位势方程

考虑方程 $-\Delta u=f$，$u:\mathbb{R}^n\supset\Omega\to\mathbb{R}$ 未知，$f:\Omega\to\mathbb{R}$ 已知.

如果 $f\equiv 0$，方程化为 $\Delta u=0$，称为 Laplace 方程.

边值条件 $\begin{cases} \text{Dirichlet}: u|_{\partial\Omega}=\varphi \\ \text{Neumann}: \frac{\partial u}{\partial n}\Big|_{\partial\Omega}=\varphi \\ \text{Robin}: \left(u+\sigma\frac{\partial u}{\partial n}\right)\Big|_{\partial\Omega}=\varphi \quad (\sigma>0) \end{cases}$

§2.1 调和函数

若 $\Delta u=0$ in $\mathbb{R}^n$ ~~Ω~~，则称 u 是 $\mathbb{R}^n$ ~~Ω~~ 上的调和函数.

若 $u(x)$ 是 $\mathbb{R}^n$ 上的调和函数，则

(1) $u(\lambda x)$ 也是调和函数.

(2) $u(x+x_0)$ 也是调和函数.

(3) 对任意正交变换 O，$u(Ox)$ 也是调和函数.

若 $u:\Omega\to\mathbb{R}$ 是二阶连续可微的，且 $\Delta u(x)=0$，$\forall x\in\Omega$，则称 u 是 Ω 上的调和函数.

· 极坐标公式：

$$\int_{\mathbb{R}^n} f(x)\,dx=\int_0^{+\infty}\int_{\partial B(x,r)} f(y)\,dS(y)\,dr \quad \left(\int_{B(x,r)} f(y)\,dy=\int_0^r\int_{\partial B(x,\rho)} f(y)\,dS(y)\,d\rho\right)$$

$$\xrightarrow{\text{对}r\text{求导}} \frac{d}{dr}\left(\int_{B(x,r)} f(y)\,dy\right)=\int_{\partial B(x,r)} f(y)\,dS(y)$$

· 散度定理（分部积分）：

$$\int_\Omega \operatorname{div}\vec{F}\,dx=\int_{\partial\Omega}\vec{F}\cdot\vec{n}\,dS(x).$$

定义：设 $u\in C(\Omega)$.

(1) 称 u 满足平均值性质，如果 $\forall B_r(x)\subset\Omega$，$u(x)=\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy$.

(2) 称 u 满足第二平均值性质，如果 $\forall B_r(x)\subset\Omega$，$u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y)$.

断言：(1) 与 (2) 等价.

（以 $n=3$ 为例）证明：(2)⇒(1)：$\int_{B_r(x)} u(y)\,dy=\int_0^r\int_{\partial B_\rho(x)} u(y)\,dS(y)\,d\rho=\int_0^r u(x)\cdot 4\pi\rho^2\,d\rho$

$=\frac{4\pi r^3}{3}u(x)=|B_r(x)|\cdot u(x)$. $\Rightarrow u(x)=\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy$.

(1)⇒(2)：将 $\frac{4\pi r^3}{3}u(x)=\int_{B_r(x)} u(y)\,dy$ 两边对 r 求导，得 $4\pi r^2 u(x)=\int_{\partial B_r(x)} u(y)\,dS(y)$

$\Rightarrow u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y)$.

再对 (2) 中等号右侧改写（$|y-x|=r$，$y-x=r\omega$）：

$$u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y)=\frac{1}{4\pi \cancel{r^2}}\int_{|\omega|=1} u(x+r\omega)\cancel{r^2}\,dS(\omega)=\frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} u(x+r\omega)\,dS(\omega).$$

定理2.2(平均值公式) 设$u\in C^2(\Omega)$是Ω上的调和函数，则对任意的闭球$B_r(x)\subset\Omega$，有$u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)dS(y)$.

证明: $0=\int_{B_r(x)}(\Delta u)(y)dy=\int_{B_r(x)}\mathrm{div}\nabla u dy\xlongequal{\text{散度定理}}\int_{\partial B_r(x)}\nabla u\cdot\vec{n}dS(y)$

$=\int_{|y-x|=r}(\nabla u)(y)\cdot\frac{y-x}{r}dS(y)\xlongequal{y=x+r\omega}\int_{|\omega|=1}\omega\cdot(\nabla u)(x+r\omega)r^2dS(\omega)$

$=r^2\int_{|\omega|=1}\frac{d}{dr}(u(x+r\omega))dS(\omega)=r^2\frac{d}{dr}\int_{|\omega|=1}u(x+r\omega)dS(\omega)$，因此

$\frac{d}{dr}\int_{|\omega|=1}u(x+r\omega)dS(\omega)=0$，从而$\int_{|\omega|=1}u(x+r\omega)dS(\omega)=\int_{|\omega|=1}u(x)dS(\omega)=u(x)\int_{|\omega|=1}dS(\omega)$

$=4\pi u(x)\Rightarrow u(x)=\frac{1}{4\pi}\int_{|\omega|=1}u(x+r\omega)dS(\omega)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)dS(y)$. □

定理2.3 假设$u\in C^2(\Omega)$满足对于任意的$B_r(x)\subset\Omega$，$u(x)=\frac{1}{|B_r(x)|}\int_{B_r(x)}u(y)dS(y)$，则$u$是调和函数.

证明: $\forall B_r(x)\subset\Omega$，$\int_{B_r(x)}(\Delta u)(y)dy=r^2\frac{d}{dr}\int_{|\omega|=1}u(x+r\omega)dS(\omega)\xlongequal{\text{第二平均值性质}}r^2\frac{d}{dr}(4\pi u(x))=0$.

断言: $\Delta u\equiv0$ in Ω. (用反证法) 否则，存在$x_0\in\Omega$，使得$(\Delta u)(x_0)=c\neq0$，不妨设$c>0$.

由$u\in C^2(\Omega)$ (从而连续)，存在r_0，使得$(\Delta u)(x)\geq\frac{c}{2}$，$\forall x\in B_{r_0}(x_0)$. 于是

$\int_{B_{r_0}(x_0)}(\Delta u)(y)dy\geq\frac{c}{2}\cdot\frac{4}{3}\pi r_0^3>0$，矛盾!

定理2.4 (Harnack不等式) 对于Ω上的任何连通紧子集V，存在一个仅与距离函数$d(V,\partial\Omega)=\min_{x\in V,y\in\partial\Omega}|x-y|$和维数有关的正常数$C$，使得$\sup_V u\leq C\inf_V u$，其中$u$是$\Omega$上的任意非负调和函数. 特别地，对任意$x,y\in V$，$\frac{1}{C}u(y)\leq u(x)\leq Cu(y)$.

证明: 令$r=\frac{1}{4}d(V,\partial\Omega)$.

(1) 先考虑$x,y\in V$，$|x-y|\leq r$. 则由距离的三角不等式可知$B_r(y)\subset B_{2r}(x)$，从而

$u(x)=\frac{1}{|B_{2r}(x)|}\int_{B_{2r}(x)}u(z)dz\geq\frac{|B_r(y)|}{|B_{2r}(x)|}\cdot\frac{1}{|B_r(y)|}\cdot\int_{B_r(y)}u(z)dz=\frac{1}{2^n}u(y)$.

(2) 由于V是连通的紧集，我们可用一串有限多个半径为r的球$\{B_i\}_{i=1}^N$覆盖它，且要求$B_i\cap B_{i-1}\neq\phi$ $(i=2,\cdots,N)$. 于是，对任意$x,y\in V$，有$u(x)\geq\frac{1}{2^{nN}}u(y)$. □

定理2.3' 若$u\in C(\Omega)$满足平均值性质，则$u\in C^\infty(\Omega)$且u是调和的.

(卷积) $(f*g)(x)=\int_{\mathbb{R}^n}f(x-y)g(y)dy$. ①若$f$和$g$有一个光滑，则$f*g$光滑. ②$\partial_i(f*g)=(\partial_if)*g=f*(\partial_ig)$.

$(n=3)$证明: 令$\varphi\in \underline{C_0^\infty(B_1(0))}$ (即: $\varphi\in C^\infty(\mathbb{R}^n)$，$\varphi\equiv1$ on $B_{\frac12}(0)$，$\varphi\equiv0$ on $\overline{B_1(0)}^c$)，$\varphi\geq0$，且要求φ是径向函数 ($\varphi(x)=\varphi(|x|)$)，$\int_{\mathbb{R}^n}\varphi(x)dx=1$.

令$\varphi_\varepsilon(x)=\frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$，则$\mathrm{supp}\,\varphi_\varepsilon\subset\overline{B_\varepsilon(0)}$，且$\int_{B_\varepsilon(0)}\varphi_\varepsilon(x)dx=1$.

由$\int_{B_1(0)}\varphi(x)dx=1$可知$1=\int_0^1\int_{|\omega|=1}\varphi(r\omega)r^2dS(\omega)dr=\int_0^1\left(\int_{|\omega|=1}dS(\omega)\right)\varphi(r)r^2dr=4\pi\int_0^1\varphi(r)r^2dr$.

$\forall x\in\Omega$, $(u*\varphi_\varepsilon)(x) \xlongequal[B_\varepsilon(x)\subset\Omega]{\varepsilon\text{充分小}} \int_{B_\varepsilon(x)} u(y)\varphi_\varepsilon(x-y)\,dy \xlongequal[dy=\varepsilon^n dz]{\frac{y-x}{\varepsilon}=z} \int_{|z|<1} u(x+\varepsilon z)\varphi(z)\,dz$

$= \int_0^1\int_{|\omega|=1} u(x+\varepsilon r\omega)\varphi(r)r^2\,dS(\omega)\,dr \xlongequal{\text{平均值性质}} \int_0^1 4\pi u(x)\varphi(r)r^2\,dr = u(x)$.

即 $u(x)=(u*\varphi_\varepsilon)(x)$ 光滑，再由定理2.3，u 是调和函数. □

定理2.7 若 $u\in C(\overline{B_R})$ 是调和的，则 $|\nabla u(x_0)|\leq \frac{n}{R}\max_{\overline{B_R}}|u(x)|$，这里 $B_R:=B_R(x_0)$.

证明：由于 u 是调和的，u 满足平均值性质，由定理2.3'，u 在 B_R 上是光滑的，且 $\Delta u=0$ in B_R. 两边对 x_i 求偏导，可得 $\Delta(\partial_{x_i}u)=0$，即 $\partial_{x_i}u$ 也是调和的，故 $\partial_{x_i}u$ 满足平均值性质，$\partial_{x_i}u(x_0)=\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}(\partial_{x_i}u)(y)\,dy=\frac{1}{|B_R(x_0)|}\int_{\partial B_R(x_0)}u\,n_i\,dS$. （$n_i$：n的第i个分量）

$|\partial_{x_i}u(x_0)|\leq\frac{1}{|B_R(x_0)|}\int_{\partial B_R(x_0)}|u|\,dS\leq\max_{\overline{B_R(x_0)}}|u|\cdot\frac{|\partial B_R(x_0)|}{|B_R(x_0)|}\xlongequal{(n=3)}\frac{n}{R}\cdot\max_{\overline{B_R}}|u|$. □

定理2.8 (Liouville定理) 设 u 是 $\mathbb{R}^n$ 上的有界调和函数，则 u 是常数.

证明：$\forall x_0\in\mathbb{R}^n$，$\forall R>0$，$u$ 是 $B_R(x_0)$ 上的调和函数，且 $u\in C(\overline{B_R(x_0)})$. 由定理2.7，$|\nabla u(x_0)|\leq\frac{n}{R}\max_{\overline{B_R(x_0)}}|u|$. 由 u 有界，存在 $M>0$，使得 $|u(x)|\leq M$，$\forall x\in\mathbb{R}^n$. 故 $|\nabla u(x_0)|\leq\frac{n}{R}M$.
令 $R\to+\infty$ 有 $|\nabla u(x_0)|=0$，由于 x_0 任意，$\nabla u(x)\equiv0$，故 u 是常数. □

定理2.9 设 u 是 Ω 上的调和函数，则 u 是 Ω 上的实解析函数.

§2.2 基本解和Green函数

$$\boxed{\Delta_{\mathbb{R}^n}=\partial_r^2+\frac{n-1}{r}\partial_r+\frac{1}{r^2}\Delta_{S^{n-1}}}$$

目标：求解Poisson方程 $\Delta u=f$.

$\Delta T=\cancel{\delta_x}\ \delta_0(x)$，$\delta_x$ (Dirac函数) 是卷积运算下的单位元：$f=f*\delta$

$\Rightarrow f=f*\delta=f*\Delta T=\Delta(f*T)\Rightarrow u=T*f$. Dirac函数是径向函数，与Laplace算子一样在正交变换下不变 $\Rightarrow$ T也是径向函数. $\delta_0(x)=\begin{cases}1, & x=0,\\ 0, & x\neq0.\end{cases}$

用极坐标写：$\partial_r^2T+\frac{n-1}{r}\partial_rT=\delta_0(r)$ （$\Delta_{S^{n-1}}$ 作用在只与 r 有关的T上为0）.

当 $r\neq0$ 时，$\partial_r^2T+\frac{n-1}{r}\partial_rT=0$. 令 $v=\partial_rT$，则 $\partial_rv+\frac{n-1}{r}v=0$. $\Rightarrow v=Cr^{-(n-1)}$.

$\Rightarrow$ T由 v 积分：$T(x)=\begin{cases}C_1\ln r+C_2, & n=2,\\ C_1r^{-(n-2)}+C_2, & n\geq3.\end{cases}$

令 $T(x)=\begin{cases}\frac{1}{2\pi}\ln|x|, & n=2,\\ -\frac{1}{4\pi}\frac{1}{|x|}, & n=3.\end{cases}$ 称为 $\Delta u=f$ 的基本解.

如果一个在 $\mathbb{R}^n$ 上的调和函数是径向对称的，则它为常数.

· Green公式：若 $u,v\in C^2(\Omega)\cap C^1(\bar{\Omega})$，则

$$\int_\Omega \Delta u\cdot v\,dx=\int_\Omega[\nabla\cdot(\nabla u v)-\nabla u\nabla v]\,dx \xlongequal{\text{散度定理}} \int_{\partial\Omega}\frac{\partial u}{\partial n}v\,dS-\int_\Omega\nabla u\cdot\nabla v\,dx \quad \cancel{\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}v-\nabla u\nabla v\right)dS}$$

同理，$\int_\Omega \Delta v\cdot u\,dx=\cancel{\int_{\partial\Omega}\left(\frac{\partial v}{\partial n}u-\nabla u\nabla v\right)dS}\ \int_{\partial\Omega}\frac{\partial v}{\partial n}u\,dS-\int_\Omega\nabla u\cdot\nabla v\,dx$. 两式作差得

$$\boxed{\int_\Omega(\Delta u\cdot v-\Delta v\cdot u)\,dx=\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}v-\frac{\partial v}{\partial n}u\right)dS}$$ （第二Green公式）

$(n=3)$ 若 $u\in C^2(\Omega)\cap C^1(\bar{\Omega})$ 满足 $\Delta u=0$ in Ω，则 $\forall x_0\in\Omega$，

$$u(x_0)=\frac{1}{4\pi}\int_{\partial\Omega}\left(-u\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{|x-x_0|}\cdot\frac{\partial u}{\partial n}\right)dS.$$

证明：不妨设 $x_0=0$.（因为 $\int_{\partial\Omega}-u\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)dS \xlongequal[\tilde{\Omega}=\Omega-x_0(\text{平移})]{y=x-x_0} \int_{\partial\tilde{\Omega}}-u(y+x_0)\frac{\partial}{\partial n}\left(\frac{1}{|y|}\right)dS(y)$，

令 $v(x)=u(x_0+x)$，则要证 $v(0)=\frac{1}{4\pi}\int_{\partial\tilde{\Omega}}\left(-u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\cdot\frac{\partial v}{\partial n}\right)dS$，且调和函数在平移下不变调和性），则只需证若 $0\in\Omega$，则 $u(0)=\frac{1}{4\pi}\int_{\partial\Omega}\left(-u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\frac{\partial u}{\partial n}\right)dS$.

令 $\Omega_\varepsilon=\Omega\setminus\overline{B_\varepsilon(0)}$，对 u 和 $v=\frac{1}{|x|}$ 在 Ω_ε 上应用第二Green公式，

$$0=\int_{\Omega_\varepsilon}\left(u\Delta\left(\frac{1}{|x|}\right)-\frac{1}{|x|}\Delta u\right)dx=\int_{\partial\Omega_\varepsilon}\left(u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)-\frac{1}{|x|}\frac{\partial u}{\partial n}\right)dS$$
$$=\int_{\partial\Omega}\left(u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)-\frac{1}{|x|}\frac{\partial u}{\partial n}\right)dS+\underbrace{\int_{|x|=\varepsilon}\left(u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)-\frac{1}{|x|}\frac{\partial u}{\partial n}\right)dS}_{(*)}$$

处理 $(*)$ 时，注意到 $n=-\frac{x}{|x|}$，$\frac{\partial u}{\partial n}=-\partial_r u$.

$$(*)=\int_{|x|=\varepsilon}\left[u\left(-\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)-\frac{1}{r}\frac{\partial u}{\partial n}\right]dS=\underbrace{\frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}u\,dS}_{(*1)}+\underbrace{\frac{1}{\varepsilon}\int_{|x|=\varepsilon}\frac{\partial u}{\partial r}dS}_{(*2)}.$$

在 $B_\varepsilon(0)$ 里看

$$(*2)=\frac{1}{\varepsilon}\int_{|x|=\varepsilon}\frac{\partial u}{\partial n}dS=\frac{1}{\varepsilon}\int_{|x|<\varepsilon}\nabla\cdot(\nabla u)\,dx=\frac{1}{\varepsilon}\int_{|x|<\varepsilon}\Delta u\,dx=0.$$

$$(*1)=4\pi\cdot\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}u(x)\,dS(x)\xlongequal{\text{平均值性质}}4\pi u(0).$$

$$\Rightarrow u(0)=\frac{1}{4\pi}\int_{\partial\Omega}\left[-u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\frac{\partial u}{\partial n}\right]dS.$$ □

用基本解 $\Gamma(x)=-\frac{1}{4\pi}\frac{1}{|x|}$ 写即 $u(x_0)=\int_{\partial\Omega}\left[u\frac{\partial}{\partial n}\Gamma(x-x_0)-\Gamma(x-x_0)\frac{\partial u}{\partial n}\right]dS$.

对 $n=2$，即用 $\Gamma(x)=\frac{1}{2\pi}\ln|x|$ 替换：$u(x_0)=\int_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln|x-x_0|\right)-\frac{1}{2\pi}\ln|x-x_0|\frac{\partial u}{\partial n}\right]dS$.

若 $\Delta u\neq 0$，有更一般的公式：

$$u(x_0)=\int_\Omega-\frac{1}{4\pi|x-x_0|}\Delta u(x)\,dx+\int_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(-\frac{1}{4\pi|x-x_0|}\right)-\left(-\frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}\right)\right]dS.$$

（事实上，这里证明用平均值性质过强，因为有

$$\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}u(x)\,dS=\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}[u(x)-u(0)]\,dS+\underbrace{\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}u(0)\,dS}_{=u(0)}$$，而

$$\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}[u(x)-u(0)]\,dS\xlongequal{\text{中值定理}}\max_{\bar{\Omega}}|\nabla u|\cdot\frac{\varepsilon}{4\pi\varepsilon^2}\cdot 4\pi\varepsilon^2=\varepsilon\cdot\max_{\bar{\Omega}}|\nabla u|,$$

用到了 $u\in C^1(\bar{\Omega})$.

令 $\varepsilon\to 0^+$ 同样可得）

考虑方程 $\begin{cases}\Delta u=0 & \text{in } \Omega\\ u=g & \text{on } \partial\Omega\end{cases}$

由前述公式，$u(x)=\frac{1}{4\pi}\int_{\partial\Omega}\left[-u(y)\frac{\partial}{\partial n}\left(\frac{1}{|x-y|}\right)+\frac{1}{|x-y|}\frac{\partial u}{\partial n}(y)\right]dS(y)$.（基本积分公式） ①

问题：无论是三类边值问题中的哪一类，都无法同时知道 u 和 $\frac{\partial u}{\partial n}$ 在 $\partial\Omega$ 上的值.

解决方法：若 $\varphi(x)$ 在 Ω 上是调和的，且 $\varphi|_{\partial\Omega}=\frac{1}{4\pi|x-y|}$，则对 u,φ 在 Ω 上用第二 Green 公式，$0=\int_{\partial\Omega}\left[u(y)\frac{\partial\varphi}{\partial n}(y)-\varphi(y)\frac{\partial u}{\partial n}(y)\right]dS(y)$ ②

①+②，得 $u(x)=\int_{\partial\Omega}u(y)\left[\frac{\partial\varphi}{\partial n}-\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-y|}\right)\right]dS(y)$

故 $\begin{cases}\Delta u=0 & \text{in } \Omega\\ u=g & \text{on } \partial\Omega\end{cases}$ 的解为 $u(x)=\int_{\partial\Omega}g(y)\frac{\partial}{\partial n}\Big(\varphi-\underbrace{\frac{1}{4\pi|x-y|}}_{\to T(x-y)}\Big)dS(y)=\int_{\partial\Omega}g(y)\frac{\partial}{\partial n}(\varphi+T(x-y))dS(y)$

令 $G(x_0,x)=\varphi(x_0,x)+T(x_0-x)$，称 $G(x_0,x)$ 是 Ω 上的算子 Δ 的 Green 函数.

(1) $G(x_0,x)$ 在 $\Omega\setminus\{x_0\}$ 上二阶连续可微且 $\Delta G=0$.

(2) $G(x_0,x)=0$ on $\partial\Omega$.

(3) $G(x_0,x)+\frac{1}{4\pi|x_0-x|}$ 在 Ω 上是调和的.

性质：$G(x_0,x)=G(x,x_0)$，$\forall x\neq x_0$.

证明：令 $u(x)=G(x,a)$，$v(x)=G(x,b)$，要证 $u(b)=v(a)$. 对 u,v 在 $\Omega_\varepsilon=\Omega\setminus(\overline{B_\varepsilon(a)}\cup\overline{B_\varepsilon(b)})$ 上应用第二 Green 公式，

$$0=\int_{\Omega_\varepsilon}(u\Delta v-v\Delta u)dx=\int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS=\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS+\underbrace{\int_{\partial B_\varepsilon(a)}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS}_{A_\varepsilon}+\underbrace{\int_{\partial B_\varepsilon(b)}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS}_{B_\varepsilon}$$

$$A_\varepsilon=\int_{\partial B_\varepsilon(a)}\left[\left(u+\frac{1}{4\pi|x-a|}\right)\frac{\partial v}{\partial n}-v\frac{\partial}{\partial n}\left(u+\frac{1}{4\pi|x-a|}\right)\right]dS-\int_{\partial B_\varepsilon(a)}\left[\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n}-v\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-a|}\right)\right]dS$$

而 $\int_{\partial B_\varepsilon(a)}\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n}dS=\frac{1}{4\pi\varepsilon}\int_{\partial B_\varepsilon(a)}\frac{\partial v}{\partial n}dS=-\frac{1}{4\pi\varepsilon}\int_{B_\varepsilon(a)}\overbrace{\Delta v}^{=0}dx=0$,

$\int_{\partial B_\varepsilon(a)}v\underbrace{\frac{\partial}{\partial n}\left(\frac{1}{|x-a|}\right)}_{-\frac{\partial}{\partial r}\left(\frac{1}{r}\right)}dS=\frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon(a)}v\,dS=4\pi v(a)$. $\Rightarrow A_\varepsilon=v(a)$. 同理，$B_\varepsilon=-u(b)$. 得证. □

解：$u(x)=\int_{\partial\Omega}g(y)\frac{\partial G(x,y)}{\partial n}dS(y)$（Poisson 公式）

Green 函数的求法

1. 半空间 $\mathbb{R}^3_+=\{x\in\mathbb{R}^3\mid x_3>0\}$.

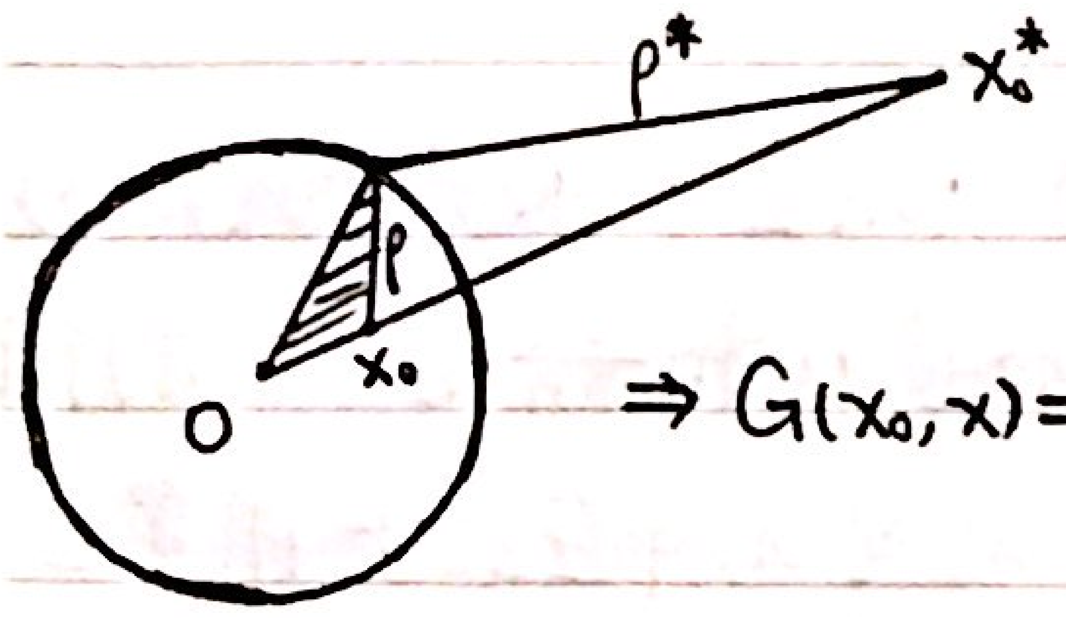
$\forall x_0\in\mathbb{R}^3_+$，$G(x_0,x)=-\frac{1}{4\pi|x-x_0|}+\frac{1}{4\pi|x-x_0^*|}$，$x_0^*$ 与 x_0 关于边界 $x_3=0$ 对称.

2. 球 $B_R(0)\subset\mathbb{R}^3$.

$\forall x_0\in B_R(0)$，$G(x_0,x)=-\frac{1}{4\pi|x-x_0|}+\frac{C}{4\pi|x-x_0^*|}$，$x_0^*\notin B_R(0)$. 记 $\rho=|x-x_0|$，$\rho^*=|x-x_0^*|$，

长度求导就是单位化

要找 x_0^*, C, 使得 $\frac{\rho^*}{\rho}=C$.



若 $\triangle$阴 $\simeq\triangle$大, 则 $\frac{|x_0|}{R}=\frac{\rho}{\rho^*}=\frac{R}{|x_0^*|}=\frac{1}{C}$.

$\Rightarrow |x_0^*|=\frac{R^2}{|x_0|}$, $x_0^*=\frac{R^2}{|x_0|^2}x_0$, $C=\frac{R}{|x_0|}$.

$$\Rightarrow G(x_0,x)=-\frac{1}{4\pi|x-x_0|}+\frac{R}{4\pi|x_0|\cdot|x-x_0^*|},\quad x_0^*=\frac{R^2}{|x_0|^2}x_0.$$

为了写出解的表达式, 只需再求 $\frac{\partial G}{\partial n}$, 即 $\nabla G\cdot\frac{x}{R}$.

$$\nabla G(x_0,x)=\frac{x-x_0}{4\pi|x-x_0|^3}-\frac{R(x-x_0^*)}{4\pi|x_0||x-x_0^*|^3}=\frac{x-x_0}{4\pi|x-x_0|^3}-\frac{1}{4\pi}\cdot\frac{R}{|x_0|}\cdot\frac{x-x_0^*}{\left(\frac{R}{|x_0|}|x-x_0|\right)^3}$$

$$=\frac{1}{4\pi|x-x_0|^3}\left(x-x_0-\left(\frac{|x_0|}{R}\right)^2(x-x_0^*)\right)=\frac{1}{4\pi|x-x_0|^3}\left(\frac{R^2-|x_0|^2}{R^2}x-x_0+\frac{|x_0|^2}{R^2}x_0^*\right)$$

$$=\frac{R^2-|x_0|^2}{4\pi R^2|x-x_0|^3}\cdot x$$

$$\Rightarrow \frac{\partial G}{\partial n}=\frac{x}{R}\cdot\nabla G=\frac{R^2-|x_0|^2}{R}\cdot\frac{1}{4\pi|x-x_0|^3}.$$

于是由 Poisson 公式, $u(x)=\frac{R^2-|x|^2}{4\pi R}\int_{|y|=R}\varphi(y)\frac{1}{|y-x|^3}dS(y)$. (*)

定理 2.4′ (Harnack 不等式) 设 u 在 $B_R(x_0)$ 内调和, $u\geq 0$, 则

$$\frac{R}{R+r}\cdot\frac{R-r}{R+r}u(x_0)\leq u(x)\leq\frac{R}{R-r}\cdot\frac{R+r}{R-r}u(x_0),\text{ 其中 } r=|x-x_0|<R.$$

证明: 不妨设 $x_0=0$, 则由 (*) 式, $u(x)=\frac{R^2-|x|^2}{4\pi R}\int_{|y|=R}u(y)\frac{1}{|y-x|^3}dS(y)$.

由于 $|y|=R$, $|x|=r$, $R-r\leq|y-x|\leq R+r$, 故

① $u(x)\leq\frac{R^2-|x|^2}{4\pi R}\cdot\frac{1}{(R-r)^3}\int_{|y|=R}u(y)dS(y)\xlongequal{\text{平均值性质}}\frac{R^2-|x|^2}{4\pi R}\cdot\frac{4\pi R^2u(0)}{(R-r)^3}=\frac{(R^2-r^2)Ru(0)}{(R-r)^3}$

$=\frac{R(R+r)}{(R-r)^2}u(0)$.

② $u(x)\geq\frac{R^2-|x|^2}{4\pi R}\cdot\frac{1}{(R+r)^3}\int_{|y|=R}u(y)dS(y)=\frac{R^2-|x|^2}{4\pi R}\cdot\frac{4\pi R^2u(0)}{(R+r)^3}=\frac{R(R-r)}{(R+r)^2}u(0)$. □

定理 2.8′ (Liouville 定理) 设 u 是 $\mathbb{R}^n$ 上的上有界(或下有界)的调和函数, 则 u 是一个常数.

证明: 只证 u 有上界的情形, 设 $u\leq M$, $\forall x\in\mathbb{R}^n$. 令 $v=M-u\geq 0$, $\Delta v=-\Delta u=0$.
由 Harnack 不等式, $\forall x, x_0$, 选 $R>|x-x_0|=r$, 有
$\frac{R}{R+r}\cdot\frac{R-r}{R+r}v(x_0)\leq v(x)\leq\frac{R}{R-r}\cdot\frac{R+r}{R-r}v(x_0)$. 令 $R\to+\infty$, 则有 $v(x)=v(x_0)$, 即 v 是常数,
因此 u 是常数. □

§2.3 极值原理和最大模估计

考虑 $Lu \triangleq -\Delta u + c(x)u = f(x)$，$x\in\Omega$，$c(x)\geq 0, \forall x\in\Omega$. (2.38)

定理2.21 假设 $c(x)\geq 0$，$f(x)<0$. 如果 $u\in C^2(\Omega)\cap C(\bar\Omega)$ 满足方程(2.38)，则 $u(x)$ 不能在 Ω 上达到它在 $\bar\Omega$ 上的非负最大值，即 $u(x)$ 只能在 $\partial\Omega$ 上达到它的非负最大值.

i.e. $\max\limits_{\bar\Omega} u \leq \max\limits_{\partial\Omega} u^+$，$u^+=\max\{u(x),0\}$.

证明：假设 $u(x)$ 在 Ω 上一点 x_0 达到非负最大值，则 $u(x_0)\geq 0$，$(\Delta u)(x_0)\leq 0$，记 $Lu\triangleq -\Delta u + c(x)u$，则 $Lu|_{x=x_0} \triangleq (-\Delta u + c(x)u)|_{x=x_0}\geq 0$，但 $Lu=f<0$，矛盾！ □

想把"$f<0$"弱化为"$f\leq 0$".

(弱极值原理)

定理2.22 假设 $c(x)\geq 0$，$f(x)\leq 0$. 如果 $u\in C^2(\Omega)\cap C(\bar\Omega)$ 满足方程(2.38)，且在 $\bar\Omega$ 上存在正的最大值，则 $u(x)$ 必在 $\partial\Omega$ 上达到它在 $\bar\Omega$ 上的最大值，且 $\max\limits_{\bar\Omega} u\leq \max\limits_{\partial\Omega} u^+$.

(与定理2.21结论相比，未对是否在 Ω 上达到最大值下结论)

证明：不妨设 $0\in\Omega$，令 $d=\operatorname{diam}\Omega$，则 $\forall x\in\Omega, |x|\leq d$. 令 $v(x)=|x|^2-d^2$，则 $v(x)\leq 0$，$Lv(x)=-2n+c(x)(|x|^2-d^2)\leq -2n<0$. 于是，对 $w(x)$ 应用定理2.21可得 (其中 $w(x):=u(x)+\varepsilon v(x)$) $\max\limits_{\bar\Omega} w\leq \max\limits_{\partial\Omega} w^+ \leq \max\limits_{\partial\Omega} u^+$. 而 $w(x)=u(x)+\varepsilon v(x)\geq u(x)-\varepsilon d^2$，所以 $\max\limits_{\bar\Omega} w\geq \max\limits_{\bar\Omega} u(x)-\varepsilon d^2$，即 $\max\limits_{\bar\Omega} u(x)\leq \max\limits_{\partial\Omega} u^+ + \varepsilon d^2$. 令 $\varepsilon\to 0^+$，则有 $\max\limits_{\bar\Omega} u\leq \max\limits_{\partial\Omega} u^+$. □

定理2.23 (Hopf引理) 设 B_R 是 $\mathbb{R}^n$ $(n\geq 2)$ 中以 R 为半径的球，在 B_R 上 $c(x)\geq 0$ 且有界. 如果 $u\in C^2(B_R)\cap C^1(\bar B_R)$ 满足

(1) $Lu=-\Delta u + c(x)u\leq 0$，$x\in B_R$；

(2) 存在 $x_0\in\partial B_R$ 使得 $u(x)$ 在 x_0 点达到在 $\bar B_R$ 上的严格非负最大值，即 $u(x_0)=\max\limits_{\bar B_R} u(x)\geq 0$，且当 $x\in B_R$ 时，$u(x)<u(x_0)$，且当 $x\in B_R$ 时，$u(x)<u(x_0)$，则

$\frac{\partial u}{\partial \nu}\Big|_{x=x_0}>0$，其中 ν 与 ∂B_R 在 x_0 点的单位外法向量 n 的夹角小于 $\frac{\pi}{2}$.

证明：$\frac{\partial u}{\partial \nu}\Big|_{x=x_0}\geq 0$ 是显然的，需证它严格大于0. 不妨设 B_R 以 0 为球心.

令 $v(x)=e^{-\alpha|x|^2}-e^{-\alpha R^2}$，$\alpha>0$，则 ① $v(x)>0, \forall x\in B_R$；且 $\forall x_0\in\partial B_R$，$v(x_0)=0$.

② $\frac{\partial v}{\partial r}\Big|_{x=x_0}=-2\alpha r e^{-\alpha r^2}\Big|_{x=x_0}=-2\alpha R e^{-\alpha R^2}<0$.

令 $w(x)=u(x)+\varepsilon v(x) - u(x_0)$，$\varepsilon>0$.

断言：$w(x)$ 在 $x_0\in\partial B_R$ 达到它在 $\bar B_R$ 上的非负最大值.

$(w(x)=u(x)+\varepsilon v(x)-u(x_0))$

事实上，当$|x|=R$时，$w(x)=u(x)-u(x_0)+\varepsilon \underbrace{v(x)}_{=0}=u(x)-u(x_0)$在$x_0$达到它在$\partial B_R$上的最大值.

$\partial_{x_i}v=-2\alpha x_i e^{-\alpha|x|^2}$，$\partial_{x_i}^2 v=-2\alpha e^{-\alpha|x|^2}+4\alpha^2 x_i^2 e^{-\alpha|x|^2}$.

$\Rightarrow \Delta v=(-2\alpha n+4\alpha^2|x|^2)e^{-\alpha|x|^2}$.

c(x)有界C

$\Rightarrow Lv=(-4\alpha^2|x|^2+2\alpha n)e^{-\alpha|x|^2}+c(x)(e^{-\alpha|x|^2}-e^{-\alpha R^2})\leq(\underline{-4\alpha^2|x|^2}+2\alpha n+C)e^{-\alpha|x|^2}$.

不能含0点

令$B_R^*=\{x\in\mathbb{R}^n \mid \frac{R}{2}<|x|<R\}$，则在$B_R^*$上，当$\alpha$充分大时，

$Lv\leq(-\alpha^2R^2+2\alpha n+C)e^{-\alpha|x|^2}<0$.

由定理2.21，$\max\limits_{\overline{B_R^*}} w(x)=\max\limits_{\partial B_R^*} w(x)$. （但注意$\partial B_R^*$由两部分构成）

当$|x|=\frac{R}{2}$时，$w(x)=u(x)-u(x_0)+\varepsilon\left(e^{-\alpha\frac{R^2}{4}}-e^{-\alpha R^2}\right)\leq \underbrace{\max\limits_{|x|=\frac{R}{2}} u(x)-u(x_0)}_{<0}+\varepsilon\left(e^{-\alpha\frac{R^2}{4}}-e^{-\alpha R^2}\right)$

$\overset{\varepsilon充分小}{<}0$. 于是$w$在$x_0$处达到它在$\overline{B_R^*}$上的严格最大值0.

故$\frac{\partial w}{\partial \nu}(x_0)\geq 0\Rightarrow \frac{\partial u}{\partial \nu}(x_0)\geq-\varepsilon\frac{\partial v}{\partial \nu}(x_0)>0$. □

定理2.24（强极值原理）假设Ω是$\mathbb{R}^n$上的有界连通开集，$c(x)\geq0$且有界. 如果$u\in C^2(\Omega)\cap C(\overline{\Omega})$在$\Omega$上满足$Lu\leq0$，且$u(x)$在$\Omega$内达到其在$\overline{\Omega}$上的非负最大值，则$u$在$\overline{\Omega}$上是常数.

证明：令$M=\max\limits_{\overline{\Omega}} u\geq0$. 令$O=\{x\in\Omega \mid u(x)=M\}$，要证$O=\Omega$.（由于$O$非空且$\Omega$连通，只需证$O$相对于$\Omega$既开又闭）

① O是闭集由$u(x)$的连续性即得.

② O相对于Ω是开集：$\forall x_0\in O$，$\exists r$，使$B_{2r}(x_0)\subset\Omega$. 若O不是Ω中开集，则存在$x_0\in\partial O$，$\tilde{x}\in B(x_0,r)$，但$\tilde{x}\notin O$. 故$|\tilde{x}-x_0|<r$. 令$d=d(\tilde{x},O)>0$，$B(\tilde{x},d)\subset B(x_0,2r)\subset\Omega$.

$\forall x\in B(\tilde{x},d)$，$u(x)<M$，$\exists y_0\in\partial B(\tilde{x},d)\cap O$，使$\forall x\in B(\tilde{x},d)$，$u(x)<M(=u(y_0))$. 由Hopf引理，$\frac{\partial u}{\partial n}(y_0)>0$. 但$u$在$y_0\in\Omega$达到最大值，$\nabla u(y_0)=0$，矛盾！ □

最大模估计

定理2.25 假设$u\in C^2(\Omega)\cap C(\overline{\Omega})$是Dirichlet问题

$$\begin{cases}\Delta u=f & \text{in } \Omega\\ u=g & \text{on } \partial\Omega\end{cases}$$

的解，则$\max\limits_{\overline{\Omega}}|u(x)|\leq G+CF$，其中$G=\max\limits_{\partial\Omega}|g(x)|$，$F=\max\limits_{\overline{\Omega}}|f(x)|$，$C$是一个仅依赖于维数$n$和$\Omega$的直径的常数.

证明：令$v(x)=u(x)-G+\frac{F}{2n}(|x|^2-d^2)$，则$\Delta v=f+F\geq0$ in Ω，$v\leq g-G\leq0$ on $\partial\Omega$.

不妨设$0\in\Omega$. 由极值原理，$\max\limits_{\overline{\Omega}} v\leq0.\Rightarrow u(x)\leq G+\frac{F}{2n}(d^2-|x|^2)\leq G+\frac{F}{2n}d^2$.

令 $\tilde{v} = -u - G + \frac{F}{2n}(|x|^2 - d^2)$，则 $\Delta\tilde{v} = -f + F \geq 0$ in Ω，$v \leq g - G \leq 0$ on $\partial\Omega$，由极值原理，$\max\limits_{\Omega} \tilde{v} \leq 0 \Rightarrow -u(x) \leq G + \frac{F}{2n}(d^2 - |x|^2) \leq G + \frac{F}{2n}d^2$. 故 $\max\limits_{\Omega}|u(x)| \leq G + \frac{d^2}{2n}F$. □

最大模估计蕴含着Dirichlet边值问题的解的唯一性(和稳定性).

方程 $\begin{cases} \Delta u = f & \text{in}\,\Omega \\ u = g & \text{on}\,\partial\Omega \end{cases}$ 的 $C^2(\Omega)\cap C^1(\bar{\Omega})$ 的解是唯一的.

证明：设 u_1, u_2 是解，令 $v = u_1 - u_2$，则

$$\begin{cases} \Delta v = 0 & \text{in}\,\Omega, \\ v = 0 & \text{on}\,\partial\Omega. \end{cases}$$

由最大模估计，$\max\limits_{\Omega}|v(x)| \leq 0$. 故 $v(x) \equiv 0$，$u_1(x) \equiv u_2(x)$，$\forall x \in \bar{\Omega}$. □

热传导方程

考虑方程 $u_t-\Delta u=f(x,t)$，$x\in\Omega\subset\mathbb{R}^n$，$t>0$，$u(x,t)$未知.

初始条件 $u(x,0)=\varphi(x)$

边界条件 $\begin{cases}\text{Dirichlet}: u|_{\partial\Omega}=g(x,t)\\ \text{Neumann}: \frac{\partial u}{\partial n}|_{\partial\Omega}=g(x,t)\\ \text{Robin}: (u+\sigma\frac{\partial u}{\partial n})|_{\partial\Omega}=g(x,t)\end{cases}$

若Ω为区间、矩形、球形，可用分离变量法解(见波方程作业).

§3.1 初值问题

$$\begin{cases}\partial_t u-\Delta u=f(x,t),\ x\in\mathbb{R}^n,\ t>0.\\ u(x,0)=\varphi(x).\end{cases}$$

注意这里$t>0$表示热方程只能往前演化，而不能逆推温度分布规律，这一点与波方程$t>0$的意义不同.

由$-\Delta(e^{-ix\cdot\xi})=|\xi|^2e^{-ix\cdot\xi}$知$\mathrm{Spec}(-\Delta)=[0,+\infty)$为连续谱，能够说明，$\mathbb{R}^n$上的$-\Delta$只有连续谱，而没有离散的特征值.

Fourier变换：若$f\in L^1(\mathbb{R}^n)$，定义$\hat{f}(\xi)=\int_{\mathbb{R}^n}f(x)e^{-2\pi ix\cdot\xi}dx$.

性质：若$f\in\mathcal{S}(\mathbb{R}^n)$(Schwartz速降函数空间：光滑，本身及任意阶导数比任意多项式衰减得快)，则有：

(1)(平移) 令$(\tau_{x_0}f)(x)=f(x-x_0)$，则$\widehat{\tau_{x_0}f}(\xi)=e^{-2\pi ix_0\cdot\xi}\hat{f}(\xi)$.

【$\widehat{\tau_{x_0}f}(\xi)=\int_{\mathbb{R}^n}f(x-x_0)e^{-2\pi ix\cdot\xi}dx\overset{y=x-x_0}{=\!=}\int_{\mathbb{R}^n}f(y)e^{-2\pi i(y+x_0)\cdot\xi}dy=e^{-2\pi ix_0\cdot\xi}\hat{f}(\xi)$】

(2)(伸缩) 令$(S_\lambda f)(x)=f(\lambda x)$，则$\widehat{S_\lambda f}(\xi)=\lambda^{-n}\hat{f}(\lambda^{-1}\xi)$.

【$\widehat{S_\lambda f}(\xi)=\int_{\mathbb{R}^n}(S_\lambda f)(x)e^{-2\pi ix\cdot\xi}dx=\int_{\mathbb{R}^n}f(\lambda x)e^{-2\pi ix\cdot\xi}dx\overset{y=\lambda x}{=\!=}\int_{\mathbb{R}^n}f(y)e^{-2\pi i\lambda^{-1}y\cdot\xi}\lambda^{-n}dy$
$=\lambda^{-n}\hat{f}(\lambda^{-1}\xi)$.】

(3)对于多重指标$\alpha=(\alpha_1,\cdots,\alpha_n)$，$|\alpha|:=\alpha_1+\cdots+\alpha_n$，$x^\alpha:=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$，$\partial^\alpha:=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$，则$\widehat{\partial^\alpha f}(\xi)=(2\pi i\xi)^\alpha\hat{f}(\xi)$.

【只验证特殊情形：$\widehat{\partial_{x_j}f}(\xi)=\int_{\mathbb{R}^n}\partial_{x_j}f(x)e^{-2\pi ix\cdot\xi}dx\ \overset{\text{Schwartz速降}}{\underset{\text{分部积分}}{=\!=\!=}}-\int_{\mathbb{R}^n}f(x)\partial_{x_j}e^{-2\pi ix\cdot\xi}dx$
$=2\pi i\xi_j\int_{\mathbb{R}^n}f(x)e^{-2\pi ix\cdot\xi}dx=2\pi i\xi_j\hat{f}(\xi)$.】 物理空间求导$\longleftrightarrow$频率空间乘法

(4) $\widehat{(-2\pi ix)^\alpha f}(\xi)=\partial_\xi^\alpha\hat{f}(\xi)$.

【只验证特殊情形：$\widehat{-2\pi ix_jf}(\xi)=\int_{\mathbb{R}^n}-2\pi ix_jf(x)e^{-2\pi ix\cdot\xi}dx=\int_{\mathbb{R}^n}f(x)\partial_{\xi_j}(e^{-2\pi ix\cdot\xi})dx$

$= \partial_{\xi_j} \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx = \partial_{\xi_j} \hat{f}(\xi)$.】物理空间乘法$\longleftrightarrow$频率空间求导

(5) $\forall f,g \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{fg}(\xi) = \hat{f} * \hat{g}$, $\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

例：令 $f(x) = e^{-|x|^2}$，则 $\hat{f}(\xi) = \sqrt{\pi}\, e^{-\pi^2|\xi|^2}$.

证明：令 $F(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx$. ~~若只考虑~~ $n=1$，则 $F'(\xi) = \int_{\mathbb{R}^n} e^{-x^2}(-2\pi i x) e^{-2\pi i x\xi}$

$= \pi i \int_{\mathbb{R}^n} (e^{-x^2})' e^{-2\pi i x\xi} dx = \pi i\, \widehat{f'}(\xi) \overset{(3)}{=} \pi i (2\pi i \xi) \hat{f}(\xi) = -2\pi^2 \xi F(\xi)$.

$$\Rightarrow \begin{cases} F'(\xi) + 2\pi^2 \xi F(\xi) = 0, \\ F(0) = \int_{\mathbb{R}^n} f(x)\, dx = \sqrt{\pi}. \end{cases} \Rightarrow F(\xi) = \sqrt{\pi}\, e^{-\pi^2\xi^2}，即 \hat{f}(\xi) = \sqrt{\pi}\, e^{-\pi^2|\xi|^2}.$$ □

例($n=1$)：$(e^{-\pi|x|^2})^\wedge(\xi) = (e^{-|\sqrt{\pi}x|^2})^\wedge(\xi) \overset{(2)}{=\!=} e^{-\pi|\xi|^2}$.

定义 f 的逆变换 $\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi} d\xi$. 若 $f \in \mathcal{S}(\mathbb{R}^n)$，则 $\check{\hat{f}} = f$.

由前例知 $e^{-|x|^2} = (\sqrt{\pi}\, e^{-\pi^2|\xi|^2})^\vee$，即 $\frac{1}{\sqrt{\pi}} e^{-|x|^2} = (e^{-\pi^2|\xi|^2})^\vee$，于是

$(e^{-\pi|\xi|^2})^\vee = (e^{-\pi^2|\frac{\xi}{\sqrt{\pi}}|^2})^\vee = \sqrt{\pi} \cdot \frac{1}{\sqrt{\pi}} e^{-|\sqrt{\pi}x|^2} = e^{-\pi|x|^2}$. 进而可得

$(e^{-4\pi^2|\xi|^2 t})^\vee = (e^{-\pi^2|2\sqrt{t}\xi|^2})^\vee = \frac{1}{2\sqrt{\pi t}} e^{-\frac{|x|^2}{4t}}$.

当 $n \geq 1$ 时，$(e^{-4\pi^2|\xi|^2 t})^\vee = \int_{\mathbb{R}^n} e^{-4\pi^2|\xi|^2 t} \cdot e^{2\pi i x\cdot\xi} d\xi = \int_{\mathbb{R}^n} \prod_{j=1}^n (e^{-4\pi^2\xi_j^2} e^{2\pi i x_j \xi_j}) d\xi_1 \cdots d\xi_n$

$= \prod_{j=1}^n \int_{\mathbb{R}} e^{-4\pi^2\xi_j^2} e^{2\pi i x_j\xi_j} d\xi_j = \prod_{j=1}^n \left(\frac{1}{2\sqrt{\pi t}} e^{-\frac{x_j^2}{4t}}\right) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$.

现考虑热方程 $\begin{cases} \partial_t u - \Delta u = 0 \\ u(x,0) = \varphi(x) \end{cases}$，两边同时关于 x 作Fourier变换可得

$$\begin{cases} \partial_t \hat{u} + 4\pi^2|\xi|^2 \hat{u}(\xi) = 0, \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi). \end{cases} \Rightarrow \hat{u}(\xi,t) = e^{-4\pi^2|\xi|^2 t} \hat{\varphi}(\xi).$$

$\Rightarrow u(x,t) = \mathcal{F}^{-1}(\hat{u})(x,t) = (e^{-4\pi^2|\xi|^2 t})^\vee * \varphi = \underbrace{\left(\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}\right)}_{\text{在}t>0\text{时光滑,因此卷积后也光滑.}} * \varphi$

$= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\, dy$. (物理意义：$t=0$ 时温度分布混乱，在开始演化后也呈光滑分布)

注记：只有在 $\mathbb{R}^n$ 上的方程才能用Fourier变换法求解，若 Ω 是有界区域应用分离变量法. 但这两者是统一的：当 Ω 有界时，$-\Delta$ 的谱是离散的，用Fourier级数；当 $\Omega = \mathbb{R}^n$ 时，$-\Delta$ 的谱是连续的，用Fourier变换.

对波动方程也可应用Fourier变换法：

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0) = \varphi(x),\ \partial_t u(x,0) = \psi(x) \end{cases} \xrightarrow{\text{关于}t,x\text{都作Fourier变换}} (-4\pi^2\tau^2 + 4\pi^2|\xi|^2)\tilde{u}(\xi,\tau) = 0.$$

$\operatorname{supp} \tilde{u}(\tau,\xi) \subset \{\tau^2 = |\xi|^2\}$ ($\mathbb{R}^{1+n}$ 维空间中的锥面) $\longrightarrow$ 调和分析

$$u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\varphi(y)\,dy\ ,\ t>0.$$

断言：$\lim\limits_{t\to 0^+} u(x,t)=\varphi(x)$，若$\varphi$连续且有界.

分析：$u(x,t)=(K_t*\varphi)(x)$，其中$K_t(x)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$.

令$K(x)=\frac{1}{(4\pi)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4}}$，则$K_t(x)=(t^{-\frac{1}{2}})^n K(t^{-\frac{1}{2}}x)$.

$$\left.\begin{array}{l}\int_{\mathbb{R}^n}K(x)\,dx=1\\ \int_{\mathbb{R}^n}|K(x)|\,dx=1\ \text{有界}\\ \forall\eta>0,\ \int_{|x|>\eta}K_t(x)\,dx\xlongequal{t^{\frac{1}{2}}x=y}\int_{|y|>t^{\frac{1}{2}}\eta}K(y)\,dy\xrightarrow{t\to 0^+}0\end{array}\right\}\{K_t(x)\}_{t>0}\text{是一族逼近恒等}$$

断言的证明：$|u(x,t)-\varphi(x)|=\left|\int_{\mathbb{R}^n}K_t(y)\varphi(x-y)\,dy-\int_{\mathbb{R}^n}K_t(y)\varphi(x)\,dy\right|$

$$=\left|\int_{\mathbb{R}^n}t^{-\frac{n}{2}}K(t^{-\frac{1}{2}}y)[\varphi(x-y)-\varphi(x)]\,dy\right|\xlongequal{z=t^{-\frac{1}{2}}y}\left|\int_{\mathbb{R}^n}K(z)[\varphi(x-t^{\frac{1}{2}}z)-\varphi(x)]\,dz\right|$$

$$\le\underbrace{\int_{|z|>R}K(z)|\varphi(x-t^{\frac{1}{2}}z)-\varphi(x)|\,dz}_{①}+\underbrace{\int_{|z|\le R}K(z)|\varphi(x-t^{\frac{1}{2}}z)-\varphi(x)|\,dz}_{②}$$

$①\le 2\|\varphi\|_\infty\int_{|z|>R}\frac{1}{(4\pi)^{\frac{n}{2}}}e^{-\frac{|z|^2}{4}}\,dz\overset{\text{选}R}{<}\varepsilon.$

② 由于φ连续，$\forall|z|<R$，当t充分小时，$|\varphi(x-t^{\frac{1}{2}}z)-\varphi(x)|<\varepsilon$，因此

  $②\le\varepsilon\int_{|z|\le R}K(z)\,dz<\varepsilon.$

故$\lim\limits_{t\to 0^+}|u(x,t)-\varphi(x)|=0.$ □

解的性质：

(1) $u\in C^\infty$，$\forall t>0$.（热方程的特殊性质）

(2) $\sup\limits_{x\in\mathbb{R}^n,t>0}|u(x,t)|\le\sup\limits_{x\in\mathbb{R}^n}|\varphi(x)|$.（因为"核"积分$=1$）（无外部热源时，开始演化后最高/低温不会比初始时最高/低温更高/低）$\rightsquigarrow$若$\varphi>0$，则$u(x,t)>0$.

(3) 无限传播速度.

(4) 只沿正向演化（$u(x,t)\to u(x,-t)$后方程变了；往后演化无唯一性）

$$\begin{cases}\partial_t u-\Delta u=f(x,t)\\ u(x,0)=\varphi(x)\end{cases}\xrightarrow{\text{两边同时关于}x\text{作Fourier变换}}\begin{cases}\partial_t\hat{u}+4\pi^2|\xi|^2\hat{u}=\hat{f}(\xi,t),\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi).\end{cases}$$

$$\Rightarrow\hat{u}(\xi,t)=e^{-4\pi^2t|\xi|^2}\hat{\varphi}(\xi)+\int_0^t e^{-4\pi^2|\xi|^2(t-\tau)}\hat{f}(\xi,\tau)\,d\tau.$$

$$\Rightarrow u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4t}}\varphi(y)\,dy+\int_0^t\int_{\mathbb{R}^n}\frac{1}{[4\pi(t-\tau)]^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4(t-\tau)}}f(y,\tau)\,dy\,d\tau.$$

唯一性之能量估计

$$\begin{cases}\partial_t u-\Delta u=f & \text{in } \Omega\times\{t\mid t>0\}\\ u(x,0)=\varphi(x) \\ u|_{\partial\Omega}=0\end{cases}$$

方程两边乘u，再关于x积分可得

$$\int_\Omega (u\partial_t u-u\Delta u)\,dx=\int_\Omega fu\,dx.$$

而由散度定理，

$$LHS=\frac{1}{2}\partial_t\int_\Omega |u|^2dx-\underbrace{\int_{\partial\Omega}u\frac{\partial u}{\partial n}dS}_{=0}+\int_\Omega|\nabla u|^2dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx+\underbrace{\int_\Omega|\nabla u|^2dx}_{\geq 0}=\int_\Omega fu\,dx\leq \frac{1}{2}\int_\Omega u^2dx+\frac{1}{2}\int_\Omega f^2dx \qquad (\star)$$

$$\Rightarrow \frac{d}{dt}\int_\Omega u^2dx\leq \int_\Omega u^2dx+\int_\Omega f^2dx.$$

由Gronwall不等式，$\int_\Omega u^2dx\leq\int_\Omega\varphi^2dx+\int_0^t\int_\Omega f^2(x,s)\,dx\,ds.$

再由$(\star)$，即$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx+\int_\Omega|\nabla u|^2dx\leq\frac{1}{2}\int_\Omega u^2dx+\frac{1}{2}\int_\Omega f^2dx$，

对t从0到T积分得

$$\frac{1}{2}\int_\Omega u^2(x,T)dx-\frac{1}{2}\int_\Omega\varphi^2dx+\int_0^T\int_\Omega|\nabla u|^2dx\,dt\leq\frac{1}{2}\int_0^T\int_\Omega f^2dx\,dt+\frac{1}{2}\int_0^T\int_\Omega u^2dx\,dt$$

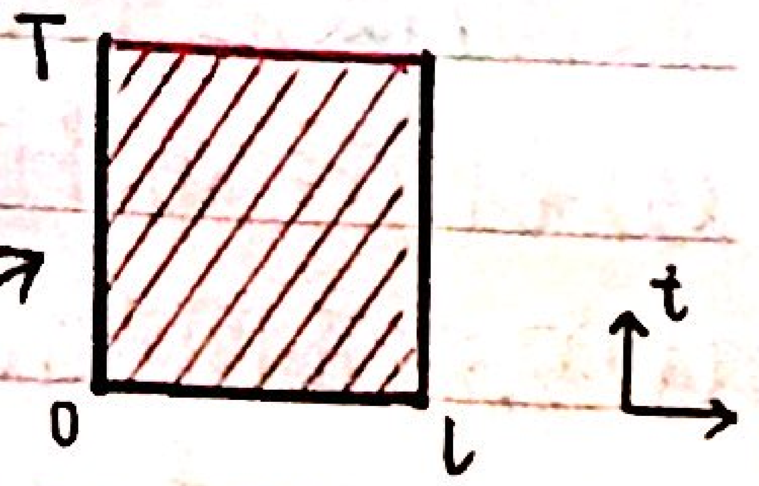
$$\leq\frac{1}{2}\int_0^T\int_\Omega f^2dx\,dt+T\left(\int_\Omega\varphi^2dx+\int_0^T\int_\Omega f^2dx\,dt\right)$$

$$\Rightarrow\frac{1}{2}\int_\Omega u^2(x,T)dx+\int_0^T\int_\Omega|\nabla u|^2dx\,dt\leq C_T\left(\int_\Omega\varphi^2dx+\int_0^T\int_\Omega|f|^2dx\,dt\right).$$

特别地，若$\varphi\equiv0$，$f\equiv0$，则$u(x,T)\equiv0$，由T的任意性即得解的唯一性.

§3.3 极值原理和最大模估计

考虑一维热传导方程$\begin{cases}\partial_t u-\partial_x^2u=0\\ u(x,0)=\varphi(x)\\ u(0,t)=g_1(t),\ u(l,t)=g_2(t).\end{cases}$



令$Q_T=(0,l)\times(0,T]$，定义其抛物边界为$\partial Q_T=\Gamma_T=\overline{Q_T}\setminus Q_T$.

(内部无热源)

定理3.8（极值原理）假设$u\in C^{2,1}(Q_T)\cap C(\overline{Q_T})$满足方程$Lu\triangleq\partial_t u-\partial_x^2u=f\leq0$，则$u(x,t)$在$\overline{Q_T}$上的最大值必在抛物边界上达到，即$\max\limits_{\overline{Q_T}}u=\max\limits_{\Gamma_T}u$.

（最高温度和最低温度只能在初始时刻或物体的边界上达到）.

证明：令 $M=\max\limits_{Q_T} u$, $m=\max\limits_{\Gamma_T} u$, 则 $M\geq m$, 用反证法，设 $M>m$.

<u>Step 1.</u> 设 $f<0$, 设存在 $(x_*,t_*)\in Q_T$, 使得 $u(x_*,t_*)=M$. 于是 $u_x(x_*,t_*)=0$, $u_t(x_*,t_*)\geq 0$, $u_{xx}(x_*,t_*)\leq 0$. 故 $0>f(x_*,t_*)=(u_t-u_{xx})(x_*,t_*)\geq 0$, 矛盾！

<u>Step 2.</u> 若 $f\leq 0$, 令 $v(x,t)=u(x,t)-\varepsilon t$, $\varepsilon>0$, 则 $\partial_t v-\partial_x^2 v=f-\varepsilon<0$, 由 <u>Step 1</u>,

$$\max_{Q_T}(u-\varepsilon T)\leq \max_{Q_T} v=\max_{\Gamma_T} v\leq \max_{\Gamma_T} u$$

$\Rightarrow \max\limits_{Q_T} u\leq \max\limits_{\Gamma_T}(u+\varepsilon T)$, 令 $\varepsilon\to 0^+$, 则 $M\leq m$, 与假设的 $M>m$ 矛盾！ □

推论 3.9. 假设 $u\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ 满足 $Lu=\partial_t u-\partial_x^2 u=f\geq 0$, 则 u 在 $\overline{Q}_T$ 上的最小值必在抛物边界取到.

推论 3.10(比较定理). 设 $u,v\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ 满足 $Lu\leq Lv$ 且 $u|_{\partial Q_T}\leq v|_{\partial Q_T}$, 则在 $\overline{Q}_T$ 上, $u(x,t)\leq v(x,t)$.

最大模估计

Dirichlet 边值问题
$$\begin{cases} u_t-u_{xx}=f(x,t), & (x,t)\in Q_T,\\ u(x,0)=\varphi(x), & x\in[0,l],\\ u(0,t)=g_1(t),\ u(l,t)=g_2(t), & t\in[0,T]. \end{cases} \quad (*)$$

定理 3.11. 设 $u\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ 是 $(*)$ 的解，则 $\max\limits_{Q_T}|u|\leq FT+B$, 其中 $F=\sup\limits_{Q_T}|f|$, $B=\max\left\{\max\limits_{0\leq x\leq l}|\varphi|, \max\limits_{0\leq t\leq T}|g_1|, \max\limits_{0\leq t\leq T}|g_2|\right\}$.

证明：令 $v(x,t)=\pm u(x,t)-Ft-B$, 则

$$\begin{cases} v_t-v_{xx}=\pm f-F\leq 0,\\ v(x,0)=\pm\varphi(x)-B\leq 0,\\ v(0,t)=\pm g_1(t)-Ft-B\leq 0,\\ v(l,t)=\pm g_2(t)-Ft-B\leq 0. \end{cases}$$

由极值原理，$\max\limits_{Q_T}(\pm u-FT-B)\leq \max\limits_{Q_T} v=\max\limits_{\Gamma_T} v\leq 0$.

$\Rightarrow \max\limits_{Q_T}\pm u\leq FT+B \Rightarrow |u(x,t)|\leq FT+B$. □

最大模估计意味着方程 $(*)$ 解的唯一性、稳定性.

Robin边值问题 $\begin{cases}\partial_t u-\partial_x^2 u=f(x,t), & (x,t)\in Q_T,\\ u(x,0)=\varphi(x), & x\in[0,l],\\ u(0,t)=g_1(t),\ (u_x+hu)(l,t)=g_2(t), & h>0,\ t\in[0,T].\end{cases}$

要证唯一性，只需证方程

$$\begin{cases}\partial_t u-\partial_x^2 u=0, & (x,t)\in Q_T,\\ u(x,0)=0, & x\in[0,l],\\ u(0,t)=0,\ (u_x+hu)(l,t)=0, & h>0.\end{cases}$$

只有零解. 用反证法，假设 u 要么有正的最大值，要么有负的最小值. 先设 u 有正的最大值，由极值原理，它在 Γ_T 取得. 由于 $u(x,0)=u(0,t)\equiv0$，它只能在 $\{(x,t)\mid x=l,\ 0<t\le T\}$ 取得. 设 u 在 (l,t_*) 达到正的最大值，则 $u_x(l,t_*)\geq 0$. 故 $(u_x+hu)(l,t_*)>0$，与边值为零矛盾! 类似可证 u 无负的最小值，故 $u\equiv0$. □

Neumann边值问题 $\begin{cases}\partial_t u-\partial_x^2 u=f(x,t), & (x,t)\in Q_T,\\ u(x,0)=\varphi(x), & x\in[0,l],\\ u(0,t)=g_1(t),\ u_x(l,t)=g_2(t), & t\in[0,T].\end{cases}$

对比前一类证法，我们尝试化其为Robin边值问题.

要证唯一性，只需证方程

$$\begin{cases}\partial_t u-\partial_x^2 u=0, & (x,t)\in Q_T,\\ u(x,0)=0, & x\in[0,l],\\ u(0,t)=0,\ u_x(l,t)=0, & t\in[0,T].\end{cases}$$

只有零解. 令 $\tilde{u}(x,t)=u(x,t)w(x)$，则 $u(x,t)=\dfrac{\tilde{u}(x,t)}{w(x)}$ (预设 $w\neq0$). 于是 $u_t=\dfrac{\tilde{u}_t}{w(x)}$,

$u_x=\dfrac{\tilde{u}_x w(x)-\tilde{u}w'(x)}{w^2(x)}=-\dfrac{w_x}{w^2}\tilde{u}+\dfrac{\tilde{u}_x}{w}$, $u_{xx}=\left(-\dfrac{w_{xx}}{w^2}+2\dfrac{(w_x)^2}{w^3}\right)\tilde{u}-2\dfrac{w_x}{w^2}\tilde{u}_x+\dfrac{1}{w}\tilde{u}_{xx}$.

代入方程得 $\dfrac{\tilde{u}_t}{w}-\dfrac{1}{w}\tilde{u}_{xx}+2\dfrac{w_x}{w^2}\tilde{u}_x+\left(\dfrac{w_{xx}}{w^2}-2\dfrac{(w_x)^2}{w^3}\right)\tilde{u}=0$.

$\Rightarrow \tilde{u}_t-u_{xx}+2\dfrac{w_x}{w}\tilde{u}_x+\left(\dfrac{w_{xx}}{w}-2\dfrac{(w_x)^2}{w^2}\right)\tilde{u}=0$.

$\tilde{u}(x,0)=0$, $\tilde{u}(0,t)=0$, $\tilde{u}_x(l,t)=\underbrace{u_x(l,t)}_{=0}w(l)+u(l,t)w_x(l)=u(l,t)w_x(l)$.

为实现化为Robin边值问题的目标，我们希望 $u(l,t)w_x(l)=u(l,t)w(l)\cdot\dfrac{w_x(l)}{w(l)}$ $=\tilde{u}(l,t)\cdot\dfrac{w_x(l)}{w(l)}=-h\,\tilde{u}(l,t)$，即 $\dfrac{w_x(l)}{w(l)}=-h$.

取 $w(x)=l+1-x$，则当 $0\leq x\leq l$ 时 $w(x)\geq1$ 且 $w_x\equiv-1$, $w(l)=1$. 于是 $\tilde{u}$ 的边界条件为 $\tilde{u}(l,t)+\tilde{u}_x(l,t)=0$.

$$\begin{cases} \tilde{u}_t-\tilde{u}_{xx}-\frac{2}{w}\tilde{u}_x-\frac{2}{w^2}\tilde{u}=0 \\ \tilde{u}(x,0)=0 \\ \tilde{u}(0,t)=0,\ \tilde{u}_x(l,t)+\tilde{u}(l,t)=0 \end{cases}$$

令 $v(x,t)=e^{-\lambda t}\tilde{u}(x,t)$，则 $v_t=-\lambda e^{-\lambda t}\tilde{u}+e^{-\lambda t}\tilde{u}_t=-\lambda v+e^{-\lambda t}\tilde{u}_t$. 故

$$\begin{cases} v_t-v_{xx}-\frac{2}{w}v_x+(\lambda-\frac{2}{w^2})v=0 \\ v(x,0)=0 \\ v(0,t)=0,\ v_x(l,t)+v(l,t)=0 \end{cases}$$

取 $\lambda>2$，则 $\lambda-\frac{2}{w^2}\geq\lambda-2>0$，$\forall\, 0<x<l$.

断言：$v(x,t)$ 在 $\overline{Q}_T$ 上的正的最大值必在边界取到.

【事实上，若 $v(x,t)$ 在 $\overline{Q}_T$ 上的正的最大值在 $(x_*,t_*)\in Q_T$ 达到，则

$v_t(x_*,t_*)\geq0$，$v_{xx}(x_*,t_*)\leq0$，$v_x(x_*,t_*)=0$，$v(x_*,t_*)>0$.

从而 $[\underbrace{v_t}_{\geq0}-\underbrace{v_{xx}}_{\leq0}-\frac{2}{w}\underbrace{v_x}_{=0}+\underbrace{(\lambda-\frac{2}{w^2})v}_{>0}](x_*,t_*)>0$，矛盾！】

由 v 满足的初边值条件，v 的正的最大值在 $\{x=0,\ 0<t\leq T\}$ 取到. 设 v 在 (l,t_*) 取到正的最大值，则 $\underbrace{v_x(l,t_*)}_{\geq0}+\underbrace{v(l,t_*)}_{>0}>0$ 与边界条件 $(v_x+v)(l,t)=0$ 矛盾！

因此 v 没有正的最大值，类似可证 v 没有负的最小值，故 $v\equiv0$. 而 $v=e^{-\lambda t}\tilde{u}$，$\tilde{u}=uw(x)$，$w(x)\neq0$，故 $u\equiv0$. □

波动方程

$\partial_t^2 u - \Delta u = f(x,t)$. $n=1$ 弦的振动；$n=2$ 鼓面振动.

在度量 $\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}_{n\times n}$ 下泛函 $\int_{\mathbb{R}^n} |\nabla u|^2 dx$ 的极小化值对应 $\Delta u=0$.

在 Minkowski 度量 $\begin{pmatrix}-1 & & \\ & 1 & \\ & & \ddots \\ & & & 1\end{pmatrix}_{(n+1)\times(n+1)}$ 下泛函 $\int_{\mathbb{R}^{1+n}} (|\partial_t u|^2 + |\nabla u|^2) dx$ 的极小化值对应 $-\partial_t^2 u + \Delta u = 0$.

考虑一阶偏微分方程 $\begin{cases}\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} + b(x,t)u = f(x,t), \\ u(x,0) = \phi(x).\end{cases}$

[解法：特征线法] 特征线 $x = x(t)$，要求

$$\begin{cases}\frac{dx(t)}{dt} = a(x(t),t), \\ x(0) = c.\end{cases}$$

令 $U(t) = u(x(t),t)$，则 $\frac{dU(t)}{dt} = \partial_t u + \frac{dx(t)}{dt}\cdot\frac{\partial u}{\partial x}(x(t),t) = (\partial_t u)(x(t),t) + a(x(t),t)\cdot\frac{\partial u}{\partial x}(x(t),t) = -b(x(t),t)U(t) + f(x(t),t)$.

$$\Rightarrow \begin{cases}\frac{dU(t)}{dt} = -b(x(t),t)U(t) + f(x(t),t), \\ U(0) = \phi(c).\end{cases}$$ （一阶线性常微分方程）

☆ 例1. $\begin{cases}\frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} = f(x,t), \\ u(x,0) = \varphi(x).\end{cases}$

特征线 $\begin{cases}\frac{dx(t)}{dt} = -a \\ x(0) = c\end{cases} \Rightarrow x(t) = -at + c.$

令 $U(t) = u(x(t),t)$，则 $\frac{dU(t)}{dt} = \frac{\partial u}{\partial t} + \frac{dx(t)}{dt}\cdot\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} = f(x(t),t)$.

$$\Rightarrow \begin{cases}\frac{dU(t)}{dt} = f(-at+c,t), \\ U(0) = \varphi(c)\end{cases} \Rightarrow \underset{u(c-at,\,t)}{\underset{\parallel}{U(t)}} = \varphi(c) + \int_0^t f(-as+c,s)ds.$$

$$\xrightarrow[c=x+at]{x=c-at} \boxed{u(x,t) = \varphi(x+at) + \int_0^t f(x+a(t-s),s)ds.}$$ 公式

例2. $\begin{cases}\frac{\partial u}{\partial t} + (x+t)\frac{\partial u}{\partial x} + u = x, \\ u(x_0,x) = x.\end{cases}$

特征线 $\begin{cases}\frac{dx(t)}{dt} = x(t) + t, \\ x(0) = c.\end{cases} \Rightarrow x(t) = ce^t + e^t - t - 1.$

令 $U(t) = u(x(t),t) = u(ce^t + e^t - t - 1, t)$，则 $\frac{dU(t)}{dt} = \frac{\partial u}{\partial t} + (x(t)+t)\frac{\partial u}{\partial x} = x(t) - U(t)$.

$$\Rightarrow\begin{cases}\dfrac{dU(t)}{dt}=-U(t)+ce^t+e^t-t-1,\\ U(0)=c.\end{cases}\Rightarrow \underset{u(ce^t+e^t-t-1,\,t)}{\overset{\|}{U(t)}}=-t+\frac{1}{2}(e^t-e^{-t})+\frac{c}{2}(e^t+e^{-t}).$$

$$\xRightarrow{x=ce^t+e^t-t-1} u(x,t)=\frac{1}{2}(x-t+1)-e^{-t}+\frac{1}{2}(x+t+1)e^{-2t}.$$

例3. $\begin{cases}\dfrac{\partial u}{\partial y}-\dfrac{1}{2}\dfrac{\partial u}{\partial x}=(y-x)e^{-y},\\ u(x,0)=2x^2.\end{cases}$

特征线 $x=x(y)$: $\begin{cases}\dfrac{dx(y)}{dy}=-\dfrac{1}{2}\\ x(0)=c\end{cases}\Rightarrow x(y)=-\frac{1}{2}y+c.$

令 $U(y)=u(-\frac{1}{2}y+c,y)$, 则 $\dfrac{dU(y)}{dy}=\dfrac{\partial u}{\partial y}-\dfrac{1}{2}\dfrac{\partial u}{\partial x}=(y-x(y))e^{-y}=(\frac{3}{2}y-c)e^{-y}$.

$$\Rightarrow\begin{cases}\dfrac{dU(y)}{dy}=(\frac{3}{2}y-c)e^{-y},\\ U(0)=2c^2\end{cases}\Rightarrow \underset{u(x(y),\,y)}{\overset{\|}{U(y)}}=2c^2+\frac{3}{2}[1-(1+y)e^{-y}]+c(e^{-y}-1).$$

$$\xRightarrow[c=x+\frac{1}{2}y]{x=-\frac{1}{2}y+c} u(x,y)=2(x+\tfrac{1}{2}y)^2+\frac{3}{2}[1-(1+y)e^{-y}]+(x+\tfrac{1}{2}y)(e^{-y}-1).$$

一维波动方程: $\partial_t^2u-\partial_x^2u=0$. 看作 $(\partial_t+\partial_x)(\partial_t-\partial_x)u=0$.

令 $v=\partial_tu-\partial_xu$, 则 $\partial_tv+\partial_xv=0$.

波方程: $u_{tt}-\Delta u=f(x,t)$, $t\in I(\ni 0)$, $x\in\Omega\subset\mathbb{R}^n$ (t称"时间", I含0时刻).

$u: I\times\Omega\to\mathbb{R}$ 未知, $f: I\times\Omega\to\mathbb{R}$ 已知.

初值: $u(x,0)=\varphi(x)$, $(\partial_tu)(x,0)=\psi(x)$, $x\in\Omega$.

边值 $\begin{cases}\text{Dirichlet边值}: u|_{\partial\Omega}=g(x,t)\\ \text{Neumann边值}: \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega}=g(x,t)\\ \text{Robin边值}: \left(u+\sigma(x,t)\dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega}=g(x,t)\end{cases}$

§4.1 初值问题

初边值问题 $\begin{cases}\partial_t^2u-\Delta u=\overset{(3)}{f(x,t)},\\ u(x,0)=\underset{(1)}{\varphi(x)},\ (\partial_tu)(x,0)=\underset{(2)}{\psi(x)},\end{cases}$ $t\in I\ni 0$, $x\in\mathbb{R}^n$.

作拆解:

(1) $\begin{cases}\partial_t^2u_1-\Delta u_1=0,\\ u_1(x,0)=\varphi(x),\ \partial_tu_1(x,0)=0.\end{cases}$ (2) $\begin{cases}\partial_t^2u_2-\Delta u_2=0,\\ u_2(x,0)=0,\ \partial_tu_2(x,0)=\psi(x).\end{cases}$ (3) $\begin{cases}\partial_t^2u_3-\Delta u_3=f(x,t),\\ u_3(x,0)=0,\ \partial_tu_3(x,0)=0.\end{cases}$

令$u=u_1+u_2+u_3$，则u是原问题的解.

定理4.1 设$u_2=M_\psi(x,t)$是初值问题(2)的解(这里M_ψ表示以ψ为初速度的(2)的解)，则

初值问题(1)的解为$u_1=\frac{\partial}{\partial t}M_\varphi(x,t)$.

初值问题(3)的解为$u_3=\int_0^t (M_{f_\tau})(x,t-\tau)\,d\tau$. (这里$f_\tau=f(x,\tau)$)

整体出现!

证明: ①令$\tilde{u}=M_\varphi$，则

$$\begin{cases}\partial_t^2\tilde{u}-\Delta\tilde{u}=0,\\ \tilde{u}(x,0)=0,\ (\partial_t\tilde{u})(x,0)=\varphi(x).\end{cases}$$

令$v=\partial_t\tilde{u}$，则

$$\begin{cases}\partial_t^2 v-\Delta v=0,\\ v(x,0)=\varphi(x),\ (\partial_t v)(x,0)=\partial_t^2\tilde{u}(x,0)=\Delta\tilde{u}(x,0)=0.\end{cases}$$

②令$\tilde{u}(x,t)=M_{f_\tau}(x,t)$，则

$$\begin{cases}\partial_t^2\tilde{u}-\Delta\tilde{u}=0,\\ \tilde{u}(x,0)=0,\ (\partial_t\tilde{u})(x,0)=f_\tau(x)=f(x,\tau).\end{cases}$$

令$v=\int_0^t\tilde{u}(x,t-\tau)\,d\tau$，则

$\partial_t v=\tilde{u}(x,0)+\int_0^t(\partial_t\tilde{u})(x,t-\tau)\,d\tau=\int_0^t(\partial_t\tilde{u})(x,t-\tau)\,d\tau.$

$\partial_t^2 v=(\partial_t\tilde{u})(x,0)+\int_0^t(\partial_t^2\tilde{u})(x,t-\tau)\,d\tau=f(x,t)+\int_0^t(\Delta\tilde{u})(x,t-\tau)\,d\tau=f(x,t)+\Delta v$

$v(x,0)=0,\ \partial_t v(x,0)=0.$ □

冲量原理/Duhamel原理(教材P152~153)的Fourier变换观点:

n维Fourier变换: $\hat{f}(\xi)=\int_{\mathbb{R}^n}f(x)e^{ix\cdot\xi}dx$.

注意到u及其导数在无穷远处消失，由分部积分,

$\int_{\mathbb{R}^n}(\Delta u)(x)e^{-ix\cdot\xi}dx=-|\xi|^2\int_{\mathbb{R}^n}u(x)e^{-ix\cdot\xi}dx=-|\xi|^2\hat{u}(\xi)$.

对$\partial_t^2u-\Delta u=f(x,t)$两边对$x$作Fourier变换可得

$$\begin{cases}\partial_t^2\hat{u}+|\xi|^2\hat{u}=\hat{f}(\xi,t),\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi),\ \partial_t\hat{u}(\xi,0)=\hat{\psi}(\xi).\end{cases}$$ (二阶常系数微分方程)

齐次方程的特征方程: $\lambda^2+|\xi|^2=0$, $\lambda=\pm|\xi|i$. 解为$\hat{u}(t,\xi)=C_1\cos(|\xi|t)+C_2\sin(|\xi|t)$

$$\Rightarrow \hat{u}(t,\xi)=\underbrace{\cos(|\xi|t)\hat{\varphi}(\xi)}_{\hat{u}_1}+\underbrace{\frac{\sin(|\xi|t)}{|\xi|}\hat{\psi}(\xi)}_{\hat{u}_2}+\underbrace{\int_0^t\frac{\sin((t-\tau)|\xi|)}{|\xi|}\hat{f}(\xi,t-\tau)\,d\tau}_{\hat{u}_3}.$$

(把$\hat{u}_2$中$\hat{\psi}$换成$\hat{\varphi}$，对t求导就得到$\hat{u}_1$ …)

· 时间反演不变性：令 $v(t,x)=u(-t,x)$，则 v 也满足自由波方程.

考虑自由波方程 $\partial_t^2 u-\Delta u=0$，$(x,t)\in\mathbb{R}^n\times\mathbb{R}$.

· 时间平移不变性：$u(t+t_0,x)$.

· 空间平移不变性：$u(t,x+x_0)$.

·（时空同步）伸缩不变性：$u(\frac{t}{\lambda},\frac{x}{\lambda})$.

· 洛伦兹变换不变性：$u\left(\frac{t-v\cdot x}{\sqrt{1-|v|^2}},\ x-x_v+\frac{x_v-vt}{\sqrt{1-|v|^2}}\right)$，$|v|<1$，$x_v:=\left(x\cdot\frac{v}{|v|}\right)\cdot\frac{v}{|v|}$ 为
（时空旋转）
x 沿 v 方向分量. 特别地，当 $v=le_1$ 时的洛伦兹变换为

$$u\left(\frac{t-lx_1}{\sqrt{1-l^2}},\frac{x_1-lt}{\sqrt{1-l^2}},x_2,\cdots,x_n\right).$$

初值问题 $\begin{cases}\partial_t^2 u-\Delta_{\mathbb{R}^n}u=f(x,t),\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x).\end{cases}$

$\boxed{n=1}$ $\begin{cases}\partial_t^2 u-\partial_x^2 u=f(x,t),\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)\end{cases}$ （W1）. (4.13)

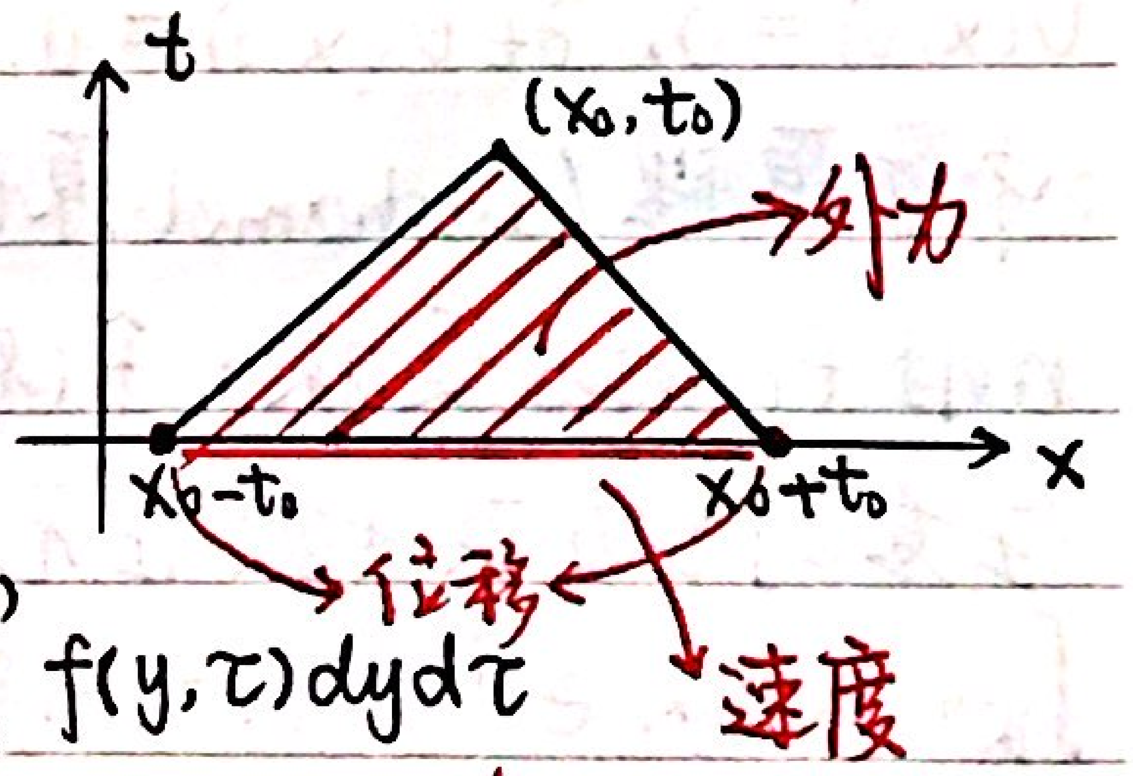
由定理4.1，关键是求解

$$\begin{cases}\partial_t^2 u-\partial_x^2 u=0,\\ u(x,0)=0,\ \partial_t u(x,0)=\psi(x).\end{cases}$$

令 $v(x,t)=\partial_t u-\partial_x u$，则 $\begin{cases}\partial_t v+\partial_x v=0,\\ v(x,0)=\psi(x)\end{cases}$ $\xRightarrow{\text{特征线法(公式)}}$ $v(x,t)=\psi(x-t)$.

于是 $\begin{cases}\partial_t u-\partial_x u=\psi(x-t)\\ u(x,0)=0\end{cases}$ $\Rightarrow u(x,t)=\int_0^t\psi(x+(t-s)-s)ds=\int_0^t\psi(x+t-2s)ds$

$$\xlongequal{y=x+t-2s}-\frac{1}{2}\int_{x+t}^{x-t}\psi(y)dy=\frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy.$$



由定理4.1，（W1）的解为

$$\boxed{u(x,t)}=\frac{d}{dt}\left(\frac{1}{2}\int_{x-t}^{x+t}\varphi(y)dy\right)+\frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy+\int_0^t\frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)dyd\tau$$

$$=\underbrace{\frac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]}_{\text{位移}}+\underbrace{\frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy}_{\text{速度}}+\underbrace{\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)dyd\tau}_{\text{外力}} \quad (4.20)$$

D'Alembert公式

（无外力）
若 $f\equiv0$，令 $F(x)=\frac{1}{2}\varphi(x)+\frac{1}{2}\int_0^x\psi(y)dy$，$G(x)=\frac{1}{2}\varphi(x)-\frac{1}{2}\int_0^x\psi(y)dy$，则
$u(x,t)=F(\overset{\text{左}}{x+t})+G(\overset{\text{右}}{x-t})$，是左行波（速度为-1）与右行波（速度为1）的叠加.

定理4.2 若$\varphi\in C^2(\mathbb{R})$，$\psi\in C^1(\mathbb{R})$，$f\in C^1(\mathbb{R}\times\mathbb{R}_+)$，则由表达式(4.20)给出的函数$u\in C^2(\mathbb{R}\times\mathbb{R}_+)$是初值问题(4.13)的解.

推论4.3 若φ，ψ及f是x的偶(或奇，或周期为l的)函数，则由表达式(4.20)给出的解u也是x的偶(或奇，或周期为l的)函数.

一维半无界问题$\begin{cases} u_{tt}-u_{xx}=f(x,t), \\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x), & x\in\mathbb{R}_+,\ t\in\mathbb{R}. \\ u(0,t)=g(t). \end{cases}$

1. $g(t)\equiv 0$ (弦的一端固定) 对称开拓法

作奇延拓，令 $\bar\varphi(x)=\begin{cases}\varphi(x), & x>0,\\ -\varphi(-x), & x<0.\end{cases}$ $\bar\psi(x)=\begin{cases}\psi(x), & x>0,\\ -\psi(-x), & x<0.\end{cases}$ $\bar f(x,t)=\begin{cases}f(x,t), & x>0,\\ -f(-x,t), & x<0.\end{cases}$

令$\bar u(x,t)$是$\begin{cases}\partial_t^2\bar u-\partial_x^2\bar u=\bar f(x,t)\\ \bar u(x,0)=\bar\varphi(x),\ \partial_t\bar u(x,0)=\bar\psi(x).\end{cases}$ $(x\in\mathbb{R},\ t\in\mathbb{R})$的解，则$\bar u(x,t)$是$x$的奇函数，故$\bar u(0,t)=0$. 由D'Alembert公式，

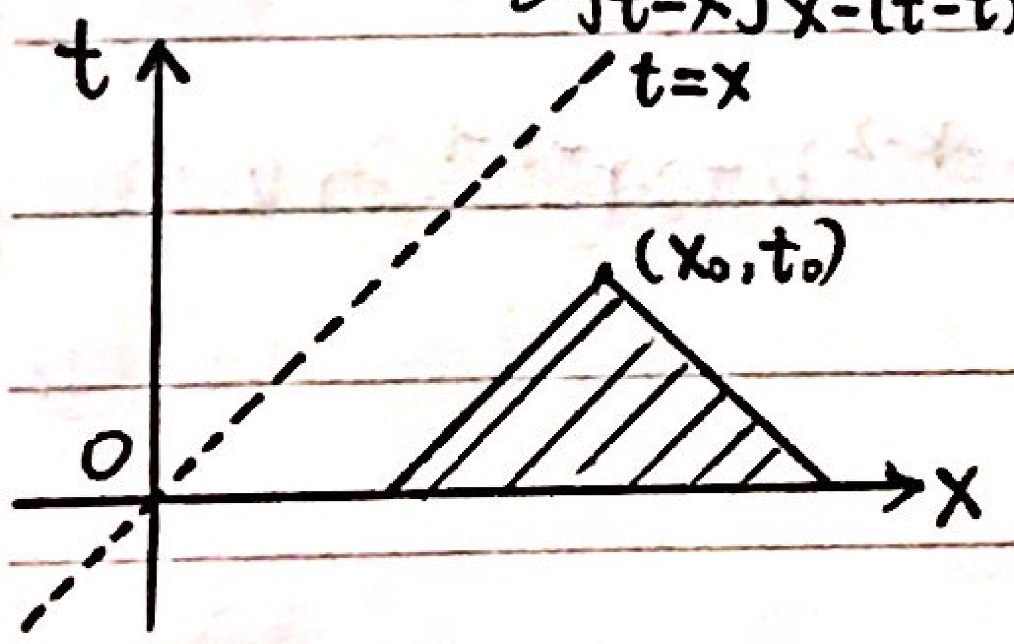
$$\bar u(x,t)=\frac12[\bar\varphi(x+t)+\bar\varphi(x-t)]+\frac12\int_{x-t}^{x+t}\bar\psi(y)dy+\frac12\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}\bar f(y,\tau)dyd\tau.$$

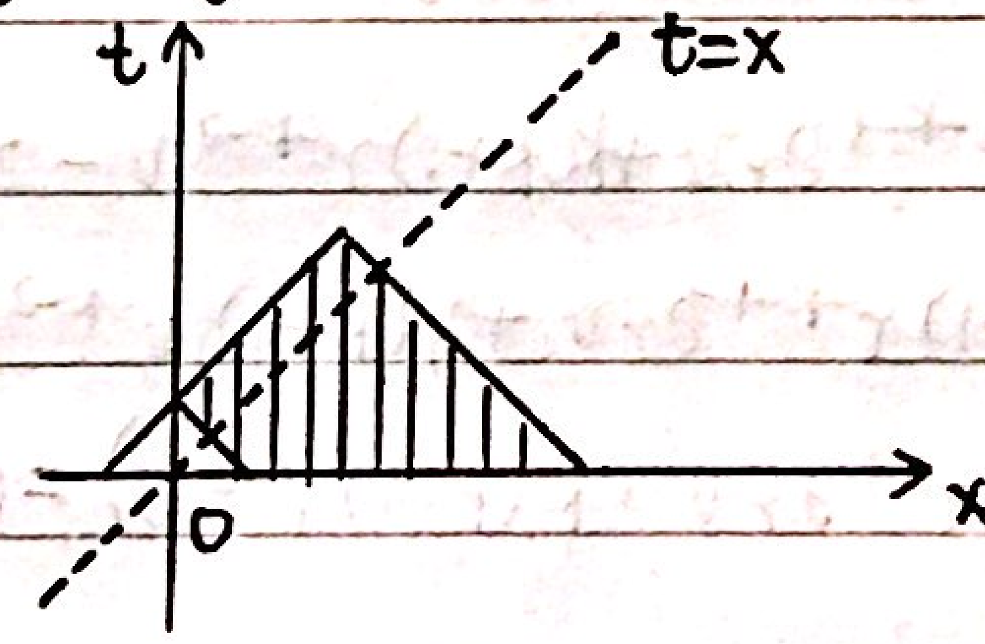
设$t>0$. 若$x\geqslant t$，则

$$u(x,t)=\frac12[\varphi(x+t)+\varphi(x-t)]+\frac12\int_{x-t}^{x+t}\psi(y)dy+\frac12\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)dyd\tau.$$

若$x<t$，则

$$u(x,t)=\frac12[\varphi(x+t)+\varphi(x-t)]+\frac12\int_{t-x}^{t+x}\psi(y)dy+\frac12\int_0^{t-x}\left(\int_{x-(t-\tau)}^{0}-f(-y,\tau)dy+\int_0^{x+(t-\tau)}f(y,\tau)dy\right)d\tau$$
$$+\frac12\int_{t-x}^{t}\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)dyd\tau.$$





2. $g(t)\not\equiv 0$. 令$v(x,t)=u(x,t)-g(t)$，则

$$\begin{cases}\partial_t^2 v-\partial_x^2 v=f(x,t)-g''(t),\\ v(x,0)=\varphi(x)-g(0),\ \partial_t v(x,0)=\psi(x)-g'(0),\\ v(0,t)=0.\end{cases}\Rightarrow \text{化为1中情形.}$$

▷初边值不能随意指定，需满足相容性条件 $\begin{cases}\varphi(0)=u(0,0)=g(0),\\ \psi(0)=\partial_t u(0,0)=g'(0),\\ g''(0)=\varphi''(0)+f(0,0).\end{cases}$

$\boxed{n=3}$ $\begin{cases}\partial_t^2 u-\Delta u=f(x,t), & x\in\mathbb{R}^3 \quad (W3)\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x).\end{cases}$

由定理4.1，关键是求解 $\begin{cases}\partial_t^2 u-\Delta u=0, & x\in\mathbb{R}^3.\\ u(x,0)=0,\ \partial_t u(x,0)=\psi(x),\end{cases}$

用极坐标 (r,ω)，由 $\Delta_{\mathbb{R}^3}\overset{(r,\omega)}{=\!=}\partial_r^2+\frac{2}{r}\partial_r+\frac{1}{r^2}\Delta_{S^2}\left(\overset{n=2时}{=\!=}\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2\right)$ 可得

$\partial_t^2 u-(\partial_r^2 u+\frac{2}{r}\partial_r u+\frac{1}{r^2}\Delta_{S^2}u)=0.$

想令 $\Delta_{S^2}u$ 这一项消失，我们采用球面平均法，在球面上积分。注意到 $\int_{S^2}\Delta_{S^2}u\,dS(\omega)=0$，这是由于 $\int_{S^2}\Delta_{S^2}u\,dS(\omega)=\int_{S^2}\mathrm{div}_{S^2}\nabla_{S^2}u\,dS(\omega)\overset{散度定理}{=\!=}\int_{\partial S^2}\frac{\partial u}{\partial n}dS(\omega)\overset{S^2无边}{=\!=}0$

令 $\overline{u}(r,t)=\frac{1}{4\pi}\int_{S^2}u(r\omega,t)\,dS(\omega)$，则

$\begin{cases}\partial_t^2\overline{u}-(\partial_r^2\overline{u}+\frac{2}{r}\partial_r\overline{u})=0, & (r>0).\\ \overline{u}(r,0)=0,\ \partial_t\overline{u}(r,0)=\overline{\psi}.\end{cases}$

令 $v(r,t)=r^k\overline{u}(r,t)$，则

$\overline{u}(r,t)=r^{-k}v(r,t),$

$\partial_r\overline{u}(r,t)=-kr^{-k-1}v(r,t)+r^{-k}\partial_r v(r,t)$

$\partial_r^2\overline{u}(r,t)=k(k+1)r^{-k-2}v(r,t)-kr^{-k-1}\partial_r v(r,t)-kr^{-k-1}\partial_r v(r,t)+r^{-k}\partial_r^2 v(r,t).$

$\quad=k(k+1)r^{-k-2}v-2kr^{-k-1}\partial_r v+r^{-k}\partial_r^2 v.$

$\Rightarrow \partial_r^2\overline{u}+\frac{2}{r}\partial_r\overline{u}=r^{-k}\partial_r^2 v-2kr^{-k-1}\partial_r v+k(k+1)r^{-k-2}v-2kr^{-k-2}v+2r^{-k-1}\partial_r v$

$\quad=r^{-k}\partial_r^2 v-2(k-1)r^{-k-1}\partial_r v+k(k-1)r^{-k-2}v.$

于是 $\cancel{r^{-k}}\partial_t^2 v-\cancel{r^{-k}}(\partial_r^2 v-2(k-1)r^{-1}\partial_r v+k(k-1)r^{-2}v)=0.$

$\xRightarrow{k=1}$ $\begin{cases}\partial_t^2 v-\partial_r^2 v=0, & r>0，其中 v(r,t)=r\overline{u}(r,t).\\ v(r,0)=0,\ \partial_t v(r,0)=r\overline{\psi}. & \hookrightarrow 半直线上波方程\end{cases}$

作 v 关于 $r>0$ 的偶延拓，仍记为 v。则由D'Alembert公式可得

$v(r,t)=\frac{1}{2}\int_{r-t}^{r+t}\rho\overline{\psi}(\rho)\,d\rho.$

问题：已知 $\overline{u}(r,t)$，如何求 $u(x,t)$？

$\underline{\bar{u}(0,t)} = \frac{1}{4\pi}\int_{S^2} u(0,t)\,dS(\omega) = \frac{u(0,t)}{4\pi}\int_{S^2} dS(\omega) = \underline{u(0,t)}.$

$u(0,t) = \bar{u}(0,t) = \partial_r(r\bar{u}(r,t))|_{r=0} = \frac{1}{2}\partial_r\left(\int_{r-t}^{r+t} \rho\bar{\psi}(\rho)d\rho\right)\Big|_{r=0}$

$= \frac{1}{2}[(r+t)\bar{\psi}(r+t) - (r-t)\bar{\psi}(r-t)]|_{r=0} = \frac{1}{2}[t\bar{\psi}(t) + t\bar{\psi}(-t)]$

$\overset{\because 偶延拓}{=} t\bar{\psi}(t) = \frac{t}{4\pi}\int_{S^2}\psi(t\omega)\,dS(\omega).$

$\xlongequal[dS(y)=t^2dS(\omega)]{y=t\omega} \frac{1}{4\pi t}\int_{|y|=t}\psi(y)\,dS(y)$

$\forall x_0\in\mathbb{R}^3$，由空间平移不变性，对 $u(\cdot+x_0,t)$ 应用上面过程，可得

$u(x_0,t) = \frac{1}{4\pi t}\int_{|y|=t}\psi(y+x_0)\,dS(y) = \frac{1}{4\pi t}\int_{|y-x_0|=t}\psi(y)\,dS(y).$

因此 $u(x,t) = \frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y).$

(W3)的解为　　　　　　　　(Kirchhoff公式)

$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)\,dS(y)\right) + \frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y) + \int_0^t \frac{1}{4\pi(t-\tau)}\int_{|y-x|=t-\tau} f(y,\tau)\,dS(y)\,d\tau$

$\boxed{n=2}\begin{cases}\partial_t^2 u - \Delta u = f(x,t), & x\in\mathbb{R}^2. \quad (W2)\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x),\end{cases}$

将二维问题看成一个特殊的三维问题.（升维法 or 降维法）

令 $\tilde{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$，类似定义 $\tilde{f}$，$\tilde{\varphi}$，$\tilde{\psi}$.

不妨设 $\tilde{f}\equiv 0$，$\tilde{\varphi}\equiv 0$，则

$\begin{cases}\partial_t^2\tilde{u} - \Delta_{\mathbb{R}^3}\tilde{u} = 0, & \tilde{x} = (x_1,x_2,x_3).\\ \tilde{u}(\tilde{x},0)=0,\ \partial_t\tilde{u}(\tilde{x},0)=\tilde{\psi}.\end{cases}$

由Kirchhoff公式，$\tilde{u}(x,t) = \frac{1}{4\pi t}\int_{|\tilde{x}-\tilde{y}|=t}\tilde{\psi}(\tilde{y})\,dS(\tilde{y})$. 于是

$u(0,0,t) = \tilde{u}(0,0,0,t) = \frac{1}{4\pi t}\int_{y_1^2+y_2^2+y_3^2=t^2}\psi(y_1,y_2)\,dS(\tilde{y})$

$= \frac{2}{4\pi t}\int_{y_3=\sqrt{t^2-y_1^2-y_2^2}}\psi(y_1,y_2)\,dS(\tilde{y}) = \frac{1}{2\pi}\int_{y_1^2+y_2^2\le t^2}\psi(y_1,y_2)\cdot\frac{1}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2$

$= \frac{1}{2\pi}\int_{|y|\le t}\frac{\psi(y)}{\sqrt{t^2-|y|^2}}\,dy.$

$\forall x_0\in\mathbb{R}^2$，对 $u(\cdot+x_0,t)$ 应用上式，可得

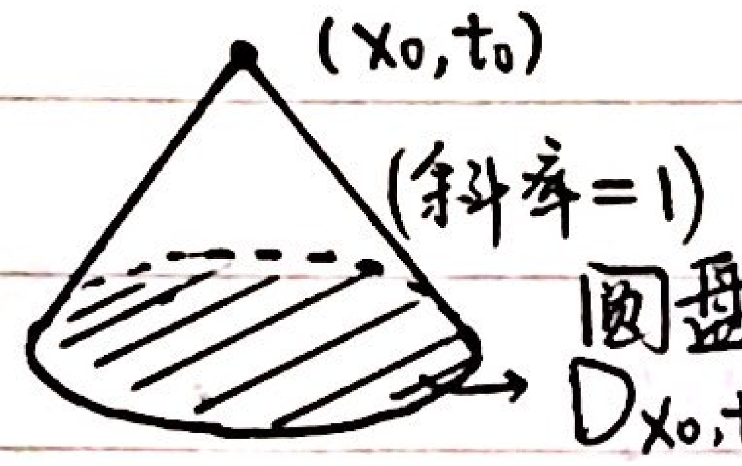
$u(x_0,t) = \frac{1}{2\pi}\int_{|y|\le t}\frac{\psi(y+x_0)}{\sqrt{t^2-|y|^2}}\,dy = \frac{1}{2\pi}\int_{|y-x_0|\le t}\frac{\psi(y)}{\sqrt{t^2-|y-x_0|^2}}\,dy$

$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{|y-x|\le t}\frac{\psi(y)}{\sqrt{t^2-|y-x|^2}}\,dy.$

(W2)的解为 (Poisson公式)

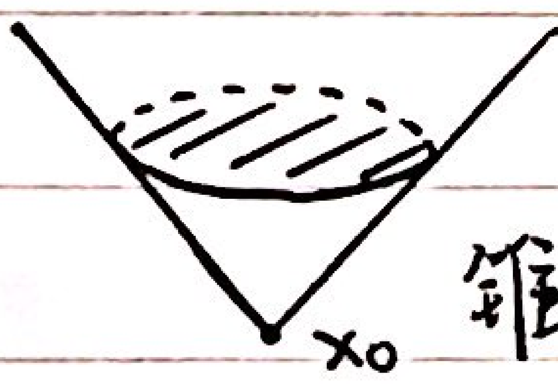
$$u(x,t)=\frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{|x-y|\le t}\frac{\varphi(y)}{\sqrt{t^2-|x-y|^2}}dy\right)+\frac{1}{2\pi}\int_{|x-y|\le t}\frac{\psi(y)}{\sqrt{t^2-|x-y|^2}}dy+\int_0^t\frac{1}{2\pi}\int_{|x-y|\le t-\tau}\frac{f(y,\tau)}{\sqrt{(t-\tau)^2-|x-y|^2}}dyd\tau.$$

$n=2$，考虑 $f\equiv 0$ 的简单情形。观察Poisson公式，

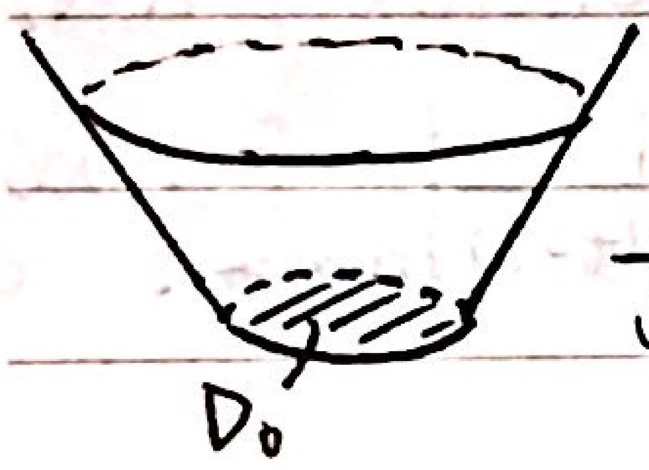


这里 $n=2$

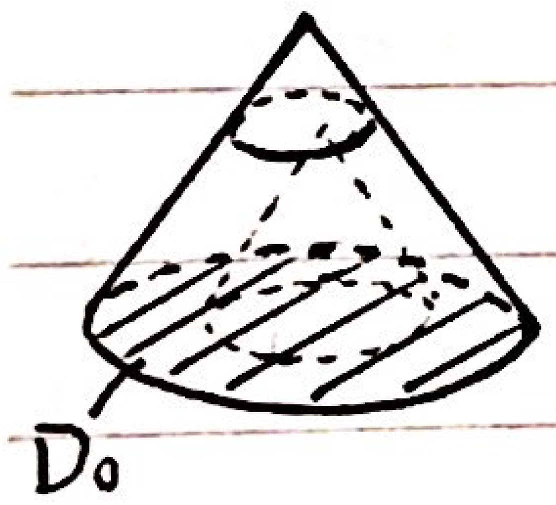
$D_{x_0,t_0}=\{x\in\mathbb{R}^n \mid |x-x_0|\le t_0\}$ 称为点 (x_0,t_0) 的依赖区域.



锥 $J_{x_0}=\{(x,t)\in\mathbb{R}^{n+1}\mid x_0\in D_{x,t}\}$ 称为点 x_0 的影响区域.



$J_{D_0}=\bigcup_{x_0\in D_0}J_{x_0}$ 称为 D_0 的影响区域.



$F_{D_0}=\{(x,t)\in\mathbb{R}^{n+1}\mid D_{x,t}\subset D_0\}$ 称为 D_0 的决定区域.

对比 $n=3$ 时的Kirchhoff公式与 $n=2$ 时的Poisson公式中等号与不等号，我们发现 $n=3$ 时波的传播有清晰的波前/波后，这称为Huygens原理或无后效现象；而 $n=2$ 时波的传播只有清晰的波前，没有清晰的波后，这称为波的弥漫或有后效现象.

§4.2 初边值问题

$$n=1\begin{cases}\partial_t^2u-\partial_x^2u=f(x,t),\ 0\le x\le l,\\ u(x,0)=\varphi(x),\ \partial_tu(x,0)=\psi(x),\\ u(0,t)=g_1(t),\ u(l,t)=g_2(t).\end{cases}$$ (一维弦振动)

1. $f\equiv 0,\ g_1\equiv 0,\ g_2\equiv 0.$

齐次边值问题 $\begin{cases} \partial_t^2 u-\partial_x^2 u=0, \\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x), \\ u(0,t)=0,\ u(l,t)=0. \end{cases}$

根据Sturm-Liouville理论，算子$-\partial_x^2$配上边值 $f(0)=0, f(l)=0$，$-\partial_x^2 f=\lambda f$有特征值 $\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$，与特征函数 $\varphi_1,\varphi_2,\cdots,\varphi_n,\cdots$，$\{\varphi_n\}$构成$L^2(0,l)$的完备正交基，任一解$v(x)$可表成 $v(x)=\sum_{n=1}^{\infty}c_n\varphi_n(x)$，故偏微分方程的解可表成

$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\varphi_n(x)$.

令$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\varphi_n(x)$，代入方程可得

$\sum_{n=1}^{\infty}T_n''(t)\varphi_n(x)+\sum_{n=1}^{\infty}\lambda_nT_n(t)\varphi_n(x)=0 \Rightarrow \sum_{n=1}^{\infty}(T_n''(t)+\lambda_nT_n(t))\varphi_n(x)=0 \xRightarrow{\text{基}} T_n''(t)+\lambda_nT_n(t)=0$

由$u(x,0)=\varphi(x)$可知 $\sum_{n=1}^{\infty}T_n(0)\varphi_n(x)=\varphi(x). \Rightarrow T_n(0)(\varphi_n,\varphi_n)=(\varphi,\varphi_n)\Rightarrow T_n(0)=\frac{(\varphi,\varphi_n)}{(\varphi_n,\varphi_n)}$.

由$u_t(x,0)=\psi(x)$可知 $\sum_{n=1}^{\infty}T_n'(0)\varphi_n(x)=\psi(x)\Rightarrow T_n'(0)(\varphi_n,\varphi_n)=(\psi,\varphi_n)\Rightarrow T_n'(0)=\frac{(\psi,\varphi_n)}{(\varphi_n,\varphi_n)}$.

解方程

设 $u(x,t)=T_n(t)X_n(x)$，则 $T_n''(t)X_n(x)-T_n(t)X_n''(x)=0$.

$\Rightarrow \frac{T_n''(t)}{T_n(t)}=\frac{X_n''(x)}{X_n(x)}\triangleq-\lambda_n$. 则 $X_n''(x)+\lambda_nX_n(x)=0$.

由$u(0,t)=0$知$T_n(t)X_n(0)=0$，所以$X_n(0)=0$. 同样地，有$X_n(l)=0$.

故$\begin{cases} X_n''+\lambda_nX_n=0, \\ X_n(0)=0,\ X_n(l)=0. \end{cases}$

①若$\lambda_n<0$：$X_n(x)=c_1e^{\sqrt{-\lambda_n}x}+c_2e^{\sqrt{-\lambda_n}x}$.

$X_n(0)=c_1+c_2=0$，$X_n(l)=c_1(e^{-\sqrt{-\lambda_n}l}-e^{-\sqrt{-\lambda_n}l})=0\Rightarrow c_1=c_2=0\Rightarrow X_n(x)\equiv 0$.

②若$\lambda_n\equiv 0$：$X_n(x)=c_1x+c_2$.

$X_n(0)=c_2=0$，$X_n(l)=c_1l=0\Rightarrow c_1=c_2=0\Rightarrow X_n(x)\equiv 0$.

③若$\lambda_n>0$：$X_n(x)=c_1\cos(\sqrt{\lambda_n}x)+c_2\sin(\sqrt{\lambda_n}x)$.

$X_n(0)=c_1=0$，$X_n(l)=c_2\sin(\sqrt{\lambda_n}l)=0\Rightarrow\sqrt{\lambda_n}l=n\pi$，$n\neq 0\Rightarrow\lambda_n=(\frac{n\pi}{l})^2, n\in\mathbb{Z}_+$.

$\Rightarrow X_n(x)=\sin(\frac{n\pi}{l}x)$，$n\in\mathbb{Z}_+$.

于是$T_n''(t)+(\frac{n\pi}{l})^2T_n(t)=0$.

令$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$，则 $\sum_{n=1}^{\infty}T_n(0)X_n(x)=\varphi(x)$，$\sum_{n=1}^{\infty}T_n'(0)X_n(x)=\psi(x)$.

$\Rightarrow T_n(0)=\underbrace{\frac{(\varphi,X_n)}{(X_n,X_n)}}_{(X_n,X_n)=\frac{l}{2}}=\frac{2}{l}\int_0^l\varphi(x)\sin(\frac{n\pi}{l}x)dx\triangleq\varphi_n$.

daolen

$T_n'(0)=\frac{(\psi,X_n)}{(X_n,X_n)}=\frac{2}{l}\int_0^l \psi(x)\sin(\frac{n\pi}{l}x)dx \triangleq \psi_n.$

故 $\begin{cases} T_n''(t)+(\frac{n\pi}{l})^2T_n(t)=0, \\ T_n(0)=\varphi_n,\ T_n'(0)=\psi_n. \end{cases} \Rightarrow T_n(t)=C_1\cos(\frac{n\pi}{l}t)+C_2\sin(\frac{n\pi}{l}t).$

$T_n(0)=C_1=\varphi_n,\ T_n'(0)=\frac{n\pi}{l}C_2=\psi_n \Rightarrow C_2=\frac{l}{n\pi}\psi_n.$

$\Rightarrow T_n(t)=\varphi_n\cos(\frac{n\pi}{l}t)+\frac{l}{n\pi}\psi_n\sin(\frac{n\pi}{l}t).$

$\Rightarrow$ 原方程的解为 $u(x,t)=\sum_{n=1}^{\infty}\left[\varphi_n\cos(\frac{n\pi}{l}t)+\frac{l}{n\pi}\psi_n\sin(\frac{n\pi}{l}t)\right]\sin(\frac{n\pi}{l}x).$

此处不关心 $u(x,t)$ 收敛与否("弱解")

2. $f\not\equiv 0,\ g_1\equiv 0,\ g_2\equiv 0.$ Sturm-Liouville 理论仍适用　教材 P184~185

$\{X_n(x)\}$ 是 $L^2(0,l)$ 的完备正交基.

将 $f(x,t),\ \varphi(x),\ \psi(x)$ 分别关于 $\{X_n(x)\}$ 展开, 系数分别为 $f_n(t),\ \varphi_n,\ \psi_n$. 则

$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$ 满足 $\sum_{n=1}^{\infty}[T_n''(t)+\lambda_nT_n(t)]X_n(x)=\sum_{n=1}^{\infty}f_n(t)X_n(x).$

$\xRightarrow{\text{基}} T_n''(t)+\lambda_nT_n(t)=f_n(t),\ T_n(0)=\varphi_n,\ T_n'(0)=\psi_n.$

$\Rightarrow T_n(t)=\varphi_n\cos(\frac{n\pi}{l}t)+\frac{l}{n\pi}\psi_n\sin(\frac{n\pi}{l}t)+\frac{l}{n\pi}\int_0^t\sin[\frac{n\pi}{l}(t-\tau)]f_n(\tau)d\tau.$ (常用ODE)

3. $f\not\equiv 0,\ g_1\not\equiv 0,\ g_2\not\equiv 0.$

令 $v(x,t)=u(x,t)-(\frac{l-x}{l}g_1(t)+\frac{x}{l}g_2(t))$, 则 $v(0,t)=0,\ v(l,t)=0$, 且

$$\begin{cases} \partial_t^2 v-\partial_x^2 v=f(x,t)-(\frac{l-x}{l}g_1''(t)+\frac{x}{l}g_2''(t)), \\ v(x,0)=\varphi(x)-(\frac{l-x}{l}g_1(0)+\frac{x}{l}g_2(0)), \\ v_t(x,0)=\psi(x)-(\frac{l-x}{l}g_1'(0)+\frac{x}{l}g_2'(0)), \\ v(0,t)=v(l,t)=0 \end{cases}$$

$\Rightarrow$ 回到第2类情形.

例. 求解方程 $\begin{cases} \partial_t u-\partial_x^2 u=0,\ 0<x<l,\ t>0. \\ u(x,0)=\varphi(x), \\ u(0,t)=0.\ u_x(l,t)+hu(l,t)=0,\ h>0. \end{cases}$

解: 令 $u(x,t)=T(t)X(x)$, 则 $T'(t)X(x)-T(t)X''(x)=0,\ T(t)X(0)=0,\ T(t)[X'(l)+hX(l)]=0.$

$\Rightarrow \frac{T'(t)}{T(t)}=\frac{X''(x)}{X(x)}\triangleq-\lambda \Rightarrow X''(x)+\lambda X(x)=0,\ X(0)=0,\ X'(l)+hX(l)=0.$

① 若 $\lambda<0$, $X(x)=C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x}$.

$X(0)=C_1+C_2=0$, $X'(l)+hX(l)=C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}l}-C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}l}+hC_1e^{\sqrt{-\lambda}l}+hC_2e^{-\sqrt{-\lambda}l}$

$=C_1[\sqrt{-\lambda}e^{\sqrt{-\lambda}l}+\sqrt{-\lambda}e^{-\sqrt{-\lambda}l}+h(e^{\sqrt{-\lambda}l}-e^{-\sqrt{-\lambda}l})]=0.\Rightarrow C_1=C_2=0.$

② 若 $\lambda=0$, 则 $X(x)=C_1x+C_2$.

$X(0)=0\Rightarrow C_2=0$, $X'(l)+hX(l)=C_1(1+hl)=0\Rightarrow C_1=0$.

③ 若 $\lambda>0$, 则 $X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$.

$X(0)=0\Rightarrow C_1=0$, $X(x)=\sin(\sqrt{\lambda}x)$, $X'(l)+hX(l)=\sqrt{\lambda}\cos(\sqrt{\lambda}l)+h\sin(\sqrt{\lambda}l)=0$

$\Leftrightarrow \tan(\sqrt{\lambda}l)=-\frac{\sqrt{\lambda}}{h}$. 令 $x=\sqrt{\lambda}l$, 则 $\tan x=-\frac{x}{hl}$. 由函数图像可知有可数个解.

故存在 $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$, 使得 $\tan(\sqrt{\lambda_k}l)=-\frac{\sqrt{\lambda_k}}{h}$.

$X_n(x)=\sin(\sqrt{\lambda_n}x)$, $T_n'(t)+\lambda_nT_n(t)=0$.

由 $u(x,0)=\varphi(x)$ 知 $\sum_{n=1}^{\infty}T_n(0)X_n(x)=\varphi(x)$. $\Rightarrow T_n(0)=\frac{(\varphi,X_n)}{(X_n,X_n)}\triangleq\varphi_n$.

$$\begin{cases}T_n'(t)+\lambda_nT_n(t)=0\\ T_n(0)=\varphi_n\end{cases}\Rightarrow T_n(t)=e^{-\lambda_nt}\cdot\varphi_n$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}\varphi_ne^{-\lambda_nt}\sin(\sqrt{\lambda_n}x).$$

例. 令 $B=\{(x,y)\mid x^2+y^2<1\}$, 考虑圆盘 B 上的 Laplace 方程

$$\begin{cases}\Delta u=0 & \text{in } B,\\ u=\varphi & \text{on } \partial B.\end{cases}$$ 要求 $u\in C^2(B)$.

用极坐标, 令 $x=r\cos\theta$, $y=r\sin\theta$, 则

$$\begin{cases}\partial_r^2u+\frac{1}{r}\partial_ru+\frac{1}{r^2}\partial_\theta^2u=0, & 0\le r<1,\\ u|_{r=1}=\varphi(\cos\theta,\sin\theta).\end{cases}$$

令 $u(r,\theta)=R(r)\Theta(\theta)$, 则 $R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0$.

$$\Rightarrow \frac{r^2R''(r)+rR'(r)}{R(r)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}\triangleq\lambda$$

于是 $\begin{cases}\Theta''(\theta)+\lambda\Theta(\theta)=0,\\ \Theta(\theta)=\Theta(\theta+2\pi).\end{cases}$ (由 θ 的意义)

① 若 $\lambda<0$, 通解为

$\Theta(\theta)=C_1e^{\sqrt{-\lambda}\theta}+C_2e^{-\sqrt{-\lambda}\theta}$, 不以 2π 为周期.

0 表示 $\lambda=0$ 对应的特征函数.

② 若 $\lambda=0$, $\Theta(\theta)=C_1\theta+C_2$, 由 $\Theta(\theta)=\Theta(\theta+2\pi)$ 知 $C_1=0$. 记 $\Theta_0(\theta)=1$.

③ 若 $\lambda>0$, 通解为 $\Theta(\theta)=C_1\cos(\sqrt{\lambda}\theta)+C_2\sin(\sqrt{\lambda}\theta)$.

$\sqrt{\lambda}\,2\pi=2k\pi \Rightarrow \lambda=k^2,\ k=1,2,\cdots$. 特征函数系为 $\cos(k\theta), \sin(k\theta)$.

故 $\Theta_k(\theta)\in\{1,\cos(k\theta),\sin(k\theta)\}_{k\in\mathbb{Z}_+}$.

由 $\dfrac{r^2R''(r)+rR'(r)}{R(r)}=k^2$ 可得 $r^2R''(r)+rR'(r)-k^2R(r)=0$ (Euler方程)

ODE笔记P51

令 $r=e^t$, 则 $\partial_t^2R-k^2R=0$

· 若 $k\neq0$, $R(e^t)=C_1e^{kt}+C_2e^{-kt}\Rightarrow R_k(r)=C_1r^k+C_2r^{-k}\notin C^2(B)$ (若 $C_2\neq0$). 故 $C_2=0$, $R_k(r)=r^k$

· 若 $k=0$, $R(e^t)=C_1t+C_2\Rightarrow R_0(r)=C_1\log r+C_2\notin C^2(B)$.

令原Laplace方程的解为 $u(r,\theta)=C_0+\sum_{k=1}^{\infty}[C_kr^k\cos(k\theta)+D_kr^k\sin(k\theta)]$,

记 $\tilde\varphi(\theta)=\varphi(\cos\theta,\sin\theta)$. 由 $u|_{r=1}=\tilde\varphi(\theta)$ 可知

$C_0+\sum_{k=1}^{\infty}[C_k\cos(k\theta)+D_k\sin(k\theta)]=\tilde\varphi(\theta)$

$$\xRightarrow{\text{Fourier 分析}}\begin{cases}C_k=\dfrac{\int_0^{2\pi}\tilde\varphi(\theta)\cos(k\theta)d\theta}{\int_0^{2\pi}\cos^2(k\theta)d\theta}=\dfrac{1}{\pi}\int_0^{2\pi}\tilde\varphi(\theta)\cos(k\theta)d\theta, & k\geq0.\\ D_k=\dfrac{\int_0^{2\pi}\tilde\varphi(\theta)\sin(k\theta)d\theta}{\int_0^{2\pi}\sin^2(k\theta)d\theta}=\dfrac{1}{\pi}\int_0^{2\pi}\tilde\varphi(\theta)\sin(k\theta)d\theta, & k\geq1.\end{cases}$$

注: 上述做法依赖于方程定义域的良好对称性(无 r,θ 混合).

能量估计

$$\text{考虑波动方程}\begin{cases}\partial_t^2u-\Delta u=f(x,t), & x\in\Omega, t>0.\\ u(x,0)=\varphi(x), & \partial_tu(x,0)=\psi(x),\\ u|_{\partial\Omega}=0.\end{cases}\qquad(\star)$$

Toy model $f\equiv0$, $\Omega=\mathbb{R}^n$.

$$\begin{cases}\partial_t^2u-\Delta u=0, & x\in\mathbb{R}^n, t>0,\\ u(x,0)=\varphi(x), & \partial_tu(x,0)=\psi(x)\end{cases}$$

(u与其导数在无穷远处消失)

方程两边同乘 ∂_tu 可得 $0=\partial_tu(\partial_t^2u-\Delta u)=\partial_t[\frac12(\partial_tu)^2+\frac12|\nabla u|^2]-\mathrm{div}(\partial_tu\nabla u)$.

$\partial_tu\,\partial_{x_i}^2u=\partial_{x_i}(\partial_tu\,\partial_{x_i}u)-\partial_t\partial_{x_i}u\,\partial_{x_i}u=\partial_{x_i}(\partial_tu\,\partial_{x_i}u)-\frac12\partial_t(\partial_{x_i}u)^2$

再在 Ω 上关于 x 积分 (假定 u 性质好, 使 ∂_tu 与积分可换序), 可得

$$0=\partial_t\int_{\mathbb{R}^n}[\tfrac12(\partial_tu)^2+\tfrac12|\nabla u|^2]dx-\underbrace{\int_{\mathbb{R}^n}\mathrm{div}(\partial_tu\nabla u)dx}_{=\int_{\partial\mathbb{R}^n}\partial_tu\frac{\partial u}{\partial n}dS(x)=0}\overset{\text{散度定理}}{=\!=}\partial_t\int_{\mathbb{R}^n}[\tfrac12(\partial_tu)^2+\tfrac12|\nabla u|^2]dx$$

令能量 $E(t)=\int_{\mathbb{R}^n}[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2]dx$，则 $\frac{dE(t)}{dt}\equiv 0\Leftrightarrow E(t)\equiv E(0),\ \forall t$，能量守恒.

上述"乘 $\partial_t u$"操作有深刻含义：每一个变换对应一个乘子，如时间平移 $u(x,t)\mapsto u(x,t+t_0)$ 构成一个单参数变换群，$\partial_t u$ 是其生成元(称为乘子)；又如空间平移 $u(x,t)\mapsto u(x+x_0,t)$ 对应于乘子 ∇u，此时，

$$0=\int_{\mathbb{R}^n}\partial_{x_i}u(\partial_t^2u-\Delta u)dx=\int_{\mathbb{R}^n}[\partial_t(\partial_{x_i}u\,\partial_t u)-\partial_t\partial_{x_i}u\,\partial_t u-\partial_{x_i}u\,\Delta u]dx$$
$$=\frac{d}{dt}\int_{\mathbb{R}^n}\partial_{x_i}u\,\partial_t u\,dx.$$

$\vec{P}(t)=\int_{\mathbb{R}^n}\partial_t u\nabla u\,dx$ 称为动量.

诺特定理：对称性 $\longleftrightarrow$ 守恒量.

在方程(☆)两边同乘 $\partial_t u$ 可得 $\partial_t u(\partial_t^2u-\Delta u)=\partial_t u\cdot f$.

$\Rightarrow \partial_t[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2]-\mathrm{div}(\partial_t u\nabla u)=\partial_t u f$.

在 Ω 上关于 x 积分，可得

$$\frac{d}{dt}\int_\Omega[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2]dx-\int_{\partial\Omega}\partial_t u\frac{\partial u}{\partial n}dS(x)=\int_\Omega\partial_t u f\,dx.$$
$$\le\underbrace{\frac{1}{2}\int_\Omega(\partial_t u)^2dx}_{\le E(t)}+\frac{1}{2}\int_\Omega|f|^2dx.$$

令 $E(t)=\int_\Omega[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2]dx$，则 $\frac{dE(t)}{dt}\le E(t)+\frac{1}{2}\int_\Omega|f|^2dx$.

$\Rightarrow \frac{d}{dt}(e^{-t}E(t))\le e^{-t}\cdot\frac{1}{2}\int_\Omega|f|^2dx\le\frac{1}{2}\int_\Omega|f|^2dx$.

$\xRightarrow{积分}\ e^{-t}E(t)-E(0)\le\frac{1}{2}\int_0^t\int_\Omega|f(x,s)|^2dx\,ds$

$\Rightarrow E(t)\le e^tE(0)+\frac{1}{2}e^t\int_0^t\int_\Omega|f|^2dx\,dS\le\underbrace{C_T[E(0)+\frac{1}{2}\int_0^t\int_\Omega|f|^2dx\,dS]}_{\triangleq G(t)}$，这里 $t\le T$.

而 $E(0)=\int_\Omega(\frac{1}{2}\psi^2+\frac{1}{2}|\nabla\varphi|^2)dx$.

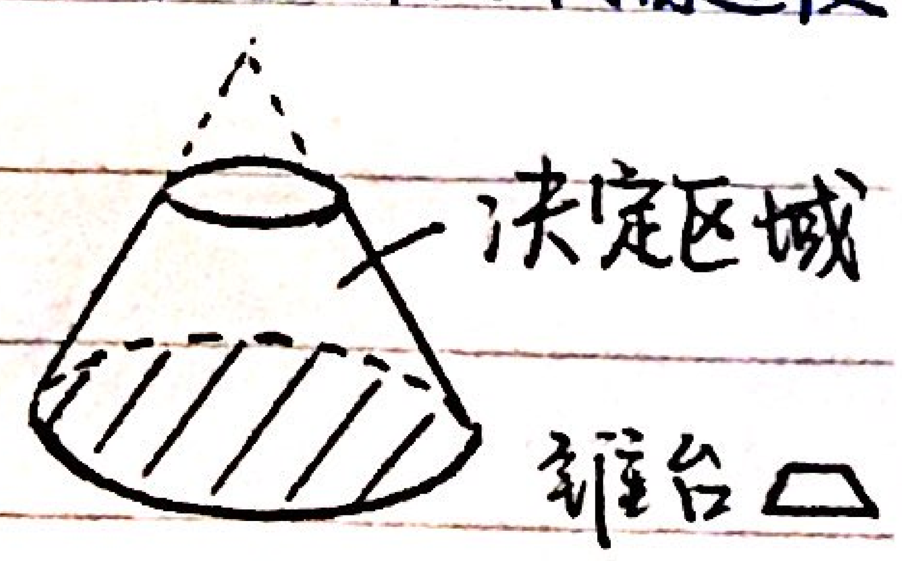
$\frac{d}{dt}\int_\Omega|u|^2dx=2\int_\Omega u\cdot u_t\,dx\le\int_\Omega u^2dx+\int_\Omega u_t^2dx\le\int_\Omega|u|^2dx+2G(t)$.

再由Gronwall不等式，$\int_\Omega|u(T)|^2dx\le C_T[\int_\Omega|\varphi|^2dx+\int_0^TG(\tau)d\tau]\le C_T[\int_\Omega|\varphi|^2dx+TG(T)]$

$\Rightarrow \int_\Omega|u(T)|^2dx\le C_T[\int_\Omega|\varphi|^2dx+\int_\Omega(|\nabla\varphi|^2+|\psi|^2)dx+\int_0^T\int_\Omega|f|^2dx\,dt]$. 教材P177

$\Rightarrow$ 由此可得解的唯一性.

能量估计与有限传播速度 (前面我们是由解的表达式得到有限传播速度)

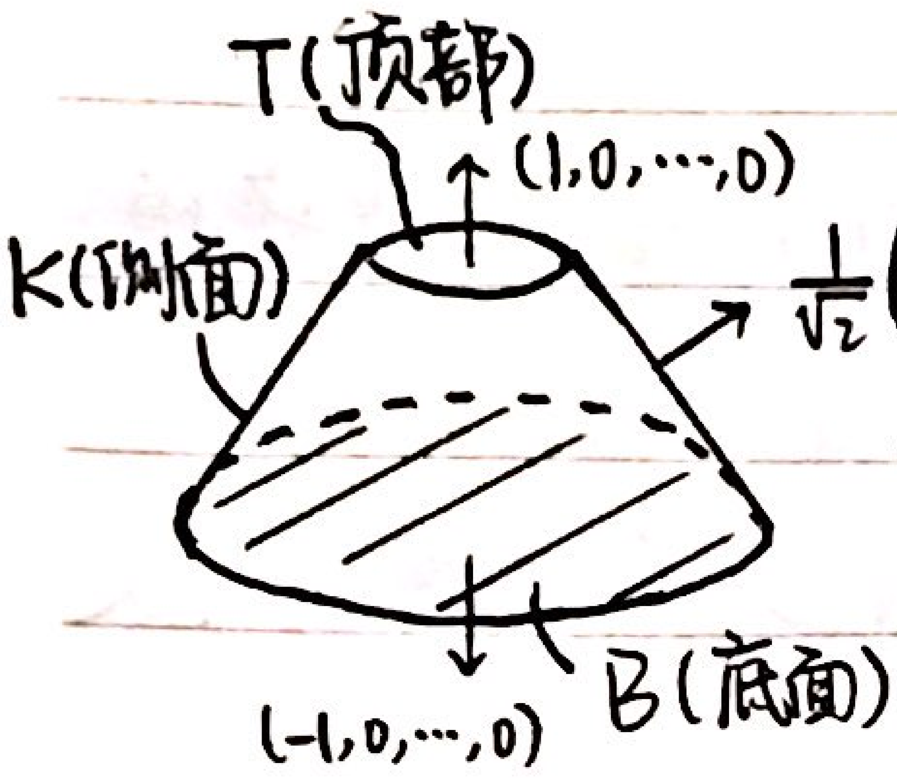


同前，仍考虑方程 $\begin{cases}\partial_t^2u-\Delta u=0,\\ u(x,0)=\varphi(x),\quad \partial_t u(x,0)=\psi(x).\end{cases}$

同能量估计，有 $\partial_t[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2]-\mathrm{div}(\partial_t u\nabla u)=0$

令能量密度 $e(t)=\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2$，在锥台 ⏢ (时空区域)上关于 t,x 积分，

$$\iint_{⏢}[\partial_t e(t)-\mathrm{div}(\partial_t u\nabla u)]dx\,dt=0.$$
$$\|$$
$$\iint_{⏢}\mathrm{div}_{t,x}(e(t),-\partial_t u\nabla u)dx\,dt$$



$\frac{1}{\sqrt{2}}\left(\frac{R-t}{|R-t|},\ \frac{x-x_0}{|x-x_0|}\right)$

$\Delta=\{(x,t)\mid |x-x_0|\le R-t,\ 0\le t\le t_0\}$，其中 $t_0<R$.

接前页，$0=\iint_{\Delta}\mathrm{div}_{t,x}(e(t),-u_t\nabla u)\,dx\,dt$

$\overset{\text{散度定理}}{=}\int_B -e(0)\,dx+\int_T e(t_0)\,dx+\int_K\left(e(t)\cdot\frac{1}{\sqrt{2}}\overset{=1}{\frac{R-t}{|R-t|}}-u_t\nabla u\cdot\frac{1}{\sqrt{2}}\cdot\frac{x-x_0}{|x-x_0|}\right)dS$

$=-\int_B e(0)\,dx+\int_T e(t_0)\,dx+\frac{1}{\sqrt{2}}\int_K\underbrace{\left(\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2-u_t\cdot\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right)}_{(\Delta)}dS$

(Δ) Flux$[0,t_0]$：从侧面流出的能量

$(\Delta)=\frac{1}{2}\left[(\partial_t u)^2-2u_t\frac{x-x_0}{|x-x_0|}\cdot\nabla u+|\nabla u|^2\right]$

$=\frac{1}{2}\left[\underbrace{\left|\partial_t u-\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2}_{\ge 0}+\underbrace{|\nabla u|^2-\left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2}_{\ge 0}\right]\ge 0$

$\Rightarrow \underbrace{\int_B e(0)\,dx}_{\text{(底面的能量)}}\ge\underbrace{\int_T e(t_0)\,dx}_{\text{(顶部的能量)}}$，说明若初始($t=0$)能量为0，则决定区域内能量$\equiv 0$.

⇒有限传播速度（影响区域的支集(即在$t=0$上投影)以速度1扩大）