Riemannian Geometry

Xiaoshuo Lin, Spring 2025

https://xiaoshuo-lin.github.io

Homework 1

Exercise 1 The sphere

$$\mathbb{S}^n = \left\{ \left(x^1, \cdots, x^{n+1}\right) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}$$

is an embedded submanifold of $\mathbb{R}^{n+1}$ with the induced metric $g_{\mathbb{S}^n}$. Consider the coordinate chart $U = \mathbb{S}^n \setminus \{(0, \cdots, 0, 1)\}$, given by the stereographic projection from the north pole:

$$\varphi(x^1, \cdots, x^{n+1}) \coloneqq \left(\frac{x^1}{1 - x^{n+1}}, \cdots, \frac{x^n}{1 - x^{n+1}} \right).$$

Write down the metric $g_{\mathbb{S}^n}$ in this chart.

Solution The inverse map $\varphi^{-1} \colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{(0, \cdots, 0, 1)\}$ is given by

$$\varphi^{-1}(u^1, \cdots, u^n) = \left(\frac{2u^1}{|u|^2 + 1}, \cdots, \frac{2u^n}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1} \right).$$

Then, we obtain the following coordinate representation of $g_{\mathbb{S}^n}$ in stereographic coordinates:

$$(\varphi^{-1})^* g_{\mathbb{S}^n} = \sum_{j=1}^n \left(\mathrm{d} \left(\frac{2u^j}{|u|^2 + 1} \right) \right)^2 + \left(\mathrm{d} \left(\frac{|u|^2 - 1}{|u|^2 + 1} \right) \right)^2.$$

If we expand each of these terms individually, we get

$$\mathrm{d} \left(\frac{2u^j}{|u|^2 + 1} \right) = \frac{2 \, \mathrm{d}u^j}{|u|^2 + 1} - \frac{4u^j}{(|u|^2 + 1)^2} \sum_{i=1}^n u^i \, \mathrm{d}u^i$$

and

$$\mathrm{d} \left(\frac{|u|^2 - 1}{|u|^2 + 1} \right) = -2 \, \mathrm{d} \left(\frac{1}{|u|^2 + 1} \right) = \frac{4}{(|u|^2 + 1)^2} \sum_{i=1}^n u^i \, \mathrm{d}u^i.$$

Therefore,

$$\begin{aligned}
(\varphi^{-1})^* g_{\mathbb{S}^n} =& \frac{4}{(|u|^2 + 1)^2} \sum_{j=1}^n (\mathrm{d}u^j)^2 - \frac{16}{(|u|^2 + 1)^3} \left(\sum_{i=1}^n u^i \, \mathrm{d}u^i \right)^2 + \frac{16|u|^2}{(|u|^2 + 1)^4} \left(\sum_{i=1}^n u^i \, \mathrm{d}u^i \right)^2 \\
&+ \frac{16}{(|u|^2 + 1)^4} \left(\sum_{i=1}^n u^i \, \mathrm{d}u^i \right)^2 \\
=& \frac{4}{(|u|^2 + 1)^2} \sum_{j=1}^n (\mathrm{d}u^j)^2 \\
=& \frac{4}{(|u|^2 + 1)^2} g_E,
\end{aligned}$$

where g_E is the Euclidean metric on $\mathbb{R}^n$. □

Exercise 2 Consider the connection ∇ defined on $\mathbb{R}^3$ so that with respect to the standard frame e_1, e_2, e_3,

$$\nabla_{e_i} e_j = e_i \times e_j,$$

where $\times$ denotes the cross product. Find the vector field X which is the parallel transport of e_2 along the e_1-axis.

Solution Write X as a linear combination of the standard basis vectors:

$$X(t) = a(t)e_1 + b(t)e_2 + c(t)e_3,$$

where t is the coordinate along the e_1-axis and $a(t), b(t), c(t)$ are functions of t. Then, the parallel transport equation is

$$\begin{aligned}0 &= \nabla_{e_1}(a(t)e_1 + b(t)e_2 + c(t)e_3)\\ &= a'(t)e_1 + a(t)\nabla_{e_1}e_1 + b'(t)e_2 + b(t)\nabla_{e_1}e_2 + c'(t)e_3 + c(t)\nabla_{e_1}e_3\\ &= a'(t)e_1 + b'(t)e_2 + c'(t)e_3 + b(t)e_3 - c(t)e_2.\end{aligned}$$

Therefore, we have the system of differential equations

$$a'(t) = 0, \quad b'(t) = c(t), \quad c'(t) = -b(t),$$

with the initial condition $X(0) = e_2$, that is,

$$a(0) = 0, \quad b(0) = 1, \quad c(0) = 0.$$

Solving this system, we find that

$$a(t) = 0, \quad b(t) = \cos t, \quad c(t) = -\sin t,$$

and hence

$$X(t) = (\cos t)e_2 - (\sin t)e_3. \qquad \square$$

In the following, all connections are Levi-Civita connections with respect to the given metrics.

Exercise 3 Let N^n be an embedded submanifold of M^m. Given a metric $\bar{g}$ on M with Levi-Civita connection $\overline{\nabla}$, we define the connection ∇ on TN by

$$\nabla_X Y = \pi_{TN}\left(\overline{\nabla}_X Y\right)$$

for any vector fields $X, Y \in \Gamma(TN)$, where π_{TN} denotes the orthogonal projection onto TN. Prove that ∇ is the Levi-Civita connection of the induced metric $g = \bar{g}|_N$ on N.

Proof For any vector fields $X, Y \in \Gamma(TN)$, we can extend them smoothly to an open neighborhood of N in M and still denote them by X, Y. It is immediate from the definition that $\nabla_X Y$ is linear over $\mathcal{C}^\infty(M)$ in X and over $\mathbb{R}$ in Y, so to show that ∇ is a connection, only the product rule needs to be checked. Let $f \in \mathcal{C}^\infty(M)$, and let $\tilde{f}$ be an extension of f to an open neighborhood of N in M. Then $\tilde{f}Y$ is a smooth extension of fY to an open neighborhood of N, so

$$\nabla_X(fY) = \pi_{TN}\left(\overline{\nabla}_X(fY)\right) = \pi_{TN}\left(\left(X\tilde{f}\right)Y\right) + \pi_{TN}\left(\tilde{f}\overline{\nabla}_X Y\right) = (Xf)Y + f\nabla_X Y.$$

Since

$$\nabla_X Y - \nabla_Y X - [X, Y] = \pi_{TN}\left(\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y]\right) = \pi_{TN}(0) = 0,$$

we have that ∇ is torsion-free. Finally, to see that ∇ is compatible with g, we compute

$$\begin{aligned}\nabla_X\langle Y,Z\rangle &= \overline{\nabla}_X\langle Y,Z\rangle = \big\langle \overline{\nabla}_X Y, Z\big\rangle + \big\langle Y, \overline{\nabla}_X Z\big\rangle \\ &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\end{aligned}$$

for any vector fields $X, Y, Z \in \Gamma(TN)$. Therefore, ∇ is the Levi-Civita connection of $g = \bar{g}|_N$. □

Exercise 4 Let (M, g) and (N, h) be Riemannian manifolds. Show that the Levi-Civita connection ∇ of $(M \times N, g \times h)$ satisfies

$$\nabla_{X_1+X_2}(Y_1+Y_2) = \nabla^g_{X_1} Y_1 + \nabla^h_{X_2} Y_2$$

for all vector fields $X_1, Y_1 \in \Gamma(TM)$ and $X_2, Y_2 \in \Gamma(TN)$.

Proof Note that vector fields from TM and TN are orthogonal, with vanishing Lie brackets between them. Therefore, for any $Z_1 \in \Gamma(TM)$ and $Z_2 \in \Gamma(TN)$, we have by Koszul's formula that

$$\begin{aligned}2\langle \nabla_{X_1+X_2}(Y_1+Y_2), Z_1+Z_2\rangle =&(X_1+X_2)\langle Y_1+Y_2, Z_1+Z_2\rangle + (Y_1+Y_2)\langle Z_1+Z_2, X_1+X_2\rangle \\ &- (Z_1+Z_2)\langle X_1+X_2, Y_1+Y_2\rangle - \langle Y_1+Y_2, [X_1+X_2, Z_1+Z_2]\rangle \\ &- \langle Z_1+Z_2, [Y_1+Y_2, X_1+X_2]\rangle + \langle X_1+X_2, [Z_1+Z_2, Y_1+Y_2]\rangle \\ =& X_1\langle Y_1, Z_1\rangle + X_2\langle Y_2, Z_2\rangle + Y_1\langle Z_1, X_1\rangle + Y_2\langle Z_2, X_2\rangle \\ &- Z_1\langle X_1, Y_1\rangle - Z_2\langle X_2, Y_2\rangle - \langle Y_1, [X_1, Z_1]\rangle - \langle Y_2, [X_2, Z_2]\rangle \\ &- \langle Z_1, [Y_1, X_1]\rangle - \langle Z_2, [Y_2, X_2]\rangle + \langle X_1, [Z_1, Y_1]\rangle + \langle X_2, [Z_2, Y_2]\rangle \\ =& 2\big\langle \nabla^g_{X_1} Y_1, Z_1\big\rangle + 2\big\langle \nabla^h_{X_2} Y_2, Z_2\big\rangle \\ =& 2\big\langle \nabla^g_{X_1} Y_1 + \nabla^h_{X_2} Y_2, Z_1+Z_2\big\rangle.\end{aligned}$$

Since this holds for all $Z_1 \in \Gamma(TM)$ and $Z_2 \in \Gamma(TN)$, the desired result follows. □

Exercise 5 Let F be an isometry of (M^n, g).

(1) Show that $\mathrm{d}F(\nabla_X Y) = \nabla_{\mathrm{d}F(X)} \mathrm{d}F(Y)$ for any vector fields $X, Y \in \Gamma(TM)$.

(2) Use this fact to show that any isometry F of $(\mathbb{R}^n, g_E)$ has the form $F(x) = Ox + b$, where $O \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$.

Proof (1) We shall show that

$$\nabla_X Y = (\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)} \mathrm{d}F(Y)\big), \quad \forall X, Y \in \Gamma(TM). \tag{5–1}$$

By the uniqueness of the Levi-Civita connection, it suffices to show that the right-hand side of (5–1) defines a connection that is compatible with g and torsion-free.

- ◇ It is a connection because it satisfies the following properties:
  - For $f_1, f_2 \in \mathcal{C}^\infty(M)$ and $X_1, X_2 \in \Gamma(TM)$,

$$\begin{aligned}&(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(f_1X_2+f_2X_2)} \mathrm{d}F(Y)\big) \\ =&(\mathrm{d}F)^{-1}\big(\nabla_{(f_1\circ F^{-1})\mathrm{d}F(X_1)+(f_2\circ F^{-1})\mathrm{d}F(X_2)} \mathrm{d}F(Y)\big) \\ =&(\mathrm{d}F)^{-1}\big(\big(f_1\circ F^{-1}\big)\nabla_{\mathrm{d}F(X_1)} \mathrm{d}F(Y) + \big(f_2\circ F^{-1}\big)\nabla_{\mathrm{d}F(X_2)} \mathrm{d}F(Y)\big) \\ =&f_1(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X_1)} \mathrm{d}F(Y)\big) + f_2(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X_2)} \mathrm{d}F(Y)\big).\end{aligned}$$

– For $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(M)$,

$$\begin{aligned}&(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(a_1Y_1+a_2Y_2)\big)\\=&(\mathrm{d}F)^{-1}\big(a_1\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y_1)+a_2\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y_2)\big)\\=&a_1(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y_1)\big)+a_2(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y_2)\big).\end{aligned}$$

– For $f \in \mathcal{C}^\infty(M)$,

$$\begin{aligned}&(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}(\mathrm{d}F(fY))\big)\\=&(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\big((f\circ F^{-1})\,\mathrm{d}F(Y)\big)\big)\\=&(\mathrm{d}F)^{-1}\big((f\circ F^{-1})\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y)+\big(\mathrm{d}F(X)(f\circ F^{-1})\big)\,\mathrm{d}F(Y)\big)\\=&f(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y)\big)+(Xf)(\mathrm{d}F)^{-1}\circ\mathrm{d}F(Y)\\=&f(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y)\big)+(Xf)Y.\end{aligned}$$

◇ To see that it is compatible with g, we use the fact that F is an isometry:

$$\begin{aligned}&\big\langle(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y)\big),Z\big\rangle+\big\langle Y,(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Z)\big)\big\rangle\\=&\big\langle\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y),\mathrm{d}F(Z)\big\rangle+\big\langle\mathrm{d}F(Y),\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Z)\big\rangle\\=&\,\mathrm{d}F(X)\langle\mathrm{d}F(Y),\mathrm{d}F(Z)\rangle\\=&X\langle Y,Z\rangle.\end{aligned}$$

◇ To see that it is torsion-free, we use the naturality of the Lie bracket:

$$\begin{aligned}&(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y)\big)-(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(Y)}\,\mathrm{d}F(X)\big)\\=&(\mathrm{d}F)^{-1}\big(\nabla_{\mathrm{d}F(X)}\,\mathrm{d}F(Y)-\nabla_{\mathrm{d}F(Y)}\,\mathrm{d}F(X)\big)\\=&(\mathrm{d}F)^{-1}[\mathrm{d}F(X),\mathrm{d}F(Y)]\\=&[X,Y].\end{aligned}$$

Therefore, the right-hand side of (5–1) is exactly the Levi-Civita connection of g, and hence (5–1) holds.

(2) Connections in $\mathbb{R}^n$ are given by the directional derivatives, so by part (1) we have

$$0=\mathrm{d}F(\mathrm{D}_{\partial_i}\partial_j)=\mathrm{D}_{\mathrm{d}F(\partial_i)}\,\mathrm{d}F(\partial_j)=\mathrm{Jac}(\mathrm{d}F(\partial_j))\,\mathrm{d}F(\partial_i),\quad\forall i,j,$$

which implies that

$$\mathrm{Jac}(\mathrm{d}F(\partial_j))\,\mathrm{Jac}(F)=0.$$

Since F is an isometry, the Jacobian $\mathrm{Jac}(F)$ is invertible at each point, we obtain

$$\mathrm{Jac}(\mathrm{d}F(\partial_j))=0,\quad\forall j.$$

Note that $\mathrm{d}F(\partial_j)$ is the j-th column of $\mathrm{Jac}(F)$, so $\mathrm{Jac}(F)$ is a constant matrix. Therefore, F is an affine transformation of the form $F(x)=Ax+b$ for some $A\in\mathrm{GL}(n,\mathbb{R})$ and $b\in\mathbb{R}^n$. Finally, since F is an isometry, A must be orthogonal. □

Exercise 6 Show that any isometry F of $(\mathbb{S}^n, g_{\mathbb{S}^n})$ can be given by $F(x) = Ox$, where $O \in \mathrm{O}(n+1)$ and $x \in \mathbb{R}^{n+1}$ with $|x| = 1$.

Proof We begin by noting that any $F \in \mathrm{Iso}(\mathbb{S}^n, g_{\mathbb{S}^n})$ preserves the $\mathbb{R}^{n+1}$-inner product of unit vectors, that is,

$$F(u) \cdot F(v) = u \cdot v, \quad \forall u, v \in \mathbb{S}^n. \tag{6–1}$$

Indeed, the inner product $u \cdot v$ can be interpreted as the cosine of the Riemannian distance between u and v on $\mathbb{S}^n$, and similarly for $F(u) \cdot F(v)$. Therefore, by the isometry invariance of the Riemannian distance function, (6–1) holds.

Now, let us consider the map

$$\widetilde{F} \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad p \mapsto \begin{cases} 0, & \text{if } p = 0, \\ |p| F\left(\frac{p}{|p|}\right), & \text{if } p \neq 0. \end{cases}$$

It is immediate that $\widetilde{F}$ preserves the $\mathbb{R}^{n+1}$-inner product:

$$\widetilde{F}(u) \cdot \widetilde{F}(v) = |u||v| F\left(\frac{u}{|u|}\right) \cdot F\left(\frac{v}{|v|}\right) = |u||v| \frac{u \cdot v}{|u||v|} = u \cdot v, \quad \forall u, v \in \mathbb{R}^{n+1} \setminus \{0\}.$$

Then, for any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}^{n+1}$, we compute

$$\begin{aligned} \left|\widetilde{F}(\lambda u + v) - \lambda \widetilde{F}(u) - \widetilde{F}(v)\right|^2 &= \left\langle \widetilde{F}(\lambda u + v) - \lambda \widetilde{F}(u) - \widetilde{F}(v), \widetilde{F}(\lambda u + v) - \lambda \widetilde{F}(u) - \widetilde{F}(v) \right\rangle \\ &= \left\langle \widetilde{F}(\lambda u + v), \widetilde{F}(\lambda u + v) \right\rangle + \text{ more such terms} \\ &= \langle \lambda u + v, \lambda u + v \rangle + \text{ more such terms} \\ &= |\lambda u + v - \lambda u - v|^2 \\ &= 0, \end{aligned}$$

which shows that $\widetilde{F}$ is linear, and is given by $F(x) = Ox$ for some $O \in \mathrm{GL}(n+1, \mathbb{R})$.

Finally, since F is the restriction of $\widetilde{F}$ to $\mathbb{S}^n$, the result follows. □

Exercise 7 Let (M, g) be a Riemannian manifold and $f \in \mathcal{C}^\infty(M)$. Show that

$$\mathcal{L}_{\mathrm{grad}\, f}\, g = 2\nabla^2 f,$$

where $\mathcal{L}$ denotes the Lie derivative.

Proof By the product rule for the Lie derivative, for any $X, Y \in \Gamma(TM)$, we have

$$(\mathcal{L}_{\mathrm{grad}\, f}\, g)(X, Y) = \mathrm{grad}\, f(\langle X, Y \rangle) - \langle [\mathrm{grad}\, f, X], Y \rangle - \langle X, [\mathrm{grad}\, f, Y] \rangle.$$

Since ∇ is compatible with g, the first term is

$$\mathrm{grad}\, f(\langle X, Y \rangle) = \left\langle \nabla_{\mathrm{grad}\, f} X, Y \right\rangle + \left\langle X, \nabla_{\mathrm{grad}\, f} Y \right\rangle.$$

And since ∇ is torsion-free, the remaining terms expand as

$$\langle [\mathrm{grad}\, f, X], Y \rangle = \left\langle \nabla_{\mathrm{grad}\, f} X, Y \right\rangle - \langle \nabla_X \,\mathrm{grad}\, f, Y \rangle,$$

$$\langle X, [\operatorname{grad} f, Y]\rangle = \left\langle X, \nabla_{\operatorname{grad} f} Y\right\rangle - \langle X, \nabla_Y \operatorname{grad} f\rangle.$$

Combining these, we obtain

$$\left(\mathcal{L}_{\operatorname{grad} f} g\right)(X, Y) = \langle \nabla_X \operatorname{grad} f, Y\rangle + \langle X, \nabla_Y \operatorname{grad} f\rangle. \tag{7–1}$$

Meanwhile, the Hessian of f is computed as

$$\begin{aligned}(\nabla^2 f)(X, Y) &= \nabla_X(\nabla_Y f) - \nabla_{\nabla_X Y} f = X(Yf) - (\nabla_X Y)f \\ &= X(\langle \operatorname{grad} f, Y\rangle) - \langle \operatorname{grad} f, \nabla_X Y\rangle \\ &= \langle \nabla_X \operatorname{grad} f, Y\rangle.\end{aligned} \tag{7–2}$$

Since $\nabla^2 f$ is a $(0,2)$-symmetric tensor, the result follows from (7–1) and (7–2). □

Exercise 8 Let (M^n, g) be a Riemannian manifold with Laplace operator Δ. For the conformal metric $\bar{g} = e^{-2f} g$, prove that

$$\Delta_{\bar{g}} \varphi = e^{2f}(\Delta_g \varphi - (n-2)\langle \operatorname{grad} f, \operatorname{grad} \varphi\rangle).$$

Proof Let (x^i) be any smooth local coordinates on an open subset of M. Then, for any $\varphi \in \mathcal{C}^\infty(M)$,

$$\begin{aligned}\Delta_{\bar{g}}\varphi &= \frac{1}{\sqrt{\det \bar{g}}}\frac{\partial}{\partial x^i}\left(\bar{g}^{ij}\sqrt{\det \bar{g}}\frac{\partial \varphi}{\partial x^j}\right) \\ &= \frac{1}{\sqrt{e^{-2nf}\det g}}\frac{\partial}{\partial x^i}\left(e^{2f}g^{ij}\sqrt{e^{-2nf}\det g}\frac{\partial \varphi}{\partial x^j}\right) \\ &= \frac{e^{nf}}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(e^{(2-n)f}g^{ij}\sqrt{\det g}\frac{\partial}{\partial x^j}\right) \\ &= \frac{e^{nf}}{\sqrt{\det g}}\left\{\left((2-n)\frac{\partial f}{\partial x^i}e^{(2-n)f}\right)\left(g^{ij}\sqrt{\det g}\frac{\partial \varphi}{\partial x^j}\right) + e^{(2-n)f}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\frac{\partial \varphi}{\partial x^j}\right)\right\} \\ &= \left((2-n)\frac{\partial f}{\partial x^i}e^{2f}\right)\left(g^{ij}\frac{\partial \varphi}{\partial x^j}\right) + \frac{e^{2f}}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\frac{\partial \varphi}{\partial x^j}\right) \\ &= e^{2f}\left(\Delta_g\varphi - (n-2)g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \varphi}{\partial x^j}\right) \\ &= e^{2f}(\Delta_g\varphi - (n-2)\langle \operatorname{grad} f, \operatorname{grad}\varphi\rangle).\end{aligned}$$

□

Homework 2

In the following, connections are assumed to be Levi-Civita connections by default.

Exercise 9 Let (M^n, g) be a Riemannian manifold. Prove that for any $p \in M$, the closure of the geodesic ball $\mathbb{B}(p, r) = \{x \in M : d_g(p, x) < r\}$ is

$$\{x \in M : d_g(p, x) \leqslant r\}.$$

Proof (1) For any x with $d_g(p, x) > r$, we can find a geodesic ball centered at x which does not intersect $\mathbb{B}(p, r)$. This implies that $x \notin \overline{\mathbb{B}(p, r)}$ and hence $\overline{\mathbb{B}(p, r)} \subset \{x \in M : d_g(p, x) \leqslant r\}$. Here we use the fact that the metric topology induced by d_g is the same as the manifold topology.

(2) Suppose $x \in M$ satisfies $d_g(p,x) \leqslant r$. For any $n \geqslant 1$, we can take $x_n \in \mathbb{B}\big(x, \frac{1}{n}\big) \cap \mathbb{B}(p,r)$, for otherwise the triangle inequality would imply that $d_g(p,x) > r$. This shows that $x \in \overline{\mathbb{B}(p,r)}$. □

Exercise 10 Let (M^n, g) be a Riemannian manifold. Prove that for any $p \in M$, there exists an open neighborhood U of p and n vector fields $E_1, \cdots, E_n \in \Gamma(TU)$, orthonormal at each point of U, such that

$$\nabla_{E_i} E_j(p) = 0.$$

Proof Let U be a normal neighborhood of p. For each $q \in U$, there is a geodesic γ_q parametrized by arc length from p to q. Take an orthonormal basis $\{v_1, \cdots, v_n\}$ of T_pM and let $\{V_1, \cdots, V_n\}$ be their parallel transport along γ_q. For each $j = 1, \cdots, n$, define the smooth vector field E_j on U by

$$E_j(q) = V_j(d_g(p,q)),$$

where d_g is the Riemannian distance function. Then the n vector fields $E_1, \cdots, E_n$ are orthonormal at each point of U. For each $i = 1, \cdots, n$, let $\gamma_i(s)$ be the geodesic such that $\gamma_i(0) = p$ and $\gamma_i'(0) = E_i(p)$. Then

$$\nabla_{E_i} E_j(p) = \nabla_{\gamma_i'(0)} E_j = \left.\frac{\mathrm{D}(E_j \circ \gamma_i)}{\mathrm{d}s}\right|_{s=0}.$$

Since $E_j \circ \gamma_i(s) = V_j(d(p, \sigma_o(s))) = V_j(s)$ is parallel along γ_i, we have

$$\nabla_{E_i} E_j(p) = \frac{\mathrm{D}V_j}{\mathrm{d}s}(0) = 0.$$

□

Exercise 11 Let M^n be a smooth manifold (Hausdorff and paracompact). Prove that there exists a countable covering $\{U_\alpha\}$ of M such that for any elements $U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_k}$ in the covering, the intersection

$$\bigcap_{i=1}^{k} U_{\alpha_i}$$

deformation retracts to a point.

Proof Endow M with a Riemannian metric. Every point in M has a *strongly* convex neighborhood (i.e., a neighborhood U in which any two points can be joined by a *unique* minimizing geodesic contained in U), and the intersection of any two such neighborhoods is again strongly convex. For any strongly convex neighborhood U of $p \in M$, we can connect any point $q \in U$ to p by a unique minimizing geodesic in U. Hence, using normal coordinates, we see that any point in $\exp_p^{-1}(U)$ can be connected to 0 by a straight line. This implies that $\exp_p^{-1}(U)$ is a star-shaped neighborhood of 0 in T_pM, which is contractible. Since U is diffeomorphic to $\exp_p^{-1}(U)$, we conclude that U is contractible. Finally, since M is second-countable, and hence Lindelöf, we can cover it with a countable collection of strongly convex neighborhoods $\{U_\alpha\}$, which gives us the desired covering. □

Exercise 12 Let (G, g) be a Lie group with a bi-invariant metric g.

(1) Prove that

$$\nabla_Y X = \frac{1}{2}[Y, X]$$

for any $X, Y \in \mathfrak{g}$, where the elements of $\mathfrak{g}$ are identified with left-invariant vector fields on G.

(2) Prove that any geodesic $\phi(t)$ from the identity element e is defined for any $t \in \mathbb{R}$ and satisfies

$$\phi(t+s) = \phi(t) \cdot \phi(s)$$

for any $t, s \in \mathbb{R}$.

Proof (1) Since g is bi-invariant, the inner product of any two left-invariant vector fields is constant. In particular, Koszul's formula simplifies to

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X,Y], Z \rangle - \langle [X,Z], Y \rangle - \langle [Y,Z], X \rangle),$$

where X, Y, Z are left-invariant vector fields. Recall that for the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(g) \coloneqq T_{\mathbb{1}} \mathrm{GL}(\mathfrak{g})$, we have $\mathrm{ad}(X)Y = [X,Y]$. Therefore,

$$\begin{aligned}
\langle \nabla_X Y, Z \rangle &= \frac{1}{2}(\langle [X,Y], Z \rangle - \langle \mathrm{ad}(X)Z, Y \rangle - \langle \mathrm{ad}(Y)Z, X \rangle) \\
&= \frac{1}{2}(\langle [X,Y], Z \rangle - \langle \mathrm{ad}^*(X)Y, Z \rangle - \langle \mathrm{ad}^*(Y)X, Z \rangle) \\
&= \frac{1}{2}\langle [X,Y] - \mathrm{ad}^*(X)Y - \mathrm{ad}^*(Y)X, Z \rangle.
\end{aligned}$$

Since Z is an arbitrary left-invariant vector field, we have

$$\nabla_X Y = \frac{1}{2}([X,Y] - \mathrm{ad}^*(X)Y - \mathrm{ad}^*(Y)X). \tag{12–1}$$

Moreover, by definition,

$$\mathrm{ad}(X)Y = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\exp(tX))Y.$$

Hence, we have

$$\begin{aligned}
0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle Y, Z \rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left\langle \left(L_{\exp(tX)}\right)_* \left(R_{\exp(-tX)}\right)_* Y, \left(L_{\exp(tX)}\right)_* \left(R_{\exp(-tX)}\right)_* Z \right\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \mathrm{Ad}(\exp(tX))Y, \mathrm{Ad}(\exp(tX))Z \rangle \\
&= \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\exp(tX))Y, Z \right\rangle + \left\langle Y, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\exp(tX))Z \right\rangle \\
&= \langle \mathrm{ad}(X)Y, Z \rangle + \langle Y, \mathrm{ad}(X)Z \rangle \\
&= \langle \mathrm{ad}(X)Y + \mathrm{ad}^*(X)Y, Z \rangle.
\end{aligned}$$

Since Y and Z are two arbitrary left-invariant vector fields, we find that

$$\mathrm{ad}(X) = -\,\mathrm{ad}^*(X).$$

With this, we obtain from (12–1) that

$$\nabla_X Y = \frac{1}{2}([X,Y] + \mathrm{ad}(X)Y + \mathrm{ad}(Y)X) = \frac{1}{2}([X,Y] + [X,Y] + [Y,X]) = \frac{1}{2}[X,Y].$$

(2) We need the following

Lemma *For a Lie group G with a bi-invariant metric g, the inversion map $i\colon G \to G$ given by $i(\varphi) = \varphi^{-1}$ is an isometry.*

Proof of the lemma Note that for any $x \in G$, we have

$$R_{\varphi^{-1}} \circ i \circ L_{\varphi^{-1}}(x) = \left(\varphi^{-1}x\right)^{-1}\varphi^{-1} = x^{-1}\varphi\varphi^{-1} = i(x).$$

Hence, using the chain rule, we get

$$\mathrm{d}i_\varphi = \mathrm{d}\left(R_{\varphi^{-1}}\right)_e \circ \mathrm{d}i_e \circ \mathrm{d}\left(L_{\varphi^{-1}}\right)_\varphi.$$

Since the differential of i at the identity element e is given by $\mathrm{d}i_e(X) = -X$, we have

$$\mathrm{d}i = -(\mathrm{d}R_\varphi)^{-1} \circ (\mathrm{d}L_\varphi)^{-1}.$$

Thus, by the bi-invariance of the metric, for any $X, Y \in T_\varphi G$,

$$\begin{aligned}\langle \mathrm{d}i(X), \mathrm{d}i(Y)\rangle_{\varphi^{-1}} &= \left\langle -(\mathrm{d}R_\varphi)^{-1} \circ (\mathrm{d}L_\varphi)^{-1}(X), -(\mathrm{d}R_\varphi)^{-1} \circ (\mathrm{d}L_\varphi)^{-1}(Y)\right\rangle_{\varphi^{-1}} \\ &= \langle X, Y\rangle_\varphi.\end{aligned}$$

This shows that i is an isometry.

By the lemma, the inversion map i is an isometry, so $i \circ \phi(t) = \phi(t)^{-1}$ is a geodesic. And since $\mathrm{d}i_e(X) = -X$, by the uniqueness of geodesics, we have $\phi(-t) = \phi(t)^{-1}$, i.e., $\phi(t)\phi(-t) = e$. For small t_0, if we define $\tilde{\phi}(t) = \phi(t_0)\phi(t)$, then $\tilde{\phi}(t)$ is a geodesic with $\tilde{\phi}(0) = \phi(t_0)$ and $\tilde{\phi}(-t_0) = e$. By the uniqueness of short geodesics, we must have $\tilde{\phi}(t) = \phi(t_0 + t)$, that is,

$$\phi(t_0)\phi(t) = \phi(t_0 + t), \tag{12–2}$$

for all t and t_0 small enough. By extending ϕ beyond any interval $[0, l]$ via $\phi(t + s) := \phi(l)\phi(s)$, we see that $\phi(t)$ can be extended to a geodesic for all $t \in \mathbb{R}$. And by a standard argument (of chopping into "small pieces"), from (12–2), we indeed have

$$\phi(t + s) = \phi(t) \cdot \phi(s), \quad \forall t, s \in \mathbb{R}.$$

□

Exercise 13 Let (M^n, g) be a Riemannian manifold. We introduce a Riemannian metric $\tilde{g}$ on the tangent bundle TM as follows. Fix $(p, v) \in TM$, and consider curves $\alpha(t) = (p(t), v(t))$ and $\beta(t) = (q(t), w(t))$ on TM such that $\alpha(0) = \beta(0) = (p, v)$. Then we define at (p, v)

$$\tilde{g}(\alpha'(0), \beta'(0)) = g(p'(0), q'(0)) + g\left(\frac{\mathrm{D}v}{\mathrm{d}t}(0), \frac{\mathrm{D}w}{\mathrm{d}t}(0)\right).$$

(1) Prove that the metric $\tilde{g}$ is well-defined and smooth.

(2) A vector field V on TM is called *horizontal* if it is orthogonal to the fiber T_pM. A curve $(p(t), v(t))$ in TM is horizontal if its tangent vector is horizontal for any t. Prove that a curve $(p(t), v(t))$ in TM is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in M.

(3) Prove that the geodesic field G is a horizontal vector field on TM.

(4) Prove that the trajectories of the geodesic field G are geodesics on TM with respect to $\tilde{g}$.

(5) Prove that with respect to $\tilde{g}$, the geodesic field G satisfies

$$\operatorname{div}(G) = 0.$$

(6) Prove that the geodesic flow preserves the Riemannian volume measure of TM.

Proof (1) The expression of $\tilde{g}$ is clearly coordinate independent. Hence, we may let $(x^1, \cdots, x^n)$ be local coordinates on M around p, and let $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ be the corresponding natural coordinates on TM near (p, v). Then we have

$$\begin{aligned} \frac{\mathrm{D}v^i}{\mathrm{d}t}(0) &= \frac{\mathrm{d}v^i}{\mathrm{d}t}(0) + \Gamma^i_{jk} v^k(0) \frac{\mathrm{d}p^j}{\mathrm{d}t}(0), \\ \frac{\mathrm{D}w^i}{\mathrm{d}t}(0) &= \frac{\mathrm{d}w^i}{\mathrm{d}t}(0) + \Gamma^i_{jk} w^k(0) \frac{\mathrm{d}q^j}{\mathrm{d}t}(0). \end{aligned} \tag{13–1}$$

Therefore, $\tilde{g}$ is well-defined in the sense that it depends only on $\alpha'(0)$ and $\beta'(0)$, and not on the choice of curves. Moreover, with (13–1), we see that $\tilde{g}$ is smooth. Finally, to check that $\tilde{g}$ is a Riemannian metric, we only need to show that $\tilde{g}(\alpha'(0), \alpha'(0)) = 0$ implies $\alpha'(0) = 0$. This is clear by taking $p'(0) = 0$ in (13–1), which then yields $v'(0) = 0$.

(2) A curve α is contained in a fiber, exactly if $\pi \circ \alpha$ is constant, which happens exactly if $\alpha'(t) \in \operatorname{Ker} \mathrm{d}\pi$ for all t. Hence, the tangent vectors parallel to the fiber are exactly those where $\mathrm{d}\pi(\alpha'(t)) = 0$. Such tangent vectors are those which can be realized as derivatives of paths $(p, w(t))$ where p is a point, and $w(t)$ is a path in T_pM. Since T_pM is a vector space, we have $\dfrac{\mathrm{D}w}{\mathrm{d}t} = w'$. Then, for any curve $(p(t), v(t))$ in TM, its inner product with the tangent vector of $(p, w(t))$ at $t = t_0$ is given by

$$\langle p'(t_0), 0 \rangle + \left\langle \frac{\mathrm{D}v}{\mathrm{d}t}(t_0), w'(t_0) \right\rangle.$$

As $w'(t_0)$ is arbitrary, this is zero for all tangent vectors to the fiber if and only if $\dfrac{\mathrm{D}v}{\mathrm{d}t}(t_0) = 0$.

(3) This follows from (2) since for any geodesic $\gamma(t)$, $\gamma'(t)$ is parallel along $\gamma(t)$.

(4) For a curve $\alpha(t) = (p(t), v(t))$ in TM, we have

$$\begin{aligned} \operatorname{Length}(\alpha) &= \int \left(\langle p'(t), p'(t) \rangle + \left\langle \frac{\mathrm{D}v}{\mathrm{d}t}(t) + \frac{\mathrm{D}v}{\mathrm{d}t}(t) \right\rangle \right)^{\frac{1}{2}} \mathrm{d}t \\ &\geqslant \int \langle p'(t), p'(t) \rangle^{\frac{1}{2}} \mathrm{d}t = \operatorname{Length}(p), \end{aligned}$$

and the equality holds if and only if $\dfrac{\mathrm{D}v}{\mathrm{d}t}(t) \equiv 0$.

Now, suppose that $\overline{\gamma}(t) = (\gamma(t), \gamma'(t))$ is a trajectory of the geodesic field G, and suppose that γ is length-minimizing between $\gamma(0)$ and $\gamma(\varepsilon)$ for some ε. Then, we have

$$\text{Length}(\overline{\gamma}) = \text{Length}(\gamma).$$

For any curve $\alpha(t) = (p(t), v(t))$ in TM joining $\overline{\gamma}(0)$ and $\overline{\gamma}(\varepsilon)$, the curve $p(t) = \pi \circ \alpha(t)$ joins $\gamma(0)$ and $\gamma(\varepsilon)$. Since γ is length-minimizing, we have

$$\text{Length}(\overline{\gamma}) = \text{Length}(\gamma) \leqslant \text{Length}(p) \leqslant \text{Length}(\alpha),$$

so $\overline{\gamma}$ is length-minimizing, which implies that $\overline{\gamma}(t)$ is a geodesic. Since being a geodesic is a local property, we conclude that $\overline{\gamma}(t)$ is a geodesic for all t.

(5) Let $p \in M$ and consider a system $(u_1, \cdots, u_n)$ of normal coordinates in an open neighborhood U of p. The Christoffel symbols all vanish at p in this coordinate system. Therefore for $X = x^i \dfrac{\partial}{\partial u_i}$, we have

$$\operatorname{div} X(p) = \sum_{i=1}^{n} \frac{\partial x^i}{\partial u^i}. \tag{13–2}$$

Now let $(u_1, \cdots, u^n, v^1, \cdots, v^n)$, $v = v^j \dfrac{\partial}{\partial u^j}$ be coordinates on TM at (q, v), where $q \in U$ and $v \in T_q M$. Note that

$$T_{(p,v)}TM \simeq T_v(T_pM) \oplus \pi^{-1}(p) \simeq T_pM \oplus T_pM.$$

Hence the volume element of $\tilde{g}$ on TM at (q, v) is the volume element of the product metric $g \times g$ on $U \times U$ at the point (q, q). Since div(G) depends only on the volume element, and by (3) G is horizontal, we can calculate div(G) in the product metric. Since

$$G(u^i) = v^i, \quad G(v^j) = -\Gamma^j_{ik} v^i v^k,$$

Since the Christoffel symbols of the product metric on $U \times U$ vanish at (p, p), by (13–2), we obtain finally, at p,

$$\operatorname{div}(G) = \sum_{i=1}^{n} \frac{\partial v^i}{\partial u^i} - \sum_{j=1}^{n} \frac{\partial}{\partial v^j} \left(\sum_{i,k=1}^{n} \Gamma^j_{ik} v^i v^k \right) = 0.$$

(6) By taking an orientable double cover, we may assume that TM is orientable. Then for Ω a volume form on TM, we have

$$\mathcal{L}_G \Omega = \operatorname{div}(G)\Omega = 0.$$

Therefore, the geodesic flow preserves the Riemannian volume measure of TM. □

Homework 3

In the following, connections are assumed to be Levi-Civita connections by default.

Exercise 14 Let (G, g) be a Lie group with a bi-invariant metric g. Prove that

$$\mathrm{Rm}(X, Y, Z, W) = \frac{1}{4}\langle [X, Y], [Z, W]\rangle$$

for any $X, Y, Z, W \in \mathfrak{g}$, where the elements of $\mathfrak{g}$ are identified with left-invariant vector fields on G.

Proof Recall from Exercise 12 (1) that $\nabla_Y Z = \frac{1}{2}[Y, Z]$ for any $Y, Z \in \mathfrak{g}$, which implies that $\nabla_Y Z$ is also a left-invariant vector field. Hence, we have

$$0 = X\langle \nabla_Y Z, W\rangle = \langle \nabla_X \nabla_Y Z, W\rangle + \langle \nabla_Y Z, \nabla_X W\rangle, \quad \forall X, Y, Z, W \in \mathfrak{g}.$$

Then

$$\begin{aligned}
\mathrm{Rm}(X, Y, Z, W) &= -\langle \mathrm{Rm}(X, Y)Z, W\rangle \\
&= -\left\langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\right\rangle \\
&= \langle \nabla_Y Z, \nabla_X W\rangle - \langle \nabla_X Z, \nabla_Y W\rangle + \left\langle \nabla_{[X,Y]} Z, W\right\rangle \\
&= \frac{1}{4}\langle [Y, Z], [X, W]\rangle - \frac{1}{4}\langle [X, Z], [Y, W]\rangle + \frac{1}{2}\langle [[X, Y], Z], W\rangle.
\end{aligned}$$

Using $\mathrm{ad}(X)Y = [X, Y]$ and the fact that $\mathrm{ad}^*(X) = -\mathrm{ad}(X)$, we find that

$$\begin{aligned}
\langle [Y, Z], [X, W]\rangle &= \langle [Y, Z], \mathrm{ad}(X)W\rangle = \langle \mathrm{ad}^*(X)[Y, Z], W\rangle \\
&= \langle -\mathrm{ad}(X)[Y, Z], W\rangle = \langle -[X, [Y, Z]], W\rangle \\
&= \langle [[Y, Z], X], W\rangle,
\end{aligned}$$

and similarly

$$\langle [X, Z], [Y, W]\rangle = \langle [[X, Z], Y], W\rangle.$$

Therefore, by the Jacobi identity, we have

$$\begin{aligned}
\mathrm{Rm}(X, Y, Z, W) &= \frac{1}{4}\langle [[Y, Z], X], W\rangle - \frac{1}{4}\langle [[X, Z], Y], W\rangle + \frac{1}{2}\langle [[X, Y], Z], W\rangle \\
&= \frac{1}{4}\langle [[Y, Z], X], W\rangle + \frac{1}{4}\langle [[Z, X], Y], W\rangle + \frac{1}{2}\langle [[X, Y], Z], W\rangle \\
&= -\frac{1}{4}\langle [[X, Y], Z], W\rangle + \frac{1}{2}\langle [[X, Y], Z], W\rangle + \frac{1}{4}\langle [[X, Y], Z], W\rangle \\
&= \frac{1}{4}\langle [[X, Y], Z], W\rangle \\
&= \frac{1}{4}\langle [X, Y], [Z, W]\rangle.
\end{aligned}$$

□

Exercise 15 Recall

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \bar{z} \end{pmatrix} : (z, w) \in \mathbb{C}^2 \text{ and } |z|^2 + |w|^2 = 1 \right\}.$$

Let $\{X_1, X_2, X_3\}$ be a basis of the Lie algebra $\mathfrak{su}(2)$ defines as

$$X_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad X_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the basis of left-invariant 1-forms dual to $\{X_1, X_2, X_3\}$. Define a left-invariant metric

$$g = \varepsilon^2 \sigma_1^2 + \sigma_2^2 + \sigma_3^2,$$

where $\varepsilon \in (0, 1)$ is a small constant.

(1) Prove that this basis satisfies the commutation relations

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

(2) Prove that the connection satisfies

$$\nabla_X Y = \frac{1}{2}([X, Y] - \mathrm{ad}^*(X)Y - \mathrm{ad}^*(Y)X)$$

for any $X, Y \in \mathfrak{su}(2)$, where ad^* is the adjoint of ad.

(3) Compute the sectional curvatures $K(X_1 \wedge X_2)$, $K(X_2 \wedge X_3)$ and $K(X_3 \wedge X_1)$.

Proof (1) Since the Lie bracket on $\mathfrak{su}(2)$ is given by the matrix commutator, we compute

$$\begin{aligned} [X_1, X_2] &= \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = 2X_3, \\ [X_2, X_3] &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2X_1, \\ [X_3, X_1] &= \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} - \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = 2X_2. \end{aligned}$$

(2) Since g is left-invariant, the inner product of any two left-invariant vector fields is constant. In particular, Koszul's formula simplifies to

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle), \quad \forall X, Y, Z \in \mathfrak{su}(2).$$

Using the fact that $\mathrm{ad}(X)Y = [X, Y]$, we have

$$\begin{aligned} \langle \nabla_X Y, Z \rangle &= \frac{1}{2}(\langle [X, Y], Z \rangle - \langle \mathrm{ad}(X)Z, Y \rangle - \langle \mathrm{ad}(Y)Z, X \rangle) \\ &= \frac{1}{2}(\langle [X, Y], Z \rangle - \langle \mathrm{ad}^*(X)Y, Z \rangle - \langle \mathrm{ad}^*(Y)X, Z \rangle) \\ &= \frac{1}{2}\langle [X, Y] - \mathrm{ad}^*(X)Y - \mathrm{ad}^*(Y)X, Z \rangle. \end{aligned}$$

Since $Z \in \mathfrak{su}(2)$ is arbitrary, we obtain the desired formula.

(3) The inner products between the basis vectors are given by

$$\langle X_1, X_1 \rangle = \varepsilon^2, \quad \langle X_2, X_2 \rangle = 1, \quad \langle X_3, X_3 \rangle = 1,$$

$$\langle X_1, X_2\rangle = \langle X_2, X_3\rangle = \langle X_3, X_1\rangle = 0.$$

Then, to use the result from (2), we need to compute $\text{ad}^*(X_i)X_j$. With the help of the commutation relations from (1), we obtain

$$\begin{aligned}
\langle \text{ad}^*(X_1)X_2, X_1\rangle &= \langle X_2, \text{ad}(X_1)X_1\rangle = \langle X_2, [X_1, X_1]\rangle = 0,\\
\langle \text{ad}^*(X_1)X_2, X_2\rangle &= \langle X_2, \text{ad}(X_1)X_2\rangle = \langle X_2, [X_1, X_2]\rangle = \langle X_2, 2X_3\rangle = 0,\\
\langle \text{ad}^*(X_1)X_2, X_3\rangle &= \langle X_2, \text{ad}(X_1)X_3\rangle = \langle X_2, [X_1, X_3]\rangle = \langle X_2, -2X_2\rangle = -2,\\
\langle \text{ad}^*(X_2)X_1, X_1\rangle &= \langle X_1, \text{ad}(X_2)X_1\rangle = \langle X_1, [X_2, X_1]\rangle = \langle X_1, -2X_3\rangle = 0,\\
\langle \text{ad}^*(X_2)X_1, X_2\rangle &= \langle X_1, \text{ad}(X_2)X_2\rangle = \langle X_1, [X_2, X_2]\rangle = 0,\\
\langle \text{ad}^*(X_2)X_1, X_3\rangle &= \langle X_1, \text{ad}(X_2)X_3\rangle = \langle X_1, [X_2, X_3]\rangle = \langle X_1, 2X_1\rangle = 2\varepsilon^2,\\
\langle \text{ad}^*(X_2)X_3, X_1\rangle &= \langle X_3, \text{ad}(X_2)X_1\rangle = \langle X_3, [X_2, X_1]\rangle = \langle X_3, -2X_3\rangle = -2,\\
\langle \text{ad}^*(X_2)X_3, X_2\rangle &= \langle X_3, \text{ad}(X_2)X_2\rangle = \langle X_3, [X_2, X_2]\rangle = 0,\\
\langle \text{ad}^*(X_2)X_3, X_3\rangle &= \langle X_3, \text{ad}(X_2)X_3\rangle = \langle X_3, [X_2, X_3]\rangle = \langle X_3, 2X_1\rangle = 0,\\
\langle \text{ad}^*(X_3)X_2, X_1\rangle &= \langle X_2, \text{ad}(X_3)X_1\rangle = \langle X_2, [X_3, X_1]\rangle = \langle X_2, 2X_2\rangle = 2,\\
\langle \text{ad}^*(X_3)X_2, X_2\rangle &= \langle X_2, \text{ad}(X_3)X_2\rangle = \langle X_2, [X_3, X_2]\rangle = \langle X_2, -2X_1\rangle = 0,\\
\langle \text{ad}^*(X_3)X_2, X_3\rangle &= \langle X_2, \text{ad}(X_3)X_3\rangle = \langle X_2, [X_3, X_3]\rangle = 0,\\
\langle \text{ad}^*(X_3)X_1, X_1\rangle &= \langle X_1, \text{ad}(X_3)X_1\rangle = \langle X_1, [X_3, X_1]\rangle = \langle X_1, 2X_2\rangle = 0,\\
\langle \text{ad}^*(X_3)X_1, X_2\rangle &= \langle X_1, \text{ad}(X_3)X_2\rangle = \langle X_1, [X_3, X_2]\rangle = \langle X_1, -2X_1\rangle = -2\varepsilon^2,\\
\langle \text{ad}^*(X_3)X_1, X_3\rangle &= \langle X_1, \text{ad}(X_3)X_3\rangle = \langle X_1, [X_3, X_3]\rangle = 0,\\
\langle \text{ad}^*(X_1)X_3, X_1\rangle &= \langle X_3, \text{ad}(X_1)X_1\rangle = \langle X_3, [X_1, X_1]\rangle = 0,\\
\langle \text{ad}^*(X_1)X_3, X_2\rangle &= \langle X_3, \text{ad}(X_1)X_2\rangle = \langle X_3, [X_1, X_2]\rangle = \langle X_3, 2X_3\rangle = 2,\\
\langle \text{ad}^*(X_1)X_3, X_3\rangle &= \langle X_3, \text{ad}(X_1)X_3\rangle = \langle X_3, [X_1, X_3]\rangle = \langle X_3, -2X_2\rangle = 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\text{ad}^*(X_1)X_2 = -2X_3, &\quad \text{ad}^*(X_2)X_1 = 2\varepsilon^2 X_3,\\
\text{ad}^*(X_2)X_3 = -\frac{2}{\varepsilon^2}X_1, &\quad \text{ad}^*(X_3)X_2 = \frac{2}{\varepsilon^2}X_1,\\
\text{ad}^*(X_3)X_1 = -2\varepsilon^2 X_2, &\quad \text{ad}^*(X_1)X_3 = 2X_2.
\end{aligned}$$

Also, it is easy to see that

$$\text{ad}^*(X_1)X_1 = \text{ad}^*(X_2)X_2 = \text{ad}^*(X_3)X_3 = 0.$$

It then follows by part (2) that

$$\begin{aligned}
\nabla_{X_1}X_2 &= \frac{1}{2}([X_1, X_2] - \text{ad}^*(X_1)X_2 - \text{ad}^*(X_2)X_1)\\
&= \frac{1}{2}\left[2X_3 - (-2X_3) - 2\varepsilon^2 X_3\right] = \left(2 - \varepsilon^2\right)X_3,\\
\nabla_{X_2}X_1 &= \frac{1}{2}([X_2, X_1] - \text{ad}^*(X_2)X_1 - \text{ad}^*(X_1)X_2)\\
&= \frac{1}{2}\left[-2X_3 - 2\varepsilon^2 X_3 - (-2X_3)\right] = -\varepsilon^2 X_3,
\end{aligned}$$

$$\begin{aligned}
\nabla_{X_2}X_3 &= \frac{1}{2}([X_2,X_3] - \mathrm{ad}^*(X_2)X_3 - \mathrm{ad}^*(X_3)X_2) \\
&= \frac{1}{2}\left[2X_1 - \left(-\frac{2}{\varepsilon^2}X_1\right) - \frac{2}{\varepsilon^2}X_1\right] = X_1, \\
\nabla_{X_3}X_2 &= \frac{1}{2}([X_3,X_2] - \mathrm{ad}^*(X_3)X_2 - \mathrm{ad}^*(X_2)X_3) \\
&= \frac{1}{2}\left[-2X_1 - \frac{2}{\varepsilon^2}X_1 - \left(-\frac{2}{\varepsilon^2}X_1\right)\right] = -X_1, \\
\nabla_{X_3}X_1 &= \frac{1}{2}([X_3,X_1] - \mathrm{ad}^*(X_3)X_1 - \mathrm{ad}^*(X_1)X_3) \\
&= \frac{1}{2}\left[2X_2 - \left(-2\varepsilon^2 X_2\right) - 2X_2\right] = \varepsilon^2 X_2, \\
\nabla_{X_1}X_3 &= \frac{1}{2}([X_1,X_3] - \mathrm{ad}^*(X_1)X_3 - \mathrm{ad}^*(X_3)X_1) \\
&= \frac{1}{2}\left[-2X_2 - 2X_2 - (-2\varepsilon^2 X_2)\right] = \left(\varepsilon^2 - 2\right)X_2,
\end{aligned}$$

and

$$\nabla_{X_i}X_i = \frac{1}{2}([X_i,X_i] - 2\,\mathrm{ad}^*(X_i)X_i) = 0, \quad i = 1,2,3.$$

Now we can compute $\mathrm{Rm}(X_i)X_j$ as follows:

$$\begin{aligned}
\mathrm{Rm}(X_1,X_2)X_1 &= \nabla_{X_1}\nabla_{X_2}X_1 - \nabla_{X_2}\nabla_{X_1}X_1 - \nabla_{[X_1,X_2]}X_1 \\
&= \nabla_{X_1}\left(-\varepsilon^2 X_3\right) - 0 - \nabla_{2X_3}X_1 \\
&= -\varepsilon^2\left(\varepsilon^2 - 2\right)X_2 - 2\varepsilon^2 X_2 \\
&= -\varepsilon^4 X_2, \\
\mathrm{Rm}(X_2,X_3)X_2 &= \nabla_{X_2}\nabla_{X_3}X_2 - \nabla_{X_3}\nabla_{X_2}X_2 - \nabla_{[X_2,X_3]}X_2 \\
&= \nabla_{X_2}(-X_1) - 0 - \nabla_{2X_1}X_2 \\
&= \varepsilon^2 X_3 - 2\left(2 - \varepsilon^2\right)X_3 \\
&= \left(3\varepsilon^2 - 4\right)X_3, \\
\mathrm{Rm}(X_3,X_1)X_3 &= \nabla_{X_3}\nabla_{X_1}X_3 - \nabla_{X_1}\nabla_{X_3}X_3 - \nabla_{[X_3,X_1]}X_3 \\
&= \nabla_{X_3}\left(\left(\varepsilon^2 - 2\right)X_2\right) - 0 - \nabla_{2X_2}X_3 \\
&= \left(\varepsilon^2 - 2\right)(-X_1) - 2X_1 \\
&= -\varepsilon^2 X_1.
\end{aligned}$$

Finally, we compute the sectional curvatures:

$$\begin{aligned}
K(X_1 \wedge X_2) &= \frac{\mathrm{Rm}(X_1,X_2,X_1,X_2)}{\langle X_1,X_1\rangle\langle X_2,X_2\rangle - \langle X_1,X_2\rangle^2} \\
&= \frac{-\langle \mathrm{Rm}(X_1,X_2)X_1,X_2\rangle}{\langle X_1,X_1\rangle\langle X_2,X_2\rangle - \langle X_1,X_2\rangle^2} \\
&= \frac{\varepsilon^4}{\varepsilon^2} = \varepsilon^2, \\
K(X_2 \wedge X_3) &= \frac{\mathrm{Rm}(X_2,X_3,X_2,X_3)}{\langle X_2,X_2\rangle\langle X_3,X_3\rangle - \langle X_2,X_3\rangle^2} \\
&= \frac{-\langle \mathrm{Rm}(X_2,X_3)X_2,X_3\rangle}{\langle X_2,X_2\rangle\langle X_3,X_3\rangle - \langle X_2,X_3\rangle^2}
\end{aligned}$$

$$
\begin{aligned}
&= 4 - 3\varepsilon^2, \\
K(X_3 \wedge X_1) &= \frac{\mathrm{Rm}(X_3, X_1, X_3, X_1)}{\langle X_3, X_3\rangle\langle X_1, X_1\rangle - \langle X_3, X_1\rangle^2} \\
&= \frac{-\langle \mathrm{Rm}(X_3, X_1)X_3, X_1\rangle}{\langle X_3, X_3\rangle\langle X_1, X_1\rangle - \langle X_3, X_1\rangle^2} \\
&= \frac{\varepsilon^4}{\varepsilon^2} = \varepsilon^2.
\end{aligned}
$$
□

Exercise 16 Given a Riemannian manifold (M^n, g), we consider the metric $\tilde{g} = e^{-2f}g$, where f is a smooth function on M. The metric $\tilde{g}$ is said to be *conformal* to g. Prove the following statements:

(1) The Christoffel symbols $\widetilde{\Gamma}_{ij}^k$ of $\tilde{g}$ satisfy

$$\widetilde{\Gamma}_{ij}^k = g^{kl}[-(\partial_i f)g_{jl} - (\partial_j f)g_{il} + (\partial_l f)g_{ij}] + \Gamma_{ij}^k.$$

(2) The curvature operator $\widetilde{\mathrm{Rm}}$ of $\tilde{g}$ as a $(0,4)$-tensor satisfies

$$\widetilde{\mathrm{Rm}} = e^{-2f}\left\{\mathrm{Rm} + \left(\nabla^2 f + \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{2}|\mathrm{grad}\, f|^2 g\right) \owedge g\right\}.$$

(3) The Ricci curvature $\widetilde{\mathrm{Ric}}$ of $\tilde{g}$ satisfies

$$\widetilde{\mathrm{Ric}} = (n-2)\left(\nabla^2 f + \frac{1}{n-2}(\Delta f)g + \mathrm{d}f \otimes \mathrm{d}f - |\mathrm{grad}\, f|^2 g\right) + \mathrm{Ric}\,.$$

(4) The scalar curvature $\widetilde{R}$ of $\tilde{g}$ satisfies

$$\widetilde{R} = e^{2f}\left\{(2n-2)\Delta f - (n-1)(n-2)|\mathrm{grad}\, f|^2 + R\right\}.$$

(5) The Weyl curvature tensor $\widetilde{W}$ of $\tilde{g}$ satisfies

$$\widetilde{W} = e^{-2f}W.$$

Proof (1) We have

$$
\begin{aligned}
\widetilde{\Gamma}_{ij}^k &= \frac{1}{2}\tilde{g}^{kl}(\partial_i \tilde{g}_{jl} + \partial_j \tilde{g}_{il} - \partial_l \tilde{g}_{ij}) \\
&= \frac{1}{2}\left(e^{2f}g^{kl}\right)\left[\partial_i\left(e^{-2f}g_{jl}\right) + \partial_j\left(e^{-2f}g_{il}\right) - \partial_l\left(e^{-2f}g_{ij}\right)\right] \\
&= \frac{1}{2}g^{kl}[-2(\partial_i f)g_{jl} + \partial_i g_{jl} - 2(\partial_j f)g_{il} + \partial_j g_{il} + 2(\partial_l f)g_{ij} + \partial_l g_{ij}] \\
&= g^{kl}[-(\partial_i f)g_{jl} - (\partial_j f)g_{il} + (\partial_l f)g_{ij}] + \Gamma_{ij}^k.
\end{aligned}
$$

(2) If we denote $f_{;i} = \partial_i f$ and $f_{;ij} = \partial_j\partial_i f$, then the formula obtained in (1) can be rewritten as

$$\widetilde{\Gamma}_{ij}^k = -f_{;i}\delta_j^k - f_{;j}\delta_i^k + g^{kl}f_{;l}g_{ij} + \Gamma_{ij}^k.$$

We can make the computations much more tractable by computing the components of the tensors at a point $p \in M$ in normal coordinates for g centered at p, so that the equations $g_{ij} = \delta_{ij}$, $\partial_k g_{ij} = 0$, and $\Gamma_{ij}^k = 0$ hold at p. This has the following consequences at p:

$$\begin{aligned}
f_{;ij} &= \partial_j \partial_i f, \\
\widetilde{\Gamma}_{ij}^k &= -f_{;i}\delta_j^k - f_{;j}\delta_i^k + g^{kl} f_{;l} g_{ij}, \\
\partial_m \widetilde{\Gamma}_{ij}^k &= -f_{;im}\delta_j^k - f_{;jm}\delta_i^k + g^{kl} f_{;lm} g_{ij} + \partial_m \Gamma_{ij}^k, \\
R_{ijk}{}^l &= \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l.
\end{aligned}$$

Inserting these relations, we obtain

$$\begin{aligned}
\widetilde{R}_{ijkl} = &- e^{-2f} g_{lm} \Big(\partial_i \widetilde{\Gamma}_{jk}^m - \partial_j \widetilde{\Gamma}_{ik}^m + \widetilde{\Gamma}_{jk}^p \widetilde{\Gamma}_{ip}^m - \widetilde{\Gamma}_{ik}^p \widetilde{\Gamma}_{jp}^m \Big) \\
= &\, e^{-2f} g_{lm} \Big\{ -\big(-f_{;ji}\delta_k^m - f_{;ki}\delta_j^m + g^{mq} f_{;qi} g_{jk} + \partial_i \Gamma_{jk}^m\big) \\
&\qquad + \big(-f_{;ij}\delta_k^m - f_{;kj}\delta_i^m + g^{mq} f_{;qj} g_{ik} + \partial_j \Gamma_{ik}^m\big) \\
&\qquad - \big(-f_{;j}\delta_k^p - f_{;k}\delta_j^p + g^{pq} f_{;q} g_{jk}\big)\big(-f_{;i}\delta_p^m - f_{;p}\delta_i^m + g^{mr} f_{;r} g_{ip}\big) \\
&\qquad + \big(-f_{;i}\delta_k^p - f_{;k}\delta_i^p + g^{pq} f_{;q} g_{ik}\big)\big(-f_{;j}\delta_p^m - f_{;p}\delta_j^m + g^{mr} f_{;r} g_{jp}\big) \Big\} \\
= &\, e^{-2f} g_{lm} \Big\{ \big(f_{;ki}\delta_j^m - f_{;kj}\delta_i^m - g^{mq} f_{;qi} g_{jk} + g^{mq} f_{;qj} g_{ik} - R_{ijk}{}^m \big) \\
&\qquad - \big(f_{;j} f_{;i} \delta_k^p \delta_p^m + f_{;j} f_{;p} \delta_k^p \delta_i^m + f_{;k} f_{;i} \delta_j^p \delta_p^m + f_{;k} f_{;p} \delta_j^p \delta_i^m \big) \\
&\qquad + \big(f_{;i} f_{;j} \delta_k^p \delta_p^m + f_{;i} f_{;p} \delta_k^p \delta_j^m + f_{;k} f_{;j} \delta_i^p \delta_p^m + f_{;k} f_{;p} \delta_i^p \delta_j^m \big) \\
&\qquad + \big(g^{pq} f_{;q} f_{;i} g_{jk} \delta_p^m + g^{pq} f_{;q} f_{;p} g_{jk} \delta_i^m - g^{pq} f_{;q} f_{;j} g_{ik} \delta_p^m - g^{pq} f_{;q} f_{;p} g_{ik} \delta_j^m \big) \\
&\qquad + \big(g^{mr} f_{;r} f_{;j} g_{ip} \delta_k^p + g^{mr} f_{;r} f_{;k} g_{ip} \delta_j^p - g^{mr} f_{;r} f_{;i} g_{jp} \delta_k^p - g^{mr} f_{;r} f_{;k} g_{jp} \delta_i^p \big) \\
&\qquad + \big(-f_{;q} f_{;r} g^{pq} g^{mr} g_{jk} g_{ip} + f_{;q} f_{;r} g^{pq} g^{mr} g_{ik} g_{jp} \big) \Big\} \\
= &\, e^{-2f} \Big\{ \big(f_{;ik} g_{jl} - f_{;jk} g_{il} - f_{;il} g_{jk} + f_{;jl} g_{ik} + R_{ijkl} \big) \\
&\qquad - \big(f_{;i} f_{;j} g_{kl} + f_{;j} f_{;k} g_{il} + f_{;i} f_{;k} g_{jl} + f_{;j} f_{;k} g_{il} \big) \\
&\qquad + \big(f_{;i} f_{;j} g_{kl} + f_{;i} f_{;k} g_{jl} + f_{;j} f_{;k} g_{il} + f_{;i} f_{;k} g_{jl} \big) \\
&\qquad + \big(f_{;i} f_{;l} g_{jk} + g^{pq} f_{;p} f_{;q} g_{il} g_{jk} - f_{;j} f_{;l} g_{ik} - g^{pq} f_{;p} f_{;q} g_{jl} g_{ik} \big) \\
&\qquad + \big(f_{;j} f_{;l} g_{ik} + f_{;k} f_{;l} g_{ij} - f_{;i} f_{;l} g_{jk} - f_{;k} f_{;l} g_{ij} \big) \\
&\qquad + \big(-f_{;i} f_{;l} g_{jk} + f_{;j} f_{;l} g_{ik} \big) \Big\} \\
= &\, e^{-2f} \Big\{ R_{ijkl} + \big(f_{;ik} g_{jl} + f_{;jl} g_{ik} - f_{;il} g_{jk} - f_{;jk} g_{il} \big) \\
&\qquad + \big(f_{;i} f_{;k} g_{jl} + f_{;j} f_{;l} g_{ik} - f_{;i} f_{;l} g_{jk} - f_{;j} f_{;k} g_{il} \big) \\
&\qquad - g^{pq} f_{;p} f_{;q} \big(g_{ik} g_{jl} - g_{il} g_{jk} \big) \Big\},
\end{aligned}$$

which is the coordinate version of

$$\widetilde{\mathrm{Rm}} = e^{-2f} \left\{ \mathrm{Rm} + \left(\nabla^2 f + \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{2} |\mathrm{grad}\, f|^2 g \right) \owedge g \right\}.$$

(3) Let tr_g denote the trace operation (with respect to g) on the second and last indices. The compo-

nents of $\widetilde{\mathrm{Ric}}$ are given by

$$\begin{aligned}\widetilde{R}_{ik} &= \tilde{g}^{jl}\widetilde{R}_{ijkl}\\ &= g^{jl}\Big\{R_{ijkl} + (f_{;ik}g_{jl} + f_{;jl}g_{ik} - f_{;il}g_{jk} - f_{;jk}g_{il})\\ &\qquad + (f_{;i}f_{;k}g_{jl} + f_{;j}f_{;l}g_{ik} - f_{;i}f_{;l}g_{jk} - f_{;j}f_{;k}g_{il})\\ &\qquad - g^{pq}f_{;p}f_{;q}(g_{ik}g_{jl} - g_{il}g_{jk})\Big\}.\end{aligned} \tag{16–1}$$

This implies that

$$\begin{aligned}\widetilde{\mathrm{Ric}} &= \mathrm{tr}_g\left\{\mathrm{Rm} + \left(\nabla^2 f + \mathrm{d}f\otimes\mathrm{d}f - \frac{1}{2}|\mathrm{grad}\, f|^2 g\right) \owedge g\right\}\\ &= \mathrm{tr}_g(\mathrm{Rm}) + \mathrm{tr}_g(\nabla^2 f \owedge g) + \mathrm{tr}_g\{(\mathrm{d}f\otimes\mathrm{d}f)\owedge g\} - \frac{1}{2}|\mathrm{grad}\, f|^2\,\mathrm{tr}(g\owedge g)\\ &= \mathrm{Ric} + (n-2)\nabla^2 f + [\mathrm{tr}_g(\nabla^2 f)]g + (n-2)\,\mathrm{d}f\otimes\mathrm{d}f + [\mathrm{tr}_g(\mathrm{d}f\otimes\mathrm{d}f)]g - (n-1)|\mathrm{grad}\, f|^2 g\\ &= \mathrm{Ric} + (n-2)\nabla^2 f + (\Delta f)g + (n-2)\,\mathrm{d}f\otimes\mathrm{d}f - (n-2)|\mathrm{grad}\, f|^2 g\\ &= (n-2)\left(\nabla^2 f + \frac{1}{n-2}(\Delta f)g + \mathrm{d}f\otimes\mathrm{d}f - |\mathrm{grad}\, f|^2 g\right) + \mathrm{Ric}\,.\end{aligned}$$

(4) From (16–1) we see that

$$\begin{aligned}\widetilde{R} &= \tilde{g}^{ik}\widetilde{R}_{ik}\\ &= e^{2f}g^{ik}g^{jl}\Big\{R_{ijkl} + (f_{;ik}g_{jl} + f_{;jl}g_{ik} - f_{;il}g_{jk} - f_{;jk}g_{il})\\ &\qquad + (f_{;i}f_{;k}g_{jl} + f_{;j}f_{;l}g_{ik} - f_{;i}f_{;l}g_{jk} - f_{;j}f_{;k}g_{il})\\ &\qquad - g^{pq}f_{;p}f_{;q}(g_{ik}g_{jl} - g_{il}g_{jk})\Big\},\end{aligned}$$

which implies that

$$\begin{aligned}\widetilde{R} &= e^{2f}\Big\{(n-2)\,\mathrm{tr}_g(\nabla^2 f) + (\Delta f)\,\mathrm{tr}_g\, g + (n-2)\,\mathrm{tr}_g(\mathrm{d}f\otimes\mathrm{d}f) - (n-2)|\mathrm{grad}\, f|^2\,\mathrm{tr}_g\, g + \mathrm{tr}_g(\mathrm{Ric})\Big\}\\ &= e^{2f}\Big\{(n-2)\Delta f + n\Delta f + (n-2)|\mathrm{grad}\, f|^2 - n(n-2)|\mathrm{grad}\, f|^2 + R\Big\}\\ &= e^{2f}\Big\{(2n-2)\Delta f - (n-1)(n-2)|\mathrm{grad}\, f|^2 + R\Big\}.\end{aligned}$$

(5) By the definition of the Weyl curvature tensor, we have for $n \geqslant 3$

$$\begin{aligned}\widetilde{W} &= \widetilde{\mathrm{Rm}} - \frac{1}{n-2}\widetilde{\mathrm{Ric}}\owedge\tilde{g} + \frac{\widetilde{R}}{2(n-1)(n-2)}\tilde{g}\owedge\tilde{g}\\ &= e^{-2f}\left\{\mathrm{Rm} + \left(\nabla^2 f + \mathrm{d}f\otimes\mathrm{d}f - \frac{1}{2}|\mathrm{grad}\, f|^2 g\right)\owedge g\right\}\\ &\quad - \frac{1}{n-2}\left\{(n-2)\left(\nabla^2 f + \frac{1}{n-2}(\Delta f)g + \mathrm{d}f\otimes\mathrm{d}f - |\mathrm{grad}\, f|^2 g\right) + \mathrm{Ric}\right\}\owedge(e^{-2f}g)\\ &\quad + \frac{e^{2f}\Big\{(2n-2)\Delta f - (n-1)(n-2)|\mathrm{grad}\, f|^2 + R\Big\}}{2(n-1)(n-2)}(e^{-2f}g)\owedge(e^{-2f}g)\\ &= e^{-2f}\left\{\mathrm{Rm} - \frac{1}{n-2}\,\mathrm{Ric}\owedge g + \frac{R}{2(n-1)(n-2)}g\owedge g\right\}\end{aligned}$$

$$= e^{-2f} W.$$ □

Exercise 17 Consider the hyperbolic space

$$\mathbb{H}^n = \left\{ \left(x^1, \cdots, x^n\right) \in \mathbb{R}^n : x^n > 0 \right\},$$

equipped with the metric

$$g_{\mathbb{H}^n} = \frac{1}{(x^n)^2} \left(\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots + \mathrm{d}x^n \otimes \mathrm{d}x^n \right).$$

Prove that $g_{\mathbb{H}^n}$ has constant sectional curvature -1.

Proof Since $g_{\mathbb{H}^n} = \frac{1}{(x^n)^2} g_E$, where g_E is the Euclidean metric, we can apply the result of Exercise 16 (2) to compute the Riemann curvature tensor Rm of $g_{\mathbb{H}^n}$. Set $f = \ln(x^n)$. Then $g_{\mathbb{H}^n} = e^{-2f} g_E$, and

$$\nabla^2 f = -\frac{1}{(x^n)^2} \mathrm{d}x^n \otimes \mathrm{d}x^n, \quad \mathrm{d}f \otimes \mathrm{d}f = \frac{1}{(x^n)^2} \mathrm{d}x^n \otimes \mathrm{d}x^n, \quad |\mathrm{grad}\, f|^2 = \frac{1}{(x^n)^2}.$$

Given any point $p \in \mathbb{H}^n$ and any 2-dimensional linear subspace σ of T_pM, if $\{X, Y\}$ is any basis of σ, then the Riemann curvature tensor Rm of $g_{\mathbb{H}^n}$ is given by

$$\begin{aligned}
\mathrm{Rm} &= \frac{1}{(x^n)^2} \left\{ 0 + \left(-\frac{1}{(x^n)^2} \mathrm{d}x^n \otimes \mathrm{d}x^n + \frac{1}{(x^n)^2} \mathrm{d}x^n \otimes \mathrm{d}x^n - \frac{1}{2} \frac{1}{(x^n)^2} g_E \right) ⃝\!\!\!\wedge\, g_E \right\} \\
&= -\frac{1}{2(x^n)^4} (g_E ⃝\!\!\!\wedge\, g_E),
\end{aligned}$$

which implies that

$$K_p(\sigma) = \frac{\mathrm{Rm}(X,Y,X,Y)}{\frac{1}{2}(g_{\mathbb{H}^n} ⃝\!\!\!\wedge\, g_{\mathbb{H}^n})(X,Y,X,Y)} = \frac{-\frac{1}{2(x^n)^4}(g_E ⃝\!\!\!\wedge\, g_E)(X,Y,X,Y)}{\frac{1}{2(x^n)^4}(g_E ⃝\!\!\!\wedge\, g_E)(X,Y,X,Y)} = -1.$$

Therefore, $g_{\mathbb{H}^n}$ has constant sectional curvature -1. □

Exercise 18 **(Bochner's formula)** Let (M^n, g) be a Riemannian manifold. For any smooth function $u \colon M \to \mathbb{R}$, prove the following identity:

$$\frac{1}{2}\Delta|\mathrm{grad}\, u|^2 = \left|\nabla^2 u\right|^2 + \mathrm{Ric}(\mathrm{grad}\, u, \mathrm{grad}\, u) + \langle \mathrm{grad}(\Delta u), \mathrm{grad}\, u \rangle.$$

Proof We can make the computations much more tractable by computing the components of the tensors at a point $p \in M$ in normal coordinates centered at p, so that the equations $g_{ij} = \delta_{ij}$, $\partial_k g_{ij} = 0$, and $\Gamma_{ij}^k = 0$ hold at p. This has the following consequence at p:

$$\begin{aligned}
\frac{1}{2}\Delta|\mathrm{grad}\, u|^2 &= \frac{1}{2} g^{kl} \left(g^{ij} u_i u_j \right)_{kl} \\
&= \frac{1}{2} g^{kl} g^{ij} (u_{i;kl} u_j + u_{i;k} u_{j;l} + u_{i;l} u_{j;k} + u_i u_{j;kl}) \\
&= g^{kl} g^{ij} u_{i;k} u_{j;l} + g^{kl} g^{ij} u_{i;kl} u_j \\
&= \left|\nabla^2 u\right|^2 + g^{kl} g^{ij} u_{i;kl} u_j \\
&= \left|\nabla^2 u\right|^2 + g^{kl} g^{ij} u_{k;il} u_j.
\end{aligned} \tag{18–1}$$

Recall that the covariant derivative of every smooth 1-form β can be computed by

$$(\nabla_X\beta)(Y) = X(\beta(Y)) - \beta(\nabla_X Y). \tag{18–2}$$

Using this repeatedly, we compute

$$\begin{aligned}(\nabla_X\nabla_Y\beta)(Z) &= X((\nabla_Y\beta)(Z)) - (\nabla_Y\beta)(\nabla_X Z)\\ &= X(Y(\beta(Z)) - \beta(\nabla_Y Z)) - (\nabla_Y\beta)(\nabla_X Z)\\ &= XY(\beta(Z)) - (\nabla_X\beta)(\nabla_Y Z) - \beta(\nabla_X\nabla_Y Z) - (\nabla_Y\beta)(\nabla_X Z).\end{aligned} \tag{18–3}$$

Reversing the roles of X and Y, we get

$$(\nabla_Y\nabla_X\beta)(Z) = YX(\beta(Z)) - (\nabla_Y\beta)(\nabla_X Z) - \beta(\nabla_Y\nabla_X Z) - (\nabla_X\beta)(\nabla_Y Z), \tag{18–4}$$

and applying (18–2) one more time yields

$$\big(\nabla_{[X,Y]}\beta\big)(Z) = [X,Y](\beta(Z)) - \beta\big(\nabla_{[X,Y]}Z\big). \tag{18–5}$$

Now subtract (18–4) and (18–5) from (18–3): all but three of the terms cancel, yielding

$$\begin{aligned}\big(\nabla_X\nabla_Y\beta - \nabla_Y\nabla_X\beta - \nabla_{[X,Y]}\beta\big)(Z) &= -\beta\big(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\big)\\ &= -\beta(\mathrm{Rm}(X,Y)Z).\end{aligned} \tag{18–6}$$

Since

$$\begin{aligned}\nabla^2_{X,Y}\beta &= \nabla_X\nabla_Y\beta - \nabla_{\nabla_X Y}\beta,\\ \nabla^2_{Y,X}\beta &= \nabla_Y\nabla_X\beta - \nabla_{\nabla_Y X}\beta,\end{aligned}$$

we see that (18–6) is equivalent to

$$\nabla^2_{X,Y}\beta - \nabla^2_{Y,X}\beta = -\,\mathrm{Rm}(X,Y)^*\beta, \tag{18–7}$$

where $\mathrm{Rm}(X,Y)^*\colon T^*M \to T^*M$ denotes the dual map to $\mathrm{Rm}(X,Y)$, defined by

$$(\mathrm{Rm}(X,Y)^*\eta)(Z) = \eta(\mathrm{Rm}(X,Y)Z).$$

In terms of any local frame, the component version of (18–7) reads

$$\beta_{j;pq} - \beta_{j;qp} = R_{pqj}{}^m\beta_m, \tag{18–8}$$

where we use a semicolon to separate indices resulting from (covariant) differentiation from the preceding indices. Now, we apply (18–8) to the 1-form grad u to obtain

$$\begin{aligned}g^{kl}g^{ij}u_{k;il}u_j &= g^{kl}g^{ij}(u_{k;li} - R_{lik}{}^m u_m)u_j\\ &= g^{ij}\big(g^{kl}u_{k;l}\big)_{;i}u_j + g^{kl}g^{ij}R_{ilk}{}^m u_m u_j\\ &= \langle \mathrm{grad}(\Delta u), \mathrm{grad}\, u\rangle + g^{ij}R_i{}^m u_m u_j\\ &= \langle \mathrm{grad}(\Delta u), \mathrm{grad}\, u\rangle + \mathrm{Ric}(\mathrm{grad}\, u, \mathrm{grad}\, u).\end{aligned} \tag{18–9}$$

Combining (18–1) and (18–9), we obtain

$$\frac{1}{2}\Delta|\text{grad}\, u|^2 = \left|\nabla^2 u\right|^2 + \text{Ric}(\text{grad}\, u, \text{grad}\, u) + \langle \text{grad}(\Delta u), \text{grad}\, u\rangle. \qquad \square$$

Exercise 19 Given a Riemannian manifold (N^{n-1}, h), we consider the warped product metric $g = dr^2 + f^2(r)h$ on $M = (0, +\infty) \times N$, where $f(r)\colon (0, +\infty) \to \mathbb{R}$ is a positive smooth function. In the following, we use indices i, j, k, l to denote the local coordinates on N. Superscripts g and h will be used to indicate the quantities computed with respect to the metrics g and h, respectively.

Prove the following statements:

(1) $R^g_{ijkl} = f^2(r)R^h_{ijkl} - f^2(r)[f'(r)]^2(h_{ik}h_{jl} - h_{il}h_{jk})$.

(2) $R^g_{ijkr} = 0$ and $R^g_{irjr} = -f(r)f''(r)h_{ij}$.

(3) $R^g_{ij} = R^h_{ij} - \left((n-2)[f'(r)]^2 + f(r)f''(r)\right)h_{ij}$.

(4) $R^g_{ir} = 0$ and $R^g_{rr} = -(n-1)[f(r)]^{-1}f''(r)$.

Proof Let us denote the Christoffel symbols of g by $\widetilde{\Gamma}^c_{ab}$ and the Christoffel symbols of h by Γ^c_{ab}. Then

$$\begin{aligned}
\widetilde{\Gamma}^k_{ij} &= \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\
&= \frac{1}{2}[f(r)]^{-2}h^{kl}\left\{\partial_i\left(f^2(r)h_{jl}\right) + \partial_j\left(f^2(r)h_{il}\right) - \partial_l\left(f^2(r)h_{ij}\right)\right\} \\
&= \frac{1}{2}h^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}) \\
&= \Gamma^k_{ij}, \\
\widetilde{\Gamma}^r_{ij} &= \frac{1}{2}g^{rr}(\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij}) \\
&= -\frac{1}{2}\partial_r\left\{f^2(r)h_{ij}\right\} \\
&= -f(r)f'(r)h_{ij}, \\
\widetilde{\Gamma}^j_{ir} &= \frac{1}{2}g^{jl}(\partial_i g_{rl} + \partial_r g_{il} - \partial_l g_{ir}) \\
&= \frac{1}{2}[f(r)]^{-2}h^{jl}\partial_r\left\{f^2(r)h_{il}\right\} \\
&= \frac{f'(r)}{f(r)}h^{jl}h_{il} \\
&= \frac{f'(r)}{f(r)}\delta^j_i, \\
\widetilde{\Gamma}^r_{ir} &= \frac{1}{2}g^{rr}(\partial_i g_{rr} + \partial_r g_{ir} - \partial_r g_{ir}) = 0.
\end{aligned}$$

(1) We compute

$$\begin{aligned}
R^g_{ijkl} =& - g_{lm}\left(\partial_i\widetilde{\Gamma}^m_{jk} - \partial_j\widetilde{\Gamma}^m_{ik} + \widetilde{\Gamma}^p_{jk}\widetilde{\Gamma}^m_{ip} - \widetilde{\Gamma}^p_{ik}\widetilde{\Gamma}^m_{jp}\right) \\
=& - f^2(r)h_{lm}\bigg\{\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{ip} + [-f(r)f'(r)h_{jk}]\left(\frac{f'(r)}{f(r)}\delta^m_i\right) \\
& -\Gamma^p_{ik}\Gamma^m_{jp} - [-f(r)f'(r)h_{ik}]\left(\frac{f'(r)}{f(r)}\delta^m_j\right)\bigg\} \\
=& - f^2(r)h_{lm}\left\{\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp} + [f'(r)]^2\left(h_{ik}\delta^m_j - h_{jk}\delta^m_i\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= f^2(r) R^h_{ijkl} - f^2(r)[f'(r)]^2 h_{lm}\big(h_{ik}\delta^m_j - h_{jk}\delta^m_i\big) \\
&= f^2(r) R^h_{ijkl} - f^2(r)[f'(r)]^2 (h_{ik}h_{jl} - h_{il}h_{jk}).
\end{aligned}
$$

(2) We compute

$$
\begin{aligned}
R^g_{ijkr} =& - g_{rr}\Big(\partial_i \widetilde{\Gamma}^r_{jk} - \partial_j \widetilde{\Gamma}^r_{ik} + \widetilde{\Gamma}^p_{jk}\widetilde{\Gamma}^r_{ip} - \widetilde{\Gamma}^p_{ik}\widetilde{\Gamma}^r_{jp}\Big) \\
=& - \Big\{\partial_i[-f(r)f'(r)h_{jk}] - \partial_j[-f(r)f'(r)h_{ik}] + \Gamma^p_{jk}[-f(r)f'(r)h_{ip}] + \widetilde{\Gamma}^r_{jk}\widetilde{\Gamma}^r_{ir} \\
& - \Gamma^p_{ik}[-f(r)f'(r)h_{jp}] - \widetilde{\Gamma}^r_{ik}\widetilde{\Gamma}^r_{jr}\Big\} \\
=& - f(r)f'(r)\Big(-\partial_i h_{jk} + \partial_j h_{ik} - \Gamma^p_{jk}h_{ip} + \Gamma^p_{ik}h_{jp}\Big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\Gamma^p_{jk}h_{ip} &= \frac{1}{2}h^{pl}(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk})h_{ip} \\
&= \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk}), \\
\Gamma^p_{ik}h_{jp} &= \frac{1}{2}h^{pl}(\partial_i h_{kl} + \partial_k h_{il} - \partial_l h_{ik})h_{jp} \\
&= \frac{1}{2}(\partial_i h_{kj} + \partial_k h_{ij} - \partial_j h_{ik}).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
R^g_{ijkr} &= -f(r)f'(r)(-\partial_i h_{jk} + \partial_j h_{ik} + \partial_i h_{jk} - \partial_j h_{ik}) \\
&= 0.
\end{aligned}
$$

Next, we compute

$$
\begin{aligned}
R^g_{irjr} &= -g_{rr}\Big(\partial_i \widetilde{\Gamma}^r_{rj} - \partial_r \widetilde{\Gamma}^r_{ij} + \widetilde{\Gamma}^p_{rj}\widetilde{\Gamma}^r_{ip} - \widetilde{\Gamma}^p_{ij}\widetilde{\Gamma}^r_{rp}\Big) \\
&= -\Big\{0 - \partial_r(-f(r)f'(r)h_{ij}) + \Big(\frac{f'(r)}{f(r)}\delta^p_j\Big)[-f(r)f'(r)h_{ip}] - 0\Big\} \\
&= -\Big\{[f'(r)]^2 + f(r)f''(r)\Big\}h_{ij} + [f'(r)]^2 h_{ij} \\
&= -f(r)f''(r)h_{ij}.
\end{aligned}
$$

(3) Using (1) and (2), we compute

$$
\begin{aligned}
R^g_{ij} &= g^{pq}R^g_{ipjq} = g^{kl}R^g_{ikjl} + g^{rr}R^g_{irjr} \\
&= [f(r)]^{-2}h^{kl}\Big\{f^2(r)R^h_{ikjl} - f^2(r)[f'(r)]^2(h_{ij}h_{kl} - h_{il}h_{kj})\Big\} - f(r)f''(r)h_{ij} \\
&= h^{kl}R^h_{ikjl} - h^{kl}[f'(r)]^2(h_{ij}h_{kl} - h_{il}h_{kj}) - f(r)f''(r)h_{ij} \\
&= R^h_{ij} - [f'(r)]^2[(n-1)h_{ij} - h_{ij}] - f(r)f''(r)h_{ij} \\
&= R^h_{ij} - \Big((n-2)[f'(r)]^2 + f(r)f''(r)\Big)h_{ij}.
\end{aligned}
$$

(4) By the first formula in (2), we see that

$$
R^g_{ir} = g^{pq}R^g_{iprq} = g^{kl}R^g_{ikrl} + g^{rr}R^g_{irrr} = 0 + 0 = 0.
$$

Finally, we use the second formula in (2) to get

$$\begin{aligned} R^g_{rr} &= g^{pq} R^g_{rprq} = g^{kl} R^g_{rkrl} + g^{rr} R^g_{rrrr} \\ &= [f(r)]^{-2} h^{kl} [-f(r) f''(r) h_{kl}] + 0 \\ &= -(n-1)[f(r)]^{-1} f''(r). \end{aligned}$$

□

Homework 4

Exercise 20 Let $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ be the unit sphere in $\mathbb{R}^3$ with induced metric g. Consider a geodesic $c\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{S}^2$ defined by $c(t) = (\cos t, 0, \sin t)$. Define a vector field X along c by $X(t) = (0, \cos t, 0)$. Prove that X is a Jacobi field.

Proof On the sphere $\mathbb{S}^2$, we have

$$\begin{aligned} \frac{\mathrm{D}X}{\mathrm{d}t} &= \pi_{T\mathbb{S}^2}(X'(t)) = (0, -\sin t, 0), \\ \frac{\mathrm{D}^2 X}{\mathrm{d}t^2} &= \pi_{T\mathbb{S}^2}\left(\frac{\mathrm{D}X}{\mathrm{d}t}\right) = (0, -\cos t, 0) = -X(t), \\ \mathrm{Rm}(X(t), c'(t))c'(t) &= \langle c'(t), c'(t) \rangle X(t) - \langle X(t), c'(t) \rangle c'(t) = X(t). \end{aligned}$$

Thus, the Jacobi equation is satisfied:

$$\frac{\mathrm{D}^2 X}{\mathrm{d}t^2} + \mathrm{Rm}(X(t), c'(t))c'(t) = 0.$$

□

Exercise 21 Given a Riemannian manifold (M, g), let $\pi\colon \widetilde{M} \to M$ be a covering map such that $\tilde{g} = \pi^* g$. Prove that g is complete if and only if $\tilde{g}$ is complete.

Proof Assume both M and $\widetilde{M}$ are connected.

($\Leftarrow$) By the assumption, π is a local isometry. Thus if $\tilde{g}$ is complete, π satisfies the hypotheses of the Ambrose theorem, which implies that g is also complete.

($\Rightarrow$) Conversely, suppose g is complete. Let $\tilde{p} \in \widetilde{M}$ and $\tilde{v} \in T_p\widetilde{M}$ be arbitrary, and let $p = \pi(\tilde{p})$ and $v = \mathrm{d}\pi_{\tilde{p}}(\tilde{v})$. Completeness of g implies that the geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = v$ is defined for all $t \in \mathbb{R}$, and then its lift $\tilde{\gamma}\colon \mathbb{R} \to \widetilde{M}$ starting at $\tilde{p}$ is a geodesic in $\widetilde{M}$ with initial velocity $\tilde{v}$, also defined for all t. □

Exercise 22 Let (M^n, g) be a complete, connected Riemannian manifold satisfying

$$\mathrm{Ric} + \nabla^2 f \geqslant Kg$$

for some constant $K > 0$. If $|\mathrm{grad}\, f| \leqslant K$ on M, prove that M is compact.

Proof Since M is complete, it follows as a consequence of the Hopf–Rinow theorem that any two points in M can be joined by a minimizing geodesic. Let $\gamma\colon [0, \ell] \to M$ be any such geodesic with unit speed.

Along γ consider the $n-1$ variational vector fields

$$V_i(t) = \sin\Big(\frac{\pi}{\ell}t\Big)E_i(t), \quad i = 1, \cdots, n-1,$$

where $E_1, \cdots, E_{n-1}$, together with $\gamma'(t)$, form an orthonormal frame for $T_{\gamma(t)}M$. Since γ is minimizing, by the second variation formula we have

$$\begin{aligned} 0 \leqslant \frac{\mathrm{d}^2 E}{\mathrm{d}s^2}\bigg|_{s=0} &= \int_0^\ell \left|\frac{\mathrm{D}V_i}{\mathrm{d}t}\right|^2 - \mathrm{Rm}(V_i, \gamma', V_i, \gamma')\,\mathrm{d}t \\ &= \int_0^\ell \Big(\frac{\pi}{\ell}\Big)^2 \cos^2\Big(\frac{\pi}{\ell}t\Big) - \sin^2\Big(\frac{\pi}{\ell}t\Big)\,\mathrm{Rm}(E_i, \gamma', E_i, \gamma')\,\mathrm{d}t. \end{aligned}$$

By adding up the contributions to the second variation formula for each variational vector field we get

$$0 \leqslant (n-1)\Big(\frac{\pi}{\ell}\Big)^2 \int_0^\ell \cos^2\Big(\frac{\pi}{\ell}t\Big)\,\mathrm{d}t - \int_0^\ell \sin^2\Big(\frac{\pi}{\ell}t\Big)\,\mathrm{Ric}(\gamma', \gamma')\,\mathrm{d}t. \tag{22–1}$$

Meanwhile, by the assumption on the Ricci curvature, we have

$$\sin^2\Big(\frac{\pi}{\ell}t\Big)\,\mathrm{Ric}(\gamma', \gamma') \geqslant -\sin^2\Big(\frac{\pi}{\ell}t\Big)\nabla^2 f(\gamma', \gamma') + K\sin^2\Big(\frac{\pi}{\ell}t\Big). \tag{22–2}$$

Since γ is a geodesic,

$$\nabla^2 f(\gamma', \gamma') = \nabla_{\gamma'}(\nabla_{\gamma'} f) - \nabla_{(\nabla_{\gamma'}\gamma')} f = (f \circ \gamma)''(t). \tag{22–3}$$

Combining (22–2) and (22–3) into (22–1) gives

$$\begin{aligned} (n-1)\Big(\frac{\pi}{\ell}\Big)^2 \int_0^\ell \cos^2\Big(\frac{\pi}{\ell}t\Big)\,\mathrm{d}t &\geqslant \int_0^\ell \sin^2\Big(\frac{\pi}{\ell}t\Big)\,\mathrm{Ric}(\gamma', \gamma')\,\mathrm{d}t \\ &= -\int_0^\ell \sin^2\Big(\frac{\pi}{\ell}t\Big)(f\circ\gamma)''(t)\,\mathrm{d}t + K\int_0^\ell \sin^2\Big(\frac{\pi}{\ell}t\Big)\,\mathrm{d}t. \end{aligned} \tag{22–4}$$

For the first integral on the right-hand side, we can integrate by parts to get

$$\begin{aligned} \int_0^\ell \sin^2\Big(\frac{\pi}{\ell}t\Big)(f\circ\gamma)''(t)\,\mathrm{d}t &= -\frac{2\pi}{\ell}\int_0^\ell \sin\Big(\frac{\pi}{\ell}t\Big)\cos\Big(\frac{\pi}{\ell}t\Big)(f\circ\gamma)'(t)\,\mathrm{d}t \\ &\leqslant \frac{\pi}{\ell}\int_0^\ell \left|\sin\Big(\frac{2\pi}{\ell}t\Big)\right| |\mathrm{grad}\, f|\,\mathrm{d}t \\ &\leqslant \frac{\pi}{\ell}\cdot\frac{2\ell}{\pi}\cdot K \\ &= 2K, \end{aligned}$$

where we used the fact that

$$|(f\circ\gamma)'(t)| = |\langle \mathrm{grad}\, f(\gamma(t)), \gamma'(t)\rangle| \leqslant |\mathrm{grad}\, f|\cdot|\gamma'| = |\mathrm{grad}\, f|\cdot 1.$$

Now, (22–4) reduces to

$$(n-1)\cdot\frac{\pi^2}{\ell^2}\cdot\frac{\ell}{2} \geqslant -2K + K\cdot\frac{\ell}{2},$$

that is,

$$\ell^2 - 4\ell - \frac{(n-1)\pi^2}{K} \leqslant 0.$$

Solving this quadratic inequality gives

$$0 \leqslant \ell \leqslant 2 + \sqrt{4 + \frac{(n-1)\pi^2}{K}}.$$

This gives a bound on diam M. By the Hopf–Rinow theorem, any closed and bounded subset of M is compact, so M is compact. □

Exercise 23 Let M^n be a smooth manifold (without boundary).

(1) Prove that for any Riemannian metric g on M, there exists a smooth function f on M, such that the conformal metric $e^{-f}g$ is complete.

(2) Prove that if every Riemannian metric on M is complete, then M is compact.

Proof (1) For each point $x \in M$, define

$$r(x) = \sup\Big\{r > 0 : \overline{\mathbb{B}(x,r)} \text{ is compact}\Big\}.$$

If $r(x) = \infty$ for some $x \in M$, then g is complete by the Hopf–Rinow theorem. Assume therefore that $r(x) < \infty$ for all $x \in M$. If $r < r(x) - d(x,y)$, then $r + d(x,y) < r(x)$, so $\overline{\mathbb{B}(x, r + d(x,y))}$ is compact. The triangle inequality ensures that $\mathbb{B}(y,r) \subset \mathbb{B}(x, r + d(x,y))$. Hence $\overline{\mathbb{B}(y,r)}$, being a closed subset of a compact set, is compact. This holds for all $r < r(x) - d(x,y)$, so we can take the supremum over r to get

$$r(y) \geqslant r(x) - d(x,y).$$

Reversing the roles of x and y, we similarly obtain

$$r(x) \geqslant r(y) - d(x,y).$$

Combining these two inequalities gives

$$|r(x) - r(y)| \leqslant d(x,y), \quad \forall x, y \in M,$$

which implies that $r(x)$ is a continuous function on M. Since M is second countable, we can choose a smooth function $\omega(x)$ such that $\omega(x) > \frac{1}{r(x)}$ for all $x \in M$. We define a conformal Riemannian metric $\tilde{g}$ by $\tilde{g}_x = [\omega(x)]^2 g_x$ at each point x.

In order to show that $\tilde{g}$ is complete, we shall show that $\widetilde{\mathbb{B}}(x, \frac{1}{3}) \subset \mathbb{B}\Big(x, \frac{r(x)}{2}\Big)$ for every x, which then implies that $\overline{\widetilde{\mathbb{B}}(x, \frac{1}{3})}$ is compact, and hence any closed and bounded subset of M is compact. For this purpose, choose y with $d(x,y) \geqslant \frac{r(x)}{2}$. For any piecewise smooth curve $c\colon [a,b] \to M$, joining x and y, its g-length L is not smaller than $d(x,y)$ and hence $L \geqslant \frac{r(x)}{2}$. We evaluate the $\tilde{g}$-length $\widetilde{L}$ of c. By a mean value theorem, we have

$$\widetilde{L} = \int_a^b \omega(c(t)) \left\| \frac{\mathrm{d}c}{\mathrm{d}t} \right\|_g \mathrm{d}t = \omega(c(\xi)) \int_a^b \left\| \frac{\mathrm{d}c}{\mathrm{d}t} \right\|_g \mathrm{d}t = \omega(c(\xi))L > \frac{L}{r(c(\xi))},$$

where ξ is a number between a and b. Since $|r(c(\xi)) - r(x)| \leqslant d(x, c(\xi)) \leqslant L$, we have $r(c(\xi)) \leqslant$

$r(x)+L$ so that

$$\widetilde{L} > \frac{L}{r(c(\xi))} \geqslant \frac{L}{r(x)+L} \geqslant \frac{\frac{r(x)}{2}}{r(x)+\frac{r(x)}{2}} = \frac{1}{3}.$$

Therefore $\tilde{d}(x,y) \geqslant \frac{1}{3}$. This proves that $\widetilde{\mathbb{B}}(x,\frac{1}{3})^c \supset \mathbb{B}\Big(x,\frac{r(x)}{2}\Big)^c$. As stated above, any closed and bounded subset of M is compact, so the conformal metric $\tilde{g}$ is complete by the Hopf–Rinow theorem.

(2) Let (M,g) be a noncompact Riemannian manifold. We shall find an incomplete Riemannian metric $\tilde{g}$ which is conformal to g. By (1), we can assume that g is complete. Fix a point $p \in M$. Since M is second countable, we can find a smooth function $\omega(x)$ on M so that $\omega(x) \geqslant d(p,x)$ for all $x \in M$. Consider the conformal metric $\tilde{g} = e^{-2\omega} g$. For any point $q \in M$, let γ be the minimizing geodesic (with respect to g) from p to q with unit speed. Then

$$\widetilde{L}(\gamma) = \int_0^{d(p,q)} e^{-\omega(t)}\,\mathrm{d}t \leqslant \int_0^{d(p,q)} e^{-d(p,\gamma(t))}\,\mathrm{d}t = \int_0^{d(p,q)} e^{-t}\,\mathrm{d}t = 1 - e^{-d(p,q)} \leqslant 1.$$

This implies that $\operatorname{diam}(M,\tilde{g}) \leqslant 2$. Thus, $(M,\tilde{g})$ is bounded but noncompact, and hence incomplete by the Hopf–Rinow theorem. □

Exercise 24 Given a Riemannian manifold (M,g), let $\gamma\colon [0,a] \to M$ be a smooth curve and

$$f(u,v,t)\colon (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times[0,a] \to M$$

be a smooth map with $f(0,0,t) = \gamma(t)$. Denote $\gamma_{u,v}(t) = f(u,v,t)$ and

$$U(t) = \frac{\partial f}{\partial u}\bigg|_{(0,0,t)}, \quad V(t) = \frac{\partial f}{\partial v}\bigg|_{(0,0,t)}.$$

Suppose γ is a geodesic, find the formula for

$$\frac{\partial^2}{\partial u\,\partial v} E(\gamma_{u,v})\bigg|_{(0,0)}.$$

Solution We compute

$$\begin{aligned}\frac{\partial}{\partial v} E(\gamma_{u,v}) &= \frac{\partial}{\partial v}\left\{\frac{1}{2}\int_0^a \langle f_t, f_t\rangle\,\mathrm{d}t\right\} = \int_0^a \left\langle \widetilde{\nabla}_{\partial_v} f_t, f_t\right\rangle \mathrm{d}t \\ &= \int_0^a \left\langle \widetilde{\nabla}_{\partial_t} f_v, f_t\right\rangle \mathrm{d}t,\end{aligned}$$

and then

$$\begin{aligned}\frac{\partial^2}{\partial u\,\partial v} E(\gamma_{u,v}) &= \int_0^a \left\langle \widetilde{\nabla}_{\partial_u}\widetilde{\nabla}_{\partial_t} f_v, f_t\right\rangle + \left\langle \widetilde{\nabla}_{\partial_t} f_v, \widetilde{\nabla}_{\partial_u} f_t\right\rangle \mathrm{d}t \\ &= \int_0^a \left\langle \widetilde{\nabla}_{\partial_u}\widetilde{\nabla}_{\partial_t} f_v, f_t\right\rangle + \left\langle \widetilde{\nabla}_{\partial_t} f_v, \widetilde{\nabla}_{\partial_t} f_u\right\rangle \mathrm{d}t \\ &= \int_0^a \langle \operatorname{Rm}(f_u,f_t) f_v, f_t\rangle + \left\langle \widetilde{\nabla}_{\partial_t}\widetilde{\nabla}_{\partial_u} f_v, f_t\right\rangle + \left\langle \widetilde{\nabla}_{\partial_t} f_v, \widetilde{\nabla}_{\partial_t} f_u\right\rangle \mathrm{d}t.\end{aligned}$$

Since $\nabla_{\dot\gamma}\dot\gamma = 0$, we have

$$\begin{aligned}\frac{\partial^2}{\partial u\,\partial v}E(\gamma_{u,v})\bigg|_{(0,0)} &= \int_0^a \langle \mathrm{Rm}(U,\dot\gamma)V,\dot\gamma\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle \nabla_U V,\dot\gamma\rangle + \langle \nabla_{\dot\gamma}U,\nabla_{\dot\gamma}V\rangle\,\mathrm{d}t \\ &= \int_0^a \langle \mathrm{Rm}(U,\dot\gamma)V,\dot\gamma\rangle + \langle \nabla_{\dot\gamma}U,\nabla_{\dot\gamma}V\rangle\,\mathrm{d}t + \langle \nabla_U V,\dot\gamma\rangle\big|_0^a.\end{aligned}$$

□

Exercise 25 Let (M^n, g) be a complete Riemannian manifold, and let $p \in M$ be a point.

(1) Suppose that along any normalized (unit-speed) geodesic γ with $\gamma(0) = p$, the sectional curvatures of M in any plane $\sigma \subset T_{\gamma(t)}M$ containing $\gamma'(t)$ is $\leqslant 1$ if $0 \leqslant t < \frac{\pi}{2}$ and $\leqslant 0$ if $t \geqslant \frac{\pi}{2}$. Show that the length of any normal Jacobi field $J(t)$ along such a geodesic γ, with $J(0) = 0$, is nondecreasing after $t = \frac{\pi}{2}$.

(2) Suppose that along any normalized (unit-speed) geodesic γ with $\gamma(0) = p$, the sectional curvatures of M in any plane $\sigma \subset T_{\gamma(t)}M$ containing $\gamma'(t)$ is $\geqslant 1$ if $0 \leqslant t < \frac{\pi}{2}$ and > 0 if $t \geqslant \frac{\pi}{2}$. Show that M is compact.

Cheat Sheet

- ◇ $\langle \nabla_X Y, Z\rangle = \dfrac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle + \langle X,[Z,Y]\rangle\}$.
- ◇ $\Gamma_{ij}^k = \dfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.
- ◇ $R_{ijk}{}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l$.
- ◇ $R_{ijkl} = -g_{lm} R_{ijk}{}^m = -g_{lm}\Big(\partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{jk}^p \Gamma_{ip}^m - \Gamma_{ik}^p \Gamma_{jp}^m\Big)$.
- ◇ $R_{ij} = R_{ikj}{}^k = g^{km} R_{ikjm}$.
- ◇ $R = g^{ij} R_{ij}$.
- ◇ In normal coordinates, $R_{ijkl}(0) = \dfrac{1}{2}(\partial_i\partial_l g_{jk} + \partial_j \partial_k g_{li} - \partial_i \partial_k g_{lj} - \partial_j \partial_l g_{ik})(0)$.
- ◇ $\mathrm{D}_t V(t) = \Big(\dot V^k(t) + \dot\gamma^i(t) V^j(t) \Gamma_{ij}^k(\gamma(t))\Big)\partial_k|_{\gamma(t)} \implies \ddot x^k(t) + \dot x^i(t)\dot x^j(t)\Gamma_{ij}^k(x(t)) = 0$.
- ◇ $(\nabla_X \operatorname{Rm})(Y,Z)W + (\nabla_Y \operatorname{Rm})(Z,X)W + (\nabla_Z \operatorname{Rm})(X,Y)W = 0$.
- ◇ For $(\mathbb{S}^n, g_{\mathbb{S}^n})$, $\operatorname{Rm}(X,Y,Z,W) = \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle$, since $\mathrm{D}_X \mathbf{n} = X$ and $\nabla_X Y = \mathrm{D}_X Y + \langle X,Y\rangle \mathbf{n}$.
- ◇ $h \owedge k(X,Y,Z,W) = h(X,Z)k(Y,W) - h(X,W)k(Y,Z) + h(Y,W)k(X,Z) - h(Y,Z)k(X,W)$.
- ◇ $\operatorname{tr}_g(h \owedge g) = (n-2)h + (\operatorname{tr}_g h)g$. In particular, $\operatorname{tr}_g(g \owedge g) = 2(n-1)g$.
- ◇ $\langle T, h \owedge g\rangle_g = 4\langle \operatorname{tr}_g T, h\rangle_g$.
- ◇ $|h \owedge g|_g^2 = 4(n-2)|h|_g^2 + 4(\operatorname{tr}_g h)^2$. In particular, $|g \owedge g|_g^2 = 8n(n-1)$.
- ◇ The Weyl tensor of g is given by $W = \operatorname{Rm} - \dfrac{1}{n-2}\operatorname{Ric} \owedge g + \dfrac{R}{2(n-1)(n-2)} g \owedge g$.
- ◇ In dimension 3, $\operatorname{Rm} = \operatorname{Ric} \owedge g - \dfrac{R}{4} g \owedge g$.
- ◇ In dimension 2, $\operatorname{Rm} = \dfrac{R}{4} g \owedge g$, $\operatorname{Ric} = \dfrac{R}{2} g$, $\operatorname{Ric} = Kg$, and $R = 2K$.
- ◇ $\mathring{\operatorname{Ric}} = \operatorname{Ric} - \dfrac{R}{n} g$.
- ◇ The decomposition $\operatorname{Rm} = W + \dfrac{1}{n-2}\mathring{\operatorname{Ric}} \owedge g + \dfrac{R}{2n(n-1)} g \owedge g$ is orthogonal.
- ◇ $|\operatorname{Rm}|^2 = |W|^2 + \dfrac{4}{n-2}|\operatorname{Ric}|^2 - \dfrac{2}{(n-1)(n-2)} R^2$.
- ◇ $K_p(\sigma) = \dfrac{\operatorname{Rm}(X,Y,X,Y)}{\frac{1}{2}(g \owedge g)(X,Y,X,Y)} = \dfrac{\operatorname{Rm}(X,Y,X,Y)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$.
- ◇ $\operatorname{div} F = \operatorname{tr}_g(\nabla F)$, where the trace is taken on the first two indices of ∇F; $\operatorname{div}(X) = \nabla_k X^k$; $(\operatorname{div}(T))_{i_1\cdots i_{k-1}} = g^{ij}\nabla_j T_{i i_1 \cdots i_{k-1}}$.
- ◇ $(\operatorname{div} \operatorname{Rm})_{kij} = (\nabla_i \operatorname{Ric})_{jk} - (\nabla_j \operatorname{Ric})_{ik}$, $\operatorname{div}(\operatorname{Ric}) = \dfrac{1}{2}\,\mathrm{d}R$.

◇ The index form of γ is $I(V,W)=\int_a^b [\langle \mathrm{D}_t V, \mathrm{D}_t W\rangle - \mathrm{Rm}(V,\gamma',W,\gamma')]\,\mathrm{d}t$.

◇ $I(V,W) = -\int_a^b \left\langle \mathrm{D}_t^2 V + \mathrm{Rm}(V,\gamma')\gamma', W\right\rangle \mathrm{d}t + \langle \mathrm{D}_t V, W\rangle\Big|_{t=a}^{t=b} - \sum_{i=1}^{k-1}\langle \Delta_i \mathrm{D}_t V, W(a_i)\rangle$, where $(a_0,\cdots,a_k)$ is an admissible partition for V and W, and $\Delta_i \mathrm{D}_t V$ is the jump in $\mathrm{D}_t V$ at $t=a_i$.

◇ The Jacobi equation is $\mathrm{D}_t^2 V + \mathrm{Rm}(V,\gamma')\gamma' = 0$.

◇ **(Bonnet–Myers)** Let (M,g) be a complete, connected Riemannian n-manifold, and suppose there is a positive constant k such that $\mathrm{Ric} \geqslant (n-1)kg$. Then $\mathrm{diam}(M) \leqslant \dfrac{\pi}{\sqrt{k}}$, and $\pi_1(M)$ is finite. In particular, M is compact.

◇ $\mathrm{D}_s \mathrm{D}_t V - \mathrm{D}_t \mathrm{D}_s V = \mathrm{Rm}(\partial_s\Gamma, \partial_t\Gamma)V$ for any smooth one-parameter family of curves $\Gamma\colon J\times I\to M$ and any smooth vector field V along Γ.

◇ $\Gamma(s,t) = \exp_{c(s)}\Big(t\big(T(s)+sW(s)\big)\Big)$ is a geodesic variation of the geodesic $\gamma(t)$, where

- $c(s)$ is a geodesic with $c(0)=\gamma(0)$ and $c'(0)=J(0)$.
- $T(s)$ is a parallel vector field along $c(s)$ with $T(0)=\gamma'(0)$.
- $W(s)$ is a parallel vector field along $c(s)$ with $W(0)=J'(0)$.

If $J(0)=0$, then $\Gamma(s,t)=\exp_{\gamma(0)}\Big(t\big(\gamma'(0)+sJ'(0)\big)\Big)$ and $J(t) = \dfrac{\partial\Gamma}{\partial s}\Big|_{s=0} = \Big(\mathrm{d}\exp_{\gamma(0)}\Big)_{t\gamma'(0)}(tJ'(0))$.

◇ $\langle J_1(t), J_2(t)\rangle = \langle J_1'(0), J_2'(0)\rangle t^2 - \dfrac{1}{3}\mathrm{Rm}(J_1'(0),\gamma'(0),J_2'(0),\gamma'(0))t^4 + O\big(t^5\big)$ when $J_1(0)=J_2(0)=0$.

◇ In normal coordinates, $g_{ij}(x) = \delta_{ij} - \dfrac{1}{3}R_{ikjl}(0)x^k x^l + O\big(|x|^3\big)$.

◇ $\displaystyle\lim_{r\to 0^+}\frac{2\pi r - L_r}{r^3} = \frac{\pi}{3}K_p(\sigma)$, $|\mathbb{B}(p,r)| = \omega_n r^n\left(1 - \dfrac{R(p)}{6(n+2)}r^2 + O\big(r^3\big)\right)$.

◇ $\mathrm{div}\left(X^i\dfrac{\partial}{\partial x^i}\right) = \dfrac{1}{\sqrt{\det g}}\dfrac{\partial}{\partial x^i}\Big(X^i\sqrt{\det g}\Big)$.

◇ $\Delta u = \dfrac{1}{\sqrt{\det g}}\dfrac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\dfrac{\partial u}{\partial x^j}\right) = g^{ij}\big(\partial_i\partial_j u - \Gamma_{ij}^k\partial_k u\big)$.

◇ Stereographic projection $\sigma(\xi,\tau) = \dfrac{R\xi}{R-\tau}$, $\sigma^{-1}(u) = \left(\dfrac{2R^2u}{|u|^2+R^2}, R\dfrac{|u|^2-R^2}{|u|^2+R^2}\right)$.

◇ $\nabla_X(\omega) = \big(X(\omega_k) - X^j\omega_i\Gamma_{jk}^i\big)\varepsilon^k$.

◇ $\nabla Y = Y^i_{\ ;j}\, E_i\otimes\varepsilon^j$, with $Y^i_{\ ;j} = E_jY^i + Y^k\Gamma_{jk}^i$.

◇ $\nabla\omega = \omega_{i;j}\varepsilon^i\otimes\varepsilon^j$, with $\omega_{i;j} = E_j\omega_i - \omega_k\Gamma_{ji}^k$.

◇ $\nabla^2_{X,Y}F = \nabla_X(\nabla_Y F) - \nabla_{(\nabla_X Y)}F$.

◇ $\nabla^2 u = u_{;ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j$, with $u_{;ij} = \partial_j\partial_i u - \Gamma_{ji}^k\partial_k u$; $\nabla^2u(X,Y) = X(Yu) - (\nabla_X Y)u = \langle Y, \nabla_X\nabla u\rangle$.

◇ $\mathrm{D}_tV(t) = \Big(\dot V^k(t) + \dot\gamma^i(t)V^j(t)\Gamma_{ij}^k(\gamma(t))\Big)\partial_k|_{\gamma(t)}$.

◇ $\ddot x^k(t) + \dot x^i(t)\dot x^j(t)\Gamma_{ij}^k(x(t)) = 0$.

- $\displaystyle\int_M (v\Delta u + \langle \nabla u, \nabla v\rangle)\,\mathrm{d}V_g = \int_{\partial\Omega} v\langle \nabla u, \mathbf{n}\rangle\,\mathrm{d}\sigma_g$.
- $\displaystyle\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L_g(\Gamma_s) = -\int_a^b \langle V, \mathrm{D}_t\gamma'\rangle\,\mathrm{d}t - \sum_{i=1}^{k-1}\langle V(a_i), \Delta_i\gamma'\rangle + \langle V(b), \gamma'(b)\rangle - \langle V(a), \gamma'(a)\rangle$, where $(a_0, \cdots, a_k)$ is an admissible partition for Γ, and $\Delta_i\gamma'$ is the jump in γ' at $t = a_i$.
- **(Ambrose)** Suppose $\left(\widetilde{M}, \tilde{g}\right)$ and (M, g) are connected Riemannian manifolds with $\widetilde{M}$ complete, and $\pi\colon \widetilde{M} \to M$ is a local isometry. Then M is complete and π is a smooth covering map.
- **(Cartan–Hadamard)** If (M, g) is a complete, connected Riemannian manifold with nonpositive sectional curvature, then for every point $p \in M$, the map $\exp_p\colon T_pM \to M$ is a smooth covering map. Thus the universal covering space of M is diffeomorphic to $\mathbb{R}^n$, and if M is simply connected, then M itself is diffeomorphic to $\mathbb{R}^n$.
- A complete, simply connected Riemannian manifold with nonpositive sectional curvature is called a Cartan–Hadamard manifold.
- Suppose M is complete and $\gamma\colon [0, \infty) \to M$ is a unit-speed geodesic from p. For any $a > 0$, $\gamma(a)$ is the cut point of p if and only if $\gamma|_{[0,a]}$ is minimizing and at least one of the following statements holds:
  - $\gamma(a)$ is conjugate to p along γ.
  - There exists another geodesic segment from p to $\gamma(a)$.
- If $q \in \mathrm{Cut}(p)$ and $d(p, q) = \mathrm{inj}_p$, then at least one of the following statements holds:
  - There exists a geodesic segment γ from p to q such that q is conjugate to p along γ.
  - There exists another geodesic segment σ from p to q so that $\gamma'(\ell) = -\sigma'(\ell)$, where $\ell = d(p, q)$.
- Suppose $c\colon [0, a] \to M$ is a geodesic segment with $c(a) \notin \mathrm{Cut}(c(0))$, and $J(t)$ is a normal Jacobi field along c with $J(0) = 0$. Then for $r(x) = f(p, x)$, we have

$$\nabla^2 r|_{c(t)}(J(t), J(t)) = \langle J(t), J'(t)\rangle, \quad t \in (0, a].$$

- On a Hadamard manifold, $f_p(x) \coloneqq \frac{1}{2}d^2(p, x)$ is strictly convex, i.e., $\nabla^2 f_p > 0$. (In fact, $\nabla^2 f_p \geqslant g$.)
- On a Riemannian manifold, $\nabla^2 f_p = g$ at p.
- **(Cartan's Fixed Point Theorem)** Let (M, g) be a Hadamard manifold. If $\varphi\colon M \to M$ is an isometry and $\varphi^k = \mathrm{Id}$ for some k, then φ has a fixed point.
- **(Cartan's Torsion Theorem)** Suppose (M, g) is a complete, connected Riemannian manifold with nonpositive sectional curvature. Then $\pi_1(M)$ is torsion-free. In particular, if $\pi_1(M) \neq \{e\}$, then $|\pi_1(M)| = \infty$.
- For an algebraic curvature tensor T, if $T(X, Y, X, Y) = 0$ for any X, Y, then $T \equiv 0$. ① $0 = T(X, Y + Z, X, Y + Z) = 2T(X, Y, X, Z)$. ② $0 = T(X + W, Y, X + W, Z) = T(X, Y, W, Z) + T(W, Y, X, Z)$. ③ Add together $T(Y, W, X, Z) = T(X, Y, W, Z)$, $T(Y, W, X, Z) = T(W, X, Y, Z)$, $T(Y, W, X, Z) = T(Y, W, X, Z)$ and use the first Bianchi identity to get $3T(Y, W, X, Z) = 0$.

◇ **(Index Lemma)** Suppose $\gamma(t)$ is not conjugate to $\gamma(0)$ for any $t \in (0,a]$. Let J be a normal Jacobi field and V a piecewise smooth vector field along γ such that $\langle V, \gamma' \rangle \equiv 0$. If $J(0) = V(0) = 0$ and $J(a) = V(a)$, then $I(J,J) \leqslant I(V,V)$, with equality holds if and only if $V \equiv J$.

◇ $\left(|J|^2\right)'(a) - \left(|J|^2\right)'(0) = 2I(J,J)$, where J is a Jacobi field along a geodesic $\gamma\colon [0,a] \to M$.

◇ **(Rauch's Comparison Theorem)** Let (M^n, g) and $\left(\widetilde{M}^n, \tilde{g}\right)$ be complete Riemannian manifolds, $\gamma\colon [0,a] \to M$ and $\tilde{\gamma}\colon [0,a] \to \widetilde{M}$ be unit-speed geodesics. If J and $\widetilde{J}$ are Jacobi fields along γ and $\tilde{\gamma}$, respectively, satisfying

- $J(0) = \widetilde{J}(0) = 0$;
- $\langle J'(0), \gamma'(0) \rangle_g = \left\langle \widetilde{J}'(0), \tilde{\gamma}'(0) \right\rangle_{\tilde{g}}$;
- $|J'(0)| = \left|\widetilde{J}'(0)\right|$;
- $\tilde{\gamma}$ has no conjugate points;
- $K_{\gamma(t)}(\sigma) \leqslant \widetilde{K}_{\tilde{\gamma}(t)}(\tilde{\sigma})$ for all $t \in [0,a]$, where σ and $\tilde{\sigma}$ are planes containing $\gamma'(t)$ and $\tilde{\gamma}'(t)$, respectively;

then $|J(t)| \geqslant \left|\widetilde{J}(t)\right|$ for all $t \in [0,a]$.

◇ **(Special Case of Rauch's Comparison)** ⑤ $\rightsquigarrow$ $\sec_M \leqslant \sec_{\widetilde{M}}$, ② $\rightsquigarrow$ normal Jacobi fields.

◇ Suppose (M,g) is a Riemannian manifold with constant sectional curvature k, and γ is a unit-speed geodesic in M. The normal Jacobi fields along γ vanishing at $t = 0$ are $J(t) = cs_k(t)E(t)$, where E is any parallel unit vector field along γ, c is an arbitrary constant. Moreover, $J'(0) = cE(0)$ and $|J(t)| = |s_k(t)||J'(0)|$.

◇ If $\sec \leqslant k$ on $\mathbb{B}\left(p, \frac{\pi}{\sqrt{k}}\right)$ for $k > 0$, then $\mathrm{d}\exp_p$ is non-singular on $\mathbb{B}\left(0, \frac{\pi}{\sqrt{k}}\right) \subset T_pM$.

◇ Let (M^n, g) and $\left(\widetilde{M}^n, \tilde{g}\right)$ be two Riemannian manifolds with $\sup K(\sigma) \leqslant \inf \widetilde{K}(\tilde{\sigma})$. Fix $p \in M$, $\tilde{p} \in \widetilde{M}$, and an isometry $i\colon T_pM \to T_{\tilde{p}}\widetilde{M}$. Take $r < \mathrm{inj}_p$ such that $\mathrm{d}\exp_p$ is non-singular on $\mathbb{B}(0,r)$ and set $\Phi = \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}|_{\mathbb{B}(p,r)}$. If $c\colon [0,a] \to \mathbb{B}(p,r)$ is a smooth curve and $\tilde{c}(s) = \Phi(c(s))$, then $L(c) \geqslant L(\tilde{c})$.

◇ $g_2 = \mathrm{d}r^2 + g^r$, where g^r is the metric on $\{r\} \times \mathbb{S}^{n-1}$.

$$
\begin{array}{ccccc}
(0,a) \times \mathbb{S}^{n-1} & \xrightarrow{\ l\ } & \mathbb{B}(0,a) \setminus \{0\} & \xrightarrow{\exp_p} & \mathbb{B}(p,a) \subset M \\
g_2 = l^* g_1 & & g_1 = \exp_p^* g & & g
\end{array}
$$

- If $\sec_M \geqslant k$, then $g_2 \leqslant \mathrm{d}r^2 + s_k^2(r) g_{\mathbb{S}^{n-1}}$.
- If $\sec_M \leqslant k$, then $g_2 \geqslant \mathrm{d}r^2 + s_k^2(r) g_{\mathbb{S}^{n-1}}$.

◇ Given a Riemannian manifold $\left(N^{n-1}, h\right)$, we consider the warped product metric $g = \mathrm{d}r^2 + f^2(r)h$ on $M = (0,+\infty) \times N$, where $f(r)\colon (0,+\infty) \to \mathbb{R}$ is a positive smooth function. In the following, we use indices i,j,k,l to denote the local coordinates on N. Superscripts g and h will be used to indicate the quantities computed with respect to the metrics g and h, respectively.

- $R^g_{ijkl} = f^2(r) R^h_{ijkl} - f^2(r)[f'(r)]^2 (h_{ik}h_{jl} - h_{il}h_{jk})$.

– $R^g_{ijkr} = 0$ and $R^g_{irjr} = -f(r)f''(r)h_{ij}$.
– $R^g_{ij} = R^h_{ij} - \Big((n-2)[f'(r)]^2 + f(r)f''(r)\Big)h_{ij}$.
– $R^g_{ir} = 0$ and $R^g_{rr} = -(n-1)[f(r)]^{-1}f''(r)$.

◇ $s_k''(r) = -ks_k(r)$, $[s_k'(r)]^2 = 1 - ks_k^2(r)$. Thus, $K(\partial_\theta, \partial_r) = \dfrac{R_{\theta r\theta r}}{|\partial_\theta|^2|\partial_r|^2} = \dfrac{-s_k(r)s_k''(r)}{s_k^2(r)} = k$, $K(\partial_{\theta_1}, \partial_{\theta_2}) = \dfrac{R_{\theta_1\theta_2\theta_1\theta_2}}{|\partial_{\theta_1}|^2|\partial_{\theta_2}|^2} = \dfrac{s_k^2(r) - s_k^2(r)[s_k'(r)]^2}{s_k^4(r)} = \dfrac{1-[s_k'(r)]^2}{s_k^2(r)} = k$. If $\rho = \dfrac{s_k}{s_k'}$, then $\rho' = 1 + k\rho^2$.

◇ **(Hessian Comparison)** If $\sec_M \leqslant k$, then for $q \in M\backslash(\{p\}\cup\mathrm{Cut}(p))$ and $X \in T_qM$ with $\langle X, \nabla r\rangle = 0$, we have
$$\nabla^2 r|_q(X,X) \geqslant \frac{s_k'(r)}{s_k(r)}|X|^2.$$
Note that $\nabla^2 r(\nabla r, \nabla r) = 0$ (since $r(c(t)) = t \implies \langle \nabla r, c'\rangle = 1 \implies \nabla^2 r(c', c') = 0$).

◇ **(Laplacian Comparison)** If $\mathrm{Ric} \geqslant (n-1)kg$, then for $q \in M \setminus (\{p\}\cup\mathrm{Cut}(p))$, we have $\Delta r|_q \leqslant (n-1)\dfrac{s_k'(r)}{s_k(r)}$.

◇ In normal coordinates, $\Sigma^r = \{|x| = r\}$ and $g_{ij}(0) = \delta_{ij}$, with volume element $\sqrt{\det(g_{ij})}\,\mathrm{d}x^1\wedge\cdots\wedge \mathrm{d}x^n$. Restricting on Σ^r, we get $\lim\limits_{r\to 0^+}\dfrac{m(r)}{\omega_{n-1}r^{n-1}} = 1$, where $m(r)$ is the volume of Σ^r and ω_{n-1} is the volume of $\mathbb{S}^{n-1}\subset\mathbb{R}^n$.

◇ **(Volume Comparison)** On a complete Riemannian manifold (M^n, g) with $\mathrm{Ric} \geqslant (n-1)kg$, for any $p \in M$, $|\mathbb{B}(p,r)| \leqslant V(n,k,r) = \displaystyle\int_0^r \rho(s)\,\mathrm{d}s = \omega_{n-1}\int_0^r s_k^{n-1}(s)\,\mathrm{d}s$, the volume of a ball of radius r in the model space with constant sectional curvature k.

◇ **(Relative Volume Comparison, Bishop–Gromov)** On a complete Riemannian manifold (M^n, g) with $\mathrm{Ric} \geqslant (n-1)kg$, $\dfrac{|\mathbb{B}(p,r)|}{V(n,k,r)}$ is decreasing for $r > 0$. In particular, $\dfrac{|\mathbb{B}(p,2r)|}{|\mathbb{B}(p,r)|} \leqslant \dfrac{V(n,k,2r)}{V(n,k,r)} \leqslant C(n,k,\Lambda)$ where $r \leqslant \Lambda$.

◇ **(Strong Maximum Principle)** Let (M,g) be connected and complete, and $f\colon (M,g)\to\mathbb{R}$ be continuous with $\Delta f \geqslant 0$ everywhere in the barrier sense. If f has a global maximum, then f is constant.

◇ **(Elliptic Regularity)** Let $f\colon (M,g)\to\mathbb{R}$ be continuous with $\Delta f \geqslant 0$ and $\Delta f \leqslant 0$ in the barrier sense. Then f is smooth and $\Delta f = 0$ in the classical sense.

◇ **(Splitting Lemma)** If $\nabla^2 f \equiv 0$ and $|\nabla f| \equiv 1$, then $(M^n, g) = (N^{n-1}, g_N)\times(\mathbb{R}, g_E)$.

◇ **(Laplacian Comparison, Calabi)** Suppose (M^n, g) is complete and $\mathrm{Ric} \geqslant (n-1)kg$. Then $\Delta r \leqslant (n-1)\dfrac{s_k'(r)}{s_k(r)}$ in the barrier sense everywhere for $r(x) = d(x,p)$.

◇ **(Splitting Theorem, Cheeger–Gromoll)** Let (M^n, g) be complete, noncompact and suppose $\mathrm{Ric} \geqslant 0$. If M contains a geodesic line, then (M^n, g) splits off a line: $(M^n, g) = (N^{n-1}, g_N)\times(\mathbb{R}, g_E)$.

◇ $\mathbb{S}^3\times\mathbb{S}^1$ does not admit a Ricci-flat metric. (If it does, then its universal cover $\big(\mathbb{S}^3\times\mathbb{R}, \tilde g\big)$ is Ricci-flat and contains a geodesic line. Then $\big(\mathbb{S}^3\times\mathbb{R}, \tilde g\big) = \big(N^3, g_N\big)\times(\mathbb{R}, g_E)$ with $\big(N^3, g_N\big)$ (Ricci-)flat. Since N is simply connected, $\big(N^3, g_N\big) = \big(\mathbb{R}^3, g_E\big)$. Thus, $\big(\mathbb{S}^3\times\mathbb{R}, \tilde g\big) = \big(\mathbb{R}^4, g_E\big)$, a contradiction.)

◇ $\big(\nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega\big)(Z) = -\omega(\mathrm{Rm}(X,Y)Z)$; $\nabla_i\nabla_j\omega_k - \nabla_j\nabla_i\omega_k = -R_{ijk}{}^l\omega_l$ ($= R_{ijkl}\omega_l$ in normal coordinates).

- $\Delta(\mathrm{d}u) = \mathrm{d}(\Delta u) + \mathrm{Ric}(\nabla u)$ as 1-forms, where $\mathrm{Ric}(\nabla u)(X) \coloneqq \mathrm{Ric}(\nabla u, X)$.
- **(Bochners' formula)** $\Delta|\nabla u|^2 = 2\left|\nabla^2 u\right|^2 + 2\,\mathrm{Ric}(\nabla u, \nabla u) + 2\langle \nabla \Delta u, \nabla u\rangle$.
- **(Structure Theorem, Cheeger–Gromoll)** Suppose (M^n, g) is compact with $\mathrm{Ric} \geqslant 0$. Then
  - The universal cover $\left(\widetilde{M}, \tilde{g}\right) = (N, g_N) \times \left(\mathbb{R}^k, g_E\right)$, where N is compact.
  - The isometry group $\mathrm{Iso}\left(\widetilde{M}, \tilde{g}\right) = \mathrm{Iso}(N, g_N) \times \mathrm{Iso}\left(\mathbb{R}^k, g_E\right)$.
- $\mathcal{L}_V(A(X_1, \cdots, X_k)) = (\mathcal{L}_V A)(X_1, \cdots, X_k) + A(\mathcal{L}_V X_1, \cdots, X_k) + \cdots + A(X_1, \cdots, \mathcal{L}_V X_k)$.