this implies that $|c| \leqslant 1$.

- ◇ If $c = 0$, then $ad - bc = ad = 1$ and $g(\tau) = \tau \pm b$. Note that b and d have opposite parity, hence b is even and $g = T_2^{\pm\frac{b}{2}}$.
- ◇ If $c = 1$, then by $|c\tau + d| \leqslant 1$ we find $|d| \leqslant 2$. Also note that c and d have opposite parity, hence $d = -2, 0, 2$.
  - If $d = -2$, then $ad - bc = -2a - b = 1$, and $g(\tau) = a - \frac{1}{\tau-2}$. Note that a and d have the same parity, hence $g = T_2^{\frac{a}{2}} \circ S \circ T_2^{-1}$.
  - If $d = 0$, then $ad - bc = -b = 1$, and $g(\tau) = a - \frac{1}{\tau}$. Note that a and d have the same parity, hence $g = T_2^{\frac{a}{2}} \circ S$.
  - If $d = 2$, then $ad - bc = 2a - b = 1$, and $g(\tau) = a - \frac{1}{\tau+2}$. Note that a and d have the same parity, hence $g = T_2^{\frac{a}{2}} \circ S \circ T_2$.
- ◇ If $c = -1$, then by $|c\tau + d| \leqslant 1$ we find $|d| \leqslant 2$. Also note that c and d have opposite parity, hence $d = -2, 0, 2$.
  - If $d = -2$, then $ad - bc = -2a + b = 1$, and $g(\tau) = -a - \frac{1}{\tau+2}$. Note that a and d have the same parity, hence $g = T_2^{-\frac{a}{2}} \circ S \circ T_2$.
  - If $d = 0$, then $ad - bc = b = 1$, and $g(\tau) = -a - \frac{1}{\tau}$. Note that a and d have the same parity, hence $g = T_2^{-\frac{a}{2}} \circ S$.
  - If $d = 2$, then $ad - bc = 2a + b = 1$, and $g(\tau) = -a - \frac{1}{\tau-2}$. Note that a and d have the same parity, hence $g = T_2^{-\frac{a}{2}} \circ S \circ T_2^{-1}$.

(3) For any $g \in G$, since $2\mathrm{i} \in \mathcal{F} \subset \mathbb{H}$, by (10.5.2–1) we have $g(2\mathrm{i}) \in \mathbb{H}$. Then by (1), there exists $h \in \langle S, T_2 \rangle$ such that $h(g(2\mathrm{i})) \in \mathcal{F}$. Now both $2\mathrm{i}$ and $h(g(2\mathrm{i}))$ are in the interior of $\mathcal{F}$. Observe that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix},$$

which implies that $h \circ g \in G$. Therefore by (2) there exists $\ell \in \langle S, T_2 \rangle$ such that $h \circ g = \ell$, and then $g = h^{-1} \circ \ell \in \langle S, T_2 \rangle$. □

Stein 10.5.4 Let G denote the group of matrices given in the previous problem. Here we give an alternate proof of Theorem 3.4, that states that a function in $\mathbb{H}$ which is holomorphic, bounded, and invariant under G must be constant.

(1) Suppose that $f : \mathbb{H} \to \mathbb{C}$ is holomorphic, bounded, and that there exists a sequence of complex numbers $\tau_k = x_k + \mathrm{i}y_k$ such that

$$f(\tau_k) = 0, \quad \sum_{k=1}^{\infty} y_k = \infty, \quad 0 < y_k \leqslant 1, \quad \text{and} \quad |x_k| \leqslant 1.$$

Then $f = 0$.

(2) Given two relatively prime integers c and d with different parity, show that there exist integers a and b such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

(3) Prove that $\sum \dfrac{1}{c^2 + d^2} = \infty$ where the sum is taken over all c and d that are relatively prime and of opposite parity.

(4) Prove that if $F : \mathbb{H} \to \mathbb{C}$ is holomorphic, bounded, and invariant under G, then F is constant.

Proof (1) Recall from Theorem 1.2 of Chapter 8 the conformal map

$$F(\tau) = \frac{\mathrm{i} - \tau}{\mathrm{i} + \tau}$$

which maps $\mathbb{H}$ onto the unit disk. Let $z_k = F(\tau_k)$, then

$$\sum_{k=1}^{\infty}(1 - |z_k|) = \sum_{k=1}^{\infty}\left(1 - \left|\frac{\mathrm{i} - x_k - \mathrm{i}y_k}{\mathrm{i} + x_k + \mathrm{i}y_k}\right|\right) = \sum_{k=1}^{\infty}\left(1 - \sqrt{\frac{x_k^2 + (1 - y_k)^2}{x_k^2 + (1 + y_k)^2}}\right) \geqslant \frac{2}{5}\sum_{k=1}^{\infty} y_k = \infty.$$

Here the inequality follows since for $|x| \leqslant 1$ and $0 < y \leqslant 1$ we have

$$x^2 + (1+y)^2 \leqslant 5 \implies \frac{1}{25}y + \frac{1}{x^2 + (1+y)^2} \geqslant \frac{1}{5} \implies \frac{4}{25}y - \frac{4}{5} \geqslant -\frac{4}{x^2 + (1+y)^2}$$
$$\implies \left(1 - \frac{2}{5}y\right)^2 \geqslant 1 - \frac{4y}{x^2 + (1+y)^2} = \frac{x^2 + (1-y)^2}{x^2 + (1+y)^2} \implies 1 - \sqrt{\frac{x^2 + (1-y)^2}{x^2 + (1+y)^2}} \geqslant \frac{2}{5}y.$$

Applying the result in Problem 5.7.1 to the function $f \circ F^{-1}$, which is holomorphic in the unit disk, bounded and with zeros at $\{z_k\}_{k=1}^{\infty}$, we conclude that $f \circ F^{-1}$ must be identically zero, and hence $f = 0$.

(2) Since c and d are relatively prime, there exist integers x_0 and y_0 such that $dx_0 - cy_0 = 1$. Moreover, all the solutions of $dx - cy = 1$ take the form $x_0 + ct$ and $y_0 + dt$ with $t \in \mathbb{Z}$.

- If c is odd and d is even, then y_0 must be odd, for otherwise $dx_0 - cy_0$ would be even.
  - If x_0 is even, then let $a = x_0$ and $b = y_0$.
  - If x_0 is odd, then let $a = x_0 + c$ and $b = y_0 + d$.
- If c is even and d is odd, then x_0 must be odd, for otherwise $dx_0 - cy_0$ would be even.

- If y_0 is even, then let $a = x_0$ and $b = y_0$.
- If y_0 is odd, then let $a = x_0 + c$ and $b = y_0 + d$.

Therefore, in all cases there exist integers a and b such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

(3) Suppose to the contrary that

$$\sum_{\substack{\gcd(c,d)=1 \\ 2\nmid(c-d)}} \frac{1}{c^2+d^2} = A < \infty,$$

then we claim that

$$\sum_{\gcd(a,b)=1} \frac{1}{a^2+b^2} < \infty.$$

To see this, note that if a and b are both odd and relatively prime, then the two numbers c and d defined by

$$c = \frac{a+b}{2} \quad \text{and} \quad d = \frac{a-b}{2}$$

are relatively prime and of opposite parity. Moreover, since

$$c^2 + d^2 = \frac{a^2+b^2}{2},$$

we see that

$$\sum_{\gcd(a,b)=1} \frac{1}{a^2+b^2} = \sum_{\substack{\gcd(a,b)=1 \\ 2\nmid(a-b)}} \frac{1}{a^2+b^2} + \sum_{\substack{\gcd(a,b)=1 \\ 2\mid(a-b)}} \frac{1}{a^2+b^2} = A + \frac{A}{2} < \infty.$$

However, this would lead to

$$\sum_{(k,\ell)\neq(0,0)} \frac{1}{k^2+\ell^2} = \sum_{n=1}^{\infty} \sum_{\gcd(a,b)=1} \frac{1}{(na)^2+(nb)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{\gcd(a,b)=1} \frac{1}{a^2+b^2} < \infty,$$

which is a contradiction.

(4) Without loss of generality, we may assume that $F(\mathrm{i}) = 0$ and prove $F = 0$, for otherwise we can replace $F(\tau)$ by $F(\tau) - F(\mathrm{i})$. For each relatively prime c and d with opposite parity, by (2) there exist integers a and b such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Then $g(\tau) = \dfrac{a\tau+b}{c\tau+d}$ satisfies

$$g(\mathrm{i}) = \frac{a\mathrm{i}+b}{c\mathrm{i}+d} = \frac{ac+bd}{c^2+d^2} + \mathrm{i}\frac{ad-bc}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{\mathrm{i}}{c^2+d^2},$$

and since

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix},$$

by composing g with finitely many T_2's or T_2^{-1}'s we see that there exists $g_{c,d} \in G$ such that $g_{c,d}(\mathrm{i}) = x_{c,d} + \frac{\mathrm{i}}{c^2+d^2}$ with $|x_{c,d}| \leqslant 1$. Using the fact that F is invariant under G, the conclusion in Problem

10.5.3 and what we proved in (3), we see that

$$F(g_{c,d}(\mathrm{i}))=0,\quad \sum_{c,d}\mathrm{Im}(g_{c,d}(\mathrm{i}))=\infty,\quad 0<\mathrm{Im}(g_{c,d}(\mathrm{i}))\leqslant 1,\quad \text{and}\quad |\mathrm{Re}(g_{c,d}(\mathrm{i}))|\leqslant 1$$

for integers c and d that are relatively prime and of opposite parity. Then by (1) we conclude that $F=0$. □

Stein 10.5.5 In Chapter 9 we proved that the Weierstrass $\wp$ function satisfies the cubic equation

$$(\wp')^2=4\wp^3-g_2\wp-g_3,$$

where $g_2=60E_4$, $g_3=140E_6$, with E_k is the Eisenstein series of order k. The discriminant of the cubic $y^2=4x^3-g_2x-g_3$ is defined by $\Delta=g_2^3-27g_3^2$. Prove that

$$\Delta(\tau)=(2\pi)^{12}\eta^{24}(\tau)\quad \text{for all } \tau\in\mathbb{H}.$$

Proof We begin with determining the Fourier expansions of $g_2(\tau)$ and $g_3(\tau)$. Recall that

$$g_2(\tau)=60\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^4},\quad g_3(\tau)=140\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^6}.$$

By (4) in Chapter 5 we have

$$\begin{aligned}\pi\cot\pi\tau&=\lim_{N\to\infty}\sum_{|n|\leqslant N}\frac{1}{\tau+n}=\lim_{N\to\infty}\left\{\frac{1}{\tau}+\sum_{n=1}^{N}\left[\left(\frac{1}{\tau+n}-\frac{1}{n}\right)+\left(\frac{1}{\tau-n}-\frac{1}{-n}\right)\right]\right\}\\&=\frac{1}{\tau}+\sum_{n\neq0}\left(\frac{1}{\tau+n}-\frac{1}{n}\right).\end{aligned}$$

Let $x=\mathrm{e}^{2\pi\mathrm{i}\tau}$. If $\tau\in\mathbb{H}$ then $|x|<1$ and we find

$$\begin{aligned}\pi\cot\pi\tau&=\pi\frac{\cos\pi\tau}{\sin\pi\tau}=\pi\mathrm{i}\frac{\mathrm{e}^{2\pi\mathrm{i}\tau}+1}{\mathrm{e}^{2\pi\mathrm{i}\tau}-1}=\pi\mathrm{i}\frac{x+1}{x-1}=-\pi\mathrm{i}\left(\frac{x}{1-x}+\frac{1}{1-x}\right)\\&=-\pi\mathrm{i}\left(\sum_{r=1}^{\infty}x^r+\sum_{r=0}^{\infty}x^r\right)=-\pi\mathrm{i}\left(1+2\sum_{r=1}^{\infty}x^r\right).\end{aligned}$$

In other words, if $\tau\in\mathbb{H}$ we have

$$\frac{1}{\tau}+\sum_{n\neq0}\left(\frac{1}{\tau+n}-\frac{1}{n}\right)=-\pi\mathrm{i}\left(1+2\sum_{r=1}^{\infty}\mathrm{e}^{2\pi\mathrm{i}r\tau}\right).$$

Differentiating repeatedly we find

$$\begin{aligned}-\frac{1}{\tau^2}-\sum_{n\neq0}\frac{1}{(\tau+n)^2}&=-(2\pi\mathrm{i})^2\sum_{r=1}^{\infty}r\mathrm{e}^{2\pi\mathrm{i}r\tau},\\-3!\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^4}&=-(2\pi\mathrm{i})^4\sum_{r=1}^{\infty}r^3\mathrm{e}^{2\pi\mathrm{i}r\tau},\end{aligned}$$

$$-5!\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^6}=-(2\pi\mathrm{i})^6\sum_{r=1}^{\infty}r^5\mathrm{e}^{2\pi\mathrm{i}r\tau}.$$

Replacing τ by $m\tau$ we obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{(n+m\tau)^4}&=\frac{8\pi^4}{3}\sum_{r=1}^{\infty}r^3\mathrm{e}^{2\pi\mathrm{i}rm\tau},\\ \sum_{n=-\infty}^{\infty}\frac{1}{(n+m\tau)^6}&=-\frac{8\pi^6}{15}\sum_{r=1}^{\infty}r^5\mathrm{e}^{2\pi\mathrm{i}rm\tau}.\end{aligned}$$

Therefore,

$$\begin{aligned}g_2(\tau)&=60\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^4}\\ &=60\left\{\sum_{n\neq 0}\frac{1}{n^4}+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\left(\frac{1}{(n+m\tau)^4}+\frac{1}{(n-m\tau)^4}\right)\right\}\\ &=60\left\{2\zeta(4)+2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(n+m\tau)^4}\right\}\\ &=60\left\{\frac{2\pi^4}{90}+\frac{16\pi^4}{3}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}r^3\mathrm{e}^{2\pi\mathrm{i}rm\tau}\right\}\\ &=\frac{4\pi^4}{3}\left\{1+240\sum_{k=1}^{\infty}\sigma_3(k)\mathrm{e}^{2\pi\mathrm{i}k\tau}\right\}\end{aligned}\tag{10.5.5–1}$$

and similarly

$$\begin{aligned}g_3(\tau)&=140\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^6}\\ &=140\left\{\sum_{n\neq 0}\frac{1}{n^6}+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\left(\frac{1}{(n+m\tau)^6}+\frac{1}{(n-m\tau)^6}\right)\right\}\\ &=140\left\{2\zeta(6)+2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(n+m\tau)^6}\right\}\\ &=140\left\{\frac{2\pi^6}{945}-\frac{16\pi^6}{15}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}r^5\mathrm{e}^{2\pi\mathrm{i}rm\tau}\right\}\\ &=\frac{8\pi^6}{27}\left\{1-504\sum_{k=1}^{\infty}\sigma_5(k)\mathrm{e}^{2\pi\mathrm{i}k\tau}\right\}.\end{aligned}$$

Let

$$x=\mathrm{e}^{2\pi\mathrm{i}\tau},\quad A=\sum_{n=1}^{\infty}\sigma_3(n)x^n,\quad B=\sum_{n=1}^{\infty}\sigma_5(n)x^n.$$

Then

$$\begin{aligned}\Delta(\tau)=g_2^3(\tau)-27g_3^2(\tau)&=\left[\frac{4\pi^4}{3}(1+240A)\right]^3-27\left[\frac{8\pi^6}{27}(1-504B)\right]^2\\ &=\frac{64\pi^{12}}{27}\left[(1+240A)^3-(1-504B)^2\right].\end{aligned}$$

Now A and B have integer coefficients, and

$$\begin{aligned}(1+240A)^3-(1-504B)^2 &= 1+720A+3(240)^2A^2+(240)^3A^3-1+1008B-(504)^2B^2\\ &= 12^2(5A+7B)+12^3\big(100A^2-147B^2+8000A^3\big).\end{aligned}$$

But

$$5A+7B=\sum_{n=1}^{\infty}[5\sigma_3(n)+7\sigma_5(n)]x^n$$

and

$$5d^3+7d^5=d^3\big(5+7d^2\big)\equiv\begin{cases}d^3\big(d^2-1\big)\equiv 0 \pmod 3,\\ d^3\big(1-d^2\big)\equiv 0 \pmod 4,\end{cases}$$

so

$$5d^3+7d^5\equiv 0 \pmod{12}.$$

Hence 12^3 is a factor of each coefficient in the power series expansion of $(1+240A)^3-(1-504B)^2$, so

$$\Delta(\tau)=\frac{64\pi^{12}}{27}\left\{12^3\sum_{n=1}^{\infty}\tau(n)\mathrm{e}^{2\pi\mathrm{i}n\tau}\right\}=(2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)\mathrm{e}^{2\pi\mathrm{i}n\tau} \qquad (10.5.5\text{–}2)$$

where the $\tau(n)$ are integers. Also note that the coefficient if x is $12^2(5+7)$, so $\tau(1)=1$ and then

$$\Delta(\tau)=(2\pi)^{12}x[1+I_1(x)] \qquad (10.5.5\text{–}3)$$

where $I_1(x)$ denotes a power series in $x=\mathrm{e}^{2\pi\mathrm{i}\tau}$ with integer coefficients. From (10.5.5–2) we see that $\Delta(\tau+1)=\Delta(\tau)$, and by Theorem 2.1 (iii) in Chapter 9, we have

$$E_4\big(-\tfrac{1}{\tau}\big)=\tau^4E_4(\tau),\quad E_6\big(-\tfrac{1}{\tau}\big)=\tau^6E_6(\tau),$$

then

$$g_2\big(-\tfrac{1}{\tau}\big)=\tau^4g_2(\tau),\quad g_3\big(-\tfrac{1}{\tau}\big)=\tau^6g_3(\tau)$$

and so

$$\Delta\big(-\tfrac{1}{\tau}\big)=g_2^3\big(-\tfrac{1}{\tau}\big)-27g_3^2\big(-\tfrac{1}{\tau}\big)=\tau^{12}\Delta(\tau).$$

Now recall the Dedekind eta function defined by

$$\eta(\tau)=\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{12}}\prod_{n=1}^{\infty}\big(1-\mathrm{e}^{2\pi\mathrm{i}n\tau}\big)$$

and note that η never vanishes in $\mathbb{H}$. Let $f(\tau)=\dfrac{\Delta(\tau)}{\eta^{24}(\tau)}$, then

$$f(\tau+1)=f(\tau)\quad\text{and}\quad f\big(-\tfrac{1}{\tau}\big)=f(\tau),$$

so f is invariant under the group $\mathfrak{g}=\langle S,T_1\rangle$ described in Problem 10.5.2. Also, f is analytic and nonzero in $\mathbb{H}$ because Δ is analytic and nonzero and η never vanishes in $\mathbb{H}$. (To see $\Delta(\tau)\neq 0$ for all $\tau\in\mathbb{H}$, one just needs to note that e_1,e_2,e_3 are distinct, which follows since $\frac{1}{2}$ is a double zero of $\wp(z)-e_1$, $\frac{\tau}{2}$ is a double zero of $\wp(z)-e_2$ and $\frac{1+\tau}{2}$ is a double zero of $\wp(z)-e_3$, then use Theorem 1.7 and Corollary 2.3

of Chapter 9 to see that the polynomial $4x^3 - g_2 x - g_3$ has distinct roots.)

Next we examine the behavior of f at $\mathrm{i}\infty$. We have

$$\eta^{24}(\tau) = \mathrm{e}^{2\pi \mathrm{i}\tau} \prod_{n=1}^{\infty} \left(1 - \mathrm{e}^{2\pi \mathrm{i} n\tau}\right)^{24} = x \prod_{n=1}^{\infty} (1 - x^n)^{24} = x[1 + I_2(x)],$$

where $I_2(x)$ denotes a power series in $x = \mathrm{e}^{2\pi \mathrm{i}\tau}$ with integer coefficients. Thus, $\eta^{24}(\tau)$ has a simple zero at $x = 0$, which corresponds to $\mathrm{Im}(\tau) \to \infty$. This observation, together with (10.5.5–3), gives the Fourier expansion of f as $\mathrm{Im}(\tau) \to \infty$:

$$f(\tau) = \frac{\Delta(\tau)}{\eta^{24}(\tau)} = \frac{(2\pi)^{12} x[1 + I_1(x)]}{x[1 + I_2(x)]} = (2\pi)^{12}[1 + I(x)], \tag{10.5.5–4}$$

so f is analytic and nonzero at $\mathrm{i}\infty$. Now f satisfies all three conditions in Theorem 3.4 of Chapter 10, so f must be constant. Moreover, (10.5.5–4) shows that this constant is $(2\pi)^{12}$. This proves the desired result. □

Stein 10.5.6 Here we will deduce the formula for $r_8(n)$, which counts the number of representations of n as a sum of eight squares. The method is parallel to that of $r_4(n)$, but the details are less delicate.
Theorem: $r_8(n) = 16\sigma_3^*(n)$.
Here $\sigma_3^*(n) = \sigma_3(n) = \sum_{d|n} d^3$, when n is odd. Also, when n is even

$$\sigma_3^*(n) = \sum_{d|n} (-1)^d d^3 = \sigma_3^{\mathrm{e}}(n) - \sigma_3^{\mathrm{o}}(n),$$

where $\sigma_3^{\mathrm{e}}(n) = \sum_{d|n,\, d \text{ even}} d^3$ and $\sigma_3^{\mathrm{o}}(n) = \sum_{d|n,\, d \text{ odd}} d^3$.

Consider the appropriate Eisenstein series

$$E_4^*(\tau) = \sum \frac{1}{(n + m\tau)^4},$$

where the sum is over integers n and m with opposite parity. Recall the standard Eisenstein series

$$E_4(\tau) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n + m\tau)^4}.$$

Notice that the series defining E_4^* is absolutely convergent, in distinction to $E_2^*(\tau)$, which arose when considering $r_4(n)$. This makes some of the considerations below quite a bit simpler.

(1) Prove that $r_8(n) = 16\sigma_3^*(n)$ is equivalent to the identity $\theta(\tau)^8 = 48\pi^{-4} E_4^*(\tau)$.

(2) Note that $E_4^*(\tau) = E_4(\tau) - 2^{-4} E_4\left(\frac{\tau - 1}{2}\right)$.

(3) $E_4^*(\tau + 2) = E_4^*(\tau)$.

(4) $E_4^*(\tau) = \tau^{-4} E_4^*\left(-\frac{1}{\tau}\right)$.

(5) $48\pi^{-4} E_4^*(\tau) \to 1$ as $\tau \to \infty$.

(6) $\left|E_4^*\left(1 - \frac{1}{\tau}\right)\right| \approx |\tau|^4 \left|\mathrm{e}^{2\pi \mathrm{i}\tau}\right|$, as $\mathrm{Im}(\tau) \to \infty$.

Since $\theta(\tau)^8$ satisfies properties similar to (3), (4), (5) and (6) above, it follows that the invariant function $48\pi^{-4}E_4^*(\tau)/\theta(\tau)^8$ is bounded and hence a constant, which must be 1. This gives the desired result.

Proof (2) $E_4(\tau) - E_4^*(\tau) = \sum_{\substack{2|(n-m)\\(n,m)\neq(0,0)}} \frac{1}{(n+m\tau)^4} = \sum_{(n,m)\neq(0,0)} \frac{1}{[(2n-m)+m\tau]^4} = 2^{-4}E_4\left(\frac{\tau-1}{2}\right).$

(1) We first relate the sequence $\{r_8(n)\}$ via its generating function to θ:

$$\theta(\tau)^8 = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^8 = \sum_{n=0}^{\infty} r_8(n)q^n, \tag{10.5.6–1}$$

where $q = \mathrm{e}^{\pi i\tau}$ with $\tau \in \mathbb{H}$. Then it suffices to prove that

$$48\pi^{-4}E_4^*(\tau) = 1 + \sum_{k=1}^{\infty} 16\sigma_3^*(k)q^k. \tag{10.5.6–2}$$

As in (10.5.5–1), one has

$$E_4(\tau) = \frac{\pi^4}{45} + \frac{(2\pi)^4}{3}\sum_{k=1}^{\infty}\sigma_3(k)\mathrm{e}^{2\pi ik\tau}. \tag{10.5.6–3}$$

Combining this with (2) gives

$$\begin{aligned}48\pi^{-4}E_4^*(\tau) &= 48\pi^{-4}\left[E_4(\tau) - 2^{-4}E_4\left(\tfrac{\tau-1}{2}\right)\right]\\
&= 48\pi^{-4}\left\{\frac{\pi^4}{45} + \frac{16\pi^4}{3}\sum_{k=1}^{\infty}\sigma_3(k)\mathrm{e}^{2\pi ik\tau} - \frac{\pi^4}{720} - \frac{\pi^4}{3}\sum_{k=1}^{\infty}(-1)^k\sigma_3(k)\mathrm{e}^{\pi ik\tau}\right\}\\
&= 1 + 256\sum_{k=1}^{\infty}\sigma_3(k)\mathrm{e}^{2\pi ik\tau} - 16\sum_{k=1}^{\infty}(-1)^k\sigma_3(k)\mathrm{e}^{\pi ik\tau}\\
&= 1 + 16\left\{\sum_{k=1}^{\infty}16\sigma_3(k)q^{2k} - \sum_{k=1}^{\infty}(-1)^k\sigma_3(k)q^k\right\}.\end{aligned} \tag{10.5.6–4}$$

Now we shall express σ_3^* in terms of σ_3. For n even, we have

$$\begin{cases}\sigma_3^{\mathrm{e}}(n) - \sigma_3^{\mathrm{o}}(n) = \sigma_3^*(n),\\ \sigma_3^{\mathrm{e}}(n) + \sigma_3^{\mathrm{o}}(n) = \sigma_3(n).\end{cases}$$

Adding these two equations gives

$$\sigma_3^*(n) + \sigma_3(n) = 2\sigma_3^{\mathrm{e}}(n) = 2\times 2^3\sigma_3\left(\tfrac{n}{2}\right) = 16\sigma_3\left(\tfrac{n}{2}\right)$$

and then

$$\sigma_3^*(n) = 16\sigma_3\left(\tfrac{n}{2}\right) - \sigma_3(n). \tag{10.5.6–5}$$

Therefore, from (10.5.6–4) and (10.5.6–5) we obtain (10.5.6–2).

(3) By the definition of E_4^*, we have

$$E_4^*(\tau+2) = \sum_{2\nmid(n-m)}\frac{1}{[n+m(\tau+2)]^4} = \sum_{2\nmid(n-m)}\frac{1}{[(n+2m)+m\tau]^4} = E_4^*(\tau)$$

(4) By the definition of E_4^*, we have

$$E_4^*(-\tfrac{1}{\tau}) = \sum_{2\nmid(n-m)} \frac{1}{\left(n-\frac{m}{\tau}\right)^4} = \sum_{2\nmid(n-m)} \frac{\tau^4}{(m-n\tau)^4} = \tau^4 E_4(\tau).$$

(5) Since $q \to 0$ as $\mathrm{Im}(\tau) \to \infty$, by (10.5.6–2) we get the desired result.

(6) Note that $n+m$ is odd if and only if $n-m$ is odd, hence

$$\sum_{2\nmid(n-m)} \frac{1}{[-m+(n+m)\tau]^4} = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^4} - \sum_{(n,m)\neq(0,0)} \frac{1}{(n+2m\tau)^4}.$$

Then

$$\begin{aligned} E_4^*(1-\tfrac{1}{\tau}) &= \sum_{2\nmid(n-m)} \frac{1}{\left[n+m\left(1-\frac{1}{\tau}\right)\right]^4} = \sum_{2\nmid(n-m)} \frac{\tau^4}{[-m+(n+m)\tau]^4} \\ &= \sum_{(n,m)\neq(0,0)} \frac{\tau^4}{(n+m\tau)^4} - \sum_{(n,m)\neq(0,0)} \frac{\tau^4}{(n+2m\tau)^4} \\ &= \tau^4[E_4(\tau) - E_4(2\tau)]. \end{aligned}$$

By (10.5.6–3) we get

$$E_4(\tau) - E_4(2\tau) = \frac{(2\pi)^4}{3}\left\{\sum_{k=1}^{\infty} \sigma_3(k) \mathrm{e}^{2\pi \mathrm{i} k\tau} - \sum_{k=1}^{\infty} \sigma_3(k) \mathrm{e}^{4\pi \mathrm{i} k\tau}\right\} \sim \frac{(2\pi)^4}{3} \mathrm{e}^{2\pi \mathrm{i}\tau} \quad \text{as } \mathrm{Im}(\tau) \to \infty,$$

which gives the desired result.

Finally, we shall show that $\theta(\tau)^8$ satisfies similar properties.

(3) By (10.5.6–1) one can verify that $\theta(\tau+2)^8 = \theta(\tau)^8$, since $q = \mathrm{e}^{\pi \mathrm{i}\tau}$ is invariant when τ is replaced by $\tau + 2$.

(4) By Corollary 1.7 in Chapter 10 we have $\theta(\tau)^8 = \tau^{-4}\theta\left(-\frac{1}{\tau}\right)^8$.

(5) $\theta(\tau)^8 \to 1$ as $\tau \to \infty$, which is obvious from its definition.

(6) By Corollary 1.8 in Chapter 10 we have $\theta\left(1-\frac{1}{\tau}\right)^8 \sim 2^8\tau^4\mathrm{e}^{2\pi \mathrm{i}\tau}$ as $\mathrm{Im}(\tau) \to \infty$. Recall the notation $x \approx y$ means both $x \lesssim y$ and $y \lesssim x$ hold. □