Differentiable Manifolds

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Homework 1

Exercise 1 Prove that, for $1 \leqslant k \leqslant n$, the Grassmannian

$$\mathrm{Gr}_{\mathbb{R}}(k,n) = \{k\text{-dimensional linear subspaces of } \mathbb{R}^n\}$$

is a smooth manifold, by explicitly constructing open cover and local charts $\left\{\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{k(n-k)}\right\}_{\alpha \in I}$.

Proof We shall use the following "*Smooth Manifold Chart Lemma*" which tells us that a set can be given a topology and a smooth structure under certain conditions:

Smooth Manifold Chart Lemma *Let M be a set, and suppose we are given a collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of subsets of M together with maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, such that the following properties are satisfied:*

(i) *For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$.*

(ii) *For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.*

(iii) *Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.*

(iv) *Countably many of the sets U_α cover M.*

(v) *Whenever p, q are distinct points in M, either there exists some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.*

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Proof of the lemma We begin by showing that

$$\mathcal{B} = \left\{\varphi_\alpha^{-1}(V) : V \text{ is open in } \mathbb{R}^n, \alpha \in \Lambda\right\}$$

is a topological basis for M. By (i) and (iv), it suffices to show that for any point p in the intersection of two basis sets $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$, there is a third basis set containing p and contained in the intersection. In fact, $\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$ is itself a basis set. To see this, note that $U_\alpha \cap U_\beta \neq \varnothing$, then (iii) implies that $\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^{-1}(W)$ is an open subset of $\varphi_\alpha(U_\alpha \cap U_\beta)$, and (ii) implies that this set is also open in $\mathbb{R}^n$. It follows that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}\left(V \cap \left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^{-1}(W)\right)$$

is also a basis set, as claimed.

Each map φ_α is then a homeomorphism onto its image, where we equip M with the topology generated by the basis $\mathcal{B}$. So M is locally Euclidean of dimension n. The Hausdorff property follows from (v), since in the case where distinct points p and q are both contained in some U_α, we can use the homeomorphism $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ to separate them with disjoint open sets. And the second countability follows from (iv) and the fact that each U_α is second countable. Finally, (iii) guarantees that the collection $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas. It is clear that this topology and smooth structure are the unique ones satisfying the conditions of the lemma.

Now let us construct charts for $\mathrm{Gr}_{\mathbb{R}}(k,n)$ and apply the smooth manifold chart lemma. Let P and Q be any complementary subspaces of $\mathbb{R}^n$ of dimensions k and $n-k$, respectively. Then $\mathbb{R}^n = P \oplus Q$. For any linear map $f \in \mathcal{L}(P, Q)$, its graph can be identified with a linear subspace of $\mathbb{R}^n$:

$$\Gamma(f) \coloneqq \{v + f(v) : v \in P\}.$$

If $\{e_1, \cdots, e_k\}$ is a basis for P, then $\{e_1 + f(e_1), \cdots, e_k + f(e_k)\}$ is a basis for $\Gamma(f)$. To see this, it suffices to prove that the set is linearly independent. Suppose that

$$\sum_{i=1}^{k} c_i[e_i + f(e_i)] = 0,$$

then we can rewrite this as

$$\sum_{i=1}^{k} c_i e_i + f\left(\sum_{i=1}^{k} c_i e_i\right) = 0.$$

Note that the first term is in P and the second term is in Q, so both must be zero. Thus $c_i = 0$ for all i, as desired. Hence $\Gamma(f)$ is a k-dimensional subspace of $\mathbb{R}^n$. Any such subspace has the property that its intersection with Q is the zero subspace. Conversely, any k-dimensional subspace $S \subset \mathbb{R}^n$ that intersects Q trivially is the graph of a unique linear map $f \in \mathcal{L}(P, Q)$, which can be constructed as follows: let $\pi_P : \mathbb{R}^n \to P$ and $\pi_Q : \mathbb{R}^n \to Q$ be the projections determined by the direct sum decomposition; then the hypothesis implies that $(\pi_P)|_S$ is an isomorphism from S to P. Therefore, $f := [(\pi_Q)|_S] \circ [(\pi_P)|_S]^{-1}$ is a well-defined linear map from P to Q whose graph $\Gamma(f)$ is S. Denote U_Q the subset of $\mathrm{Gr}_{\mathbb{R}}(k, n)$ consisting of k-dimensional subspaces whose intersections with Q are trivial, then we have a bijection

$$\mathcal{L}(P, Q) \xleftrightarrow{1:1} U_Q$$

$$f \xmapsto{\quad\Gamma\quad} \Gamma(f)$$

$$[(\pi_Q)|_S] \circ [(\pi_P)|_S]^{-1} \xleftarrow{\varphi} S$$

By choosing bases for P and Q, we can identify $\mathcal{L}(P, Q)$ with $M_{(n-k)\times k}(\mathbb{R})$ and hence with $\mathbb{R}^{k(n-k)}$, and thus we can think of $\left(U_Q, \varphi := \Gamma^{-1}\right)$ as a coordinate chart. Since the image of each such chart is all of $\mathcal{L}(P, Q)$, condition (i) of the lemma is clearly satisfied.

Now let (P', Q') be any other such pair of subspaces, and let $\pi_{P'}, \pi_{Q'}$ be the corresponding projections and $\varphi' : U_{Q'} \to \mathcal{L}(P', Q')$ the corresponding map. We shall prove that $\varphi(U_Q \cap U_{Q'})$ is open in $\mathcal{L}(P, Q)$, which will establish condition (ii) of the lemma. For each $f \in \varphi(U_Q)$, define the map

$$I_f : P \to \mathbb{R}^n, \quad v \mapsto v + f(v),$$

which is a bijection from P to $\Gamma(f)$. Note that $\Gamma(f) = \operatorname{Im} I_f$ and $Q' = \operatorname{Ker} \pi_{P'}$, hence

$$f \in \varphi(U_Q \cap U_{Q'}) \iff \Gamma(f) \cap Q' = \varnothing \iff \operatorname{Im} I_f \cap \operatorname{Ker} \pi_{P'} = \varnothing,$$

and by linear algebra the last condition is equivalent to

$$\operatorname{rank}(\pi_{P'} \circ I_f) = \operatorname{rank}(I_f),$$

namely, the map $\pi_{P'} \circ I_f$ has full rank k. Therefore, the corresponding matrix A of $\pi_{P'} \circ I_f$ is a non-singular $k \times k$ matrix, i.e. $A \in \mathrm{GL}(k, \mathbb{R})$. Arrows in the reverse direction then show that f has an open neighborhood contained in $\varphi(U_Q \cap U_{Q'})$, which means $\varphi(U_Q \cap U_{Q'})$ is open in $\mathcal{L}(P, Q)$. Thus property (ii) in the lemma holds.

We need to show that the transition map $\varphi' \circ \varphi^{-1}$ is smooth on $\varphi(U_Q \cap U_{Q'})$. For any $f \in \varphi(U_Q \cap U_{Q'})$, let S denote the subspace $\Gamma(f) \subset \mathbb{R}^n$. If we put $f' \coloneqq \varphi' \circ \varphi^{-1}(f)$, then as above, $f' = [(\pi_{Q'})|_S] \circ [(\pi_{P'})|_S]^{-1}$. Recall that $I_f : P \to S$ is an isomorphism, so we can write

$$f' = [(\pi_{Q'})|_S] \circ I_f \circ (I_f)^{-1} \circ [(\pi_{P'})|_S]^{-1} = (\pi_{Q'} \circ I_f) \circ (\pi_{P'} \circ I_f)^{-1}.$$

To see that this depends smoothly on f, define linear maps

$$g = (\pi_{P'})|_P, \quad h = (\pi_{Q'})|_P, \quad j = (\pi_{P'})|_Q, \quad k = (\pi_{Q'})|_Q.$$

Then for any $v \in P$ we have

$$(\pi_{P'} \circ I_f)v = (g + j \circ f)v, \quad (\pi_{Q'} \circ I_f)v = (h + k \circ f)v,$$

from which it follows that

$$f' = (h + k \circ f) \circ (g + j \circ f)^{-1}.$$

Once we choose bases for P, Q, P', Q', all of these linear maps are represented by matrices, say F, F' and G, H, J, K, respectively. Then

$$F' = (H + KF)(G + JF)^{-1}.$$

By Cramer's rule, the entries of $(G + JF)^{-1}$ are rational functions of those of $G + JF$, hence the entries of F' depend smoothly on those of F. This proves that $\varphi' \circ \varphi^{-1}$ is a smooth map, so the charts we have constructed satisfy condition (iii) of the lemma.

To check condition (iv), we just note that $\mathrm{Gr}_{\mathbb{R}}(k, n)$ can in fact be covered by finitely many of the sets U_Q. Let $(e_1, \cdots, e_n)$ be a basis for $\mathbb{R}^n$, and consider those $(n-k)$-dimensional spaces Q that are spanned by $n-k$ of them. There are $\binom{n}{n-k}$ such spaces. For any k-dimensional subspace $S \subset \mathbb{R}^n$, suppose $(f_1, \cdots, f_k)$ is a basis of S. Then by the Steinitz exchange lemma, we can replace k of the e_i, without loss of generality, say $e_1, \cdots, e_k$, by $(f_1, \cdots, f_k)$, such that $(f_1, \cdots, f_k, e_{k+1}, \cdots, e_n)$ is a basis for $\mathbb{R}^n$. Then the $(n-k)$-dimensional subspace Q spanned by $e_{k+1}, \cdots, e_n$ is such that $S \in U_Q$. Thus, these $\binom{n}{n-k}$ charts cover $\mathrm{Gr}_{\mathbb{R}}(k, n)$.

Finally, the Hausdorff condition (v) can be verified by noting that for any two k-dimensional subspaces $P, P' \subset \mathbb{R}^n$, one can find a subspace Q of dimension $n-k$ whose intersections with both P and P' are trivial, and then P and P' are both contained in U_Q. In fact, in the case $k > 0$, since a real vector space cannot be a finite union of its proper subspaces, $P \cup P' \neq \mathbb{R}^n$. Hence there exists $v_1 \in \mathbb{R}^n \setminus (P \cup P')$. If $k < n-1$, we can find $v_2 \in \mathbb{R}^n \setminus ((P \oplus \mathrm{Span}(v_1)) \cup (P' \cup \mathrm{Span}(v_1)))$, and so on. This process terminates at some $v_n - k$ with

$$v_{n-k} \in \mathbb{R}^n \setminus ((P \oplus \mathrm{Span}(v_1, \cdots, v_{n-k-1})) \cup (P' \oplus \mathrm{Span}(v_1, \cdots, v_{n-k-1}))).$$

The process of choosing $v_1, \cdots, v_{n-k}$ implies that they are linearly independent, so the subspace Q spanned by them has the desired properties. □

Exercise 2 Let M be a smooth manifold and $\phi \in \mathrm{Diff}(M)$. Prove that its graph

$$\mathrm{Graph}(\phi) \coloneqq \{(x, \phi(x)) : x \in M\}$$

is a smooth manifold.

Proof Define the map

$$F : M \to \text{Graph}(\phi), \quad x \mapsto (x, \phi(x)).$$

Since F is the product of the identity map Id_M and the diffeomorphism $x \mapsto \phi(x)$, it is smooth. Moreover, it is clear that F is a bijection, and its inverse is just the projection onto the first factor, which is smooth. Therefore, F is a diffeomorphism, $\text{Graph}(\phi) \simeq M$ is a smooth manifold. □

Exercise 3 Let M be a closed smooth manifold and $\phi \in \text{Diff}(M)$. Prove that the *mapping torus* defined by

$$T_\phi(M) := [0,1] \times M / \sim$$

is a smooth manifold, where $(0, x)$ is identified with $(1, \phi(x))$ for any $x \in M$.

Proof Consider the $\mathbb{Z}$-action on $\mathbb{R} \times M$ defined by

$$n.(r, x) = (r + n, \phi^n(x)).$$

In the sense of quotient topology, the mapping torus $T_\phi(M)$ is just the orbit space $(\mathbb{R} \times M)/\mathbb{Z}$ under this action. It is clear that this discrete Lie group action is smooth and free. Moreover, since M is compact, the action is proper. To verify this, we need to show that the preimage of any compact set under the action map

$$F : \mathbb{Z} \times (\mathbb{R} \times M) \to (\mathbb{R} \times M) \times (\mathbb{R} \times M), \quad (n, (r, x)) \mapsto ((r + n, \phi^n(x)), (r, x))$$

is compact. Suppose $K \subset (\mathbb{R} \times M) \times (\mathbb{R} \times M)$ is compact, and let $K_1 = \pi_1(K)$ and $K_2 = \pi_2(K)$, where π_1, π_2 are the projections onto the first and second factors, respectively. Then both K_1 and K_2 are compact in $\mathbb{R} \times M$. The projection of K_1 onto $\mathbb{R}$ is compact, so $(r + n, \phi^n(x)) \in K_1$ holds for only finitely many integers n. Thus $F^{-1}(K)$ is compact in $\mathbb{Z} \times (\mathbb{R} \times M)$ as desired. By the quotient manifold theorem, $T_\phi(M)$ is a smooth manifold. □

Exercise 4 Prove that the following set of matrices

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

is a Lie group. Here "H" stands for Heisenberg.

Proof Let us first show that H is group under matrix multiplication. The product of two Heisenberg matrices is given by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + u & z + xv + w \\ 0 & 1 & y + v \\ 0 & 0 & 1 \end{pmatrix}.$$

The neutral element of the Heisenberg group is the identity matrix, and inverses are given by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

We can naturally identify H with $\mathbb{R}^3$, and define the multiplication map on $\mathbb{R}^3$ by

$$\mu : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad ((x, y, z), (u, v, w)) \mapsto (x + u, y + v, z + xv + w)$$

and the inverse map by

$$i : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (-x, -y, xy - z).$$

Both μ and i are smooth maps, so H is a Lie group. □

Exercise 5 Assume the orthogonal group $\mathrm{O}(n) = \left\{A \in M_{n\times n}(\mathbb{R}) : AA^{\mathsf{T}} = \mathbb{1}\right\}$ is a compact Lie group of dimension $\frac{1}{2}n(n-1)$. Prove that the special orthogonal group

$$\mathrm{SO}(n) := \{A \in \mathrm{O}(n) : \det(A) = 1\}$$

is a compact Lie group and calculate its dimension.

Proof Consider the determinant map $\det : \mathrm{O}(n) \to \mathbb{R}$. Since $\mathrm{SO}(n) = \det^{-1}(1) = \det^{-1}(\mathbb{R}_{>0})$, it is a clopen subgroup of $\mathrm{O}(n)$. By openness, $\mathrm{SO}(n)$ has the same dimension as $\mathrm{O}(n)$; and since $\mathrm{SO}(n)$ is closed in the compact Lie group $\mathrm{O}(n)$, it is itself compact. Therefore, $\mathrm{SO}(n)$ is a compact Lie group of dimension $\frac{1}{2}n(n-1)$. □

Exercise 6 Prove that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}P^3$ as two smooth manifolds.

Proof Any element in $\mathrm{SO}(3)$ is a rotation. It can be represented by a pair (v, θ), where $v \in \mathbb{S}^2$ is a unit vector along the axis of rotation and $\theta \in [0, 2\pi]$ is the angle of rotation about v. Note that this rotation is equivalent to the rotation about $-v$ by the angle $2\pi - \theta$. Therefore we have

$$\mathrm{SO}(3) \simeq \frac{\mathbb{S}^2 \times [0, 2\pi]}{(v, \theta) \sim (-v, 2\pi - \theta) \text{ and } (v, 0) \sim (w, 0)}.$$

In this identification, we define the map

$$\varphi : \mathrm{SO}(3) \to \mathbb{R}P^3 \simeq \frac{\mathbb{S}^3}{-x \sim x}, \quad [(v, \theta)] \mapsto \left[\left(v \sin\tfrac{\theta}{2}, \cos\tfrac{\theta}{2}\right)\right].$$

It is well-defined, since

$$(v, \theta) \sim (-v, 2\pi - \theta) \text{ in } \mathrm{SO}(3) \leftrightsquigarrow \left(v \sin\tfrac{\theta}{2}, \cos\tfrac{\theta}{2}\right) \sim \left(-v \sin\tfrac{2\pi-\theta}{2}, \cos\tfrac{2\pi-\theta}{2}\right) \text{ in } \mathbb{R}P^3,$$
$$(v, 0) \sim (w, 0) \text{ in } \mathrm{SO}(3) \leftrightsquigarrow (v \sin 0, \cos 0) \sim (w \sin 0, \cos 0) \text{ in } \mathbb{R}P^3.$$

It is straightforward to check that φ is a diffeomorphism. □

Exercise 7 Identify $\mathbb{C}P^n$ with the set of equivalence classes in $\left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*$. Consider the map $S : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^3$ by

$$([(w_0, w_1)], [(z_0, z_1)]) \mapsto [(w_0z_0, w_0z_1, w_1z_0, w_1z_1)].$$

Prove that S is a smooth map. Here, "S" stands for Segre.

Proof The map S is well-defined, since the product w_iz_j $(i, j = 0, 1)$ are all homogeneous of degree 2 in the variables w_0, w_1, z_0, z_1. Take the standard charts (U_0, φ_0) and (U_1, φ_1) on $\mathbb{C}P^1$ given by

$$U_0 = \{[w_0, w_1] : w_0 \neq 0\}, \quad U_1 = \{[w_0, w_1] : w_1 \neq 0\}$$

with local coordinates $\varphi_0([(w_0, w_1)]) = \frac{w_1}{w_0}$ on U_0 and $\varphi_1([(w_0, w_1)]) = \frac{w_0}{w_1}$ on U_1. Similarly, we choose charts for $\mathbb{C}P^3$, denoted by (V_i, ψ_i) for $i = 0, 1, 2, 3$.

- If $w_0 z_0 \neq 0$, then we can choose charts $(U_0 \times U_0, \varphi_0 \times \varphi_0)$ for $([(w_0, w_1)], [(z_0, z_1)])$ and (V_0, ψ_0) for $[(w_0 z_0, w_0 z_1, w_1 z_0, w_1 z_1)]$. Clearly $S(U_0 \times U_0) \subset V_0$. The composite map $\psi_0 \circ S \circ (\varphi_0 \times \varphi_0)^{-1}$ is given by

$$(x, y) \xrightarrow{(\varphi_0 \times \varphi_0)^{-1}} ([(1, x)], [(1, y)]) \xrightarrow{S} [(1, y, x, xy)] \xrightarrow{\psi_0} (y, x, xy),$$

  which is clearly smooth.

- If $w_0 z_1 \neq 0$, then we can choose charts $(U_0 \times U_1, \varphi_0 \times \varphi_1)$ for $([(w_0, w_1)], [(z_0, z_1)])$ and (V_1, ψ_1) for $[(w_0 z_0, w_0 z_1, w_1 z_0, w_1 z_1)]$. Clearly $S(U_0 \times U_1) \subset V_1$. The composite map $\psi_1 \circ S \circ (\varphi_0 \times \varphi_1)^{-1}$ is given by

$$(x, y) \xrightarrow{(\varphi_0 \times \varphi_1)^{-1}} ([(1, x)], [(y, 1)]) \xrightarrow{S} [(y, 1, xy, x)] \xrightarrow{\psi_1} (y, xy, x),$$

  which is clearly smooth.

- If $w_1 z_0 \neq 0$, then we can choose charts $(U_1 \times U_0, \varphi_1 \times \varphi_0)$ for $([(w_0, w_1)], [(z_0, z_1)])$ and (V_2, ψ_2) for $[(w_0 z_0, w_0 z_1, w_1 z_0, w_1 z_1)]$. Clearly $S(U_1 \times U_0) \subset V_2$. The composite map $\psi_2 \circ S \circ (\varphi_1 \times \varphi_0)^{-1}$ is given by

$$(x, y) \xrightarrow{(\varphi_1 \times \varphi_0)^{-1}} ([(x, 1)], [(1, y)]) \xrightarrow{S} [(x, xy, 1, y)] \xrightarrow{\psi_2} (x, xy, y),$$

  which is clearly smooth.

- If $w_1 z_1 \neq 0$, then we can choose charts $(U_1 \times U_1, \varphi_1 \times \varphi_1)$ for $([(w_0, w_1)], [(z_0, z_1)])$ and (V_3, ψ_3) for $[(w_0 z_0, w_0 z_1, w_1 z_0, w_1 z_1)]$. Clearly $S(U_1 \times U_1) \subset V_3$. The composite map $\psi_3 \circ S \circ (\varphi_1 \times \varphi_1)^{-1}$ is given by

$$(x, y) \xrightarrow{(\varphi_1 \times \varphi_1)^{-1}} ([(x, 1)], [(y, 1)]) \xrightarrow{S} [(xy, x, y, 1)] \xrightarrow{\psi_3} (xy, x, y),$$

  which is clearly smooth.

Therefore, S is a smooth map. □

Exercise 8 Consider group $E(n) \coloneqq \mathbb{R}^n \rtimes \mathrm{O}(n)$ where the multiplication is given by

$$(v, A) \cdot (w, B) = (v + Aw, AB)$$

where "E" stands for Euclidean. Note that $E(n)$ is a Lie group. Meanwhile, a representation of $E(n)$ is a Lie group homomorphism from $E(n)$ to $\mathrm{GL}(k, \mathbb{R})$ for some $k > 0$. Construct a non-trivial representation of $E(n)$ that is injective.

Proof We have already seen in elementary geometry that $E(n)$ is just the isometry group of the n-dimensional Euclidean space, and $E(n)$ can be viewed as the product manifold $\mathbb{R}^n \times \mathrm{O}(n)$. So we are left to verify that the group operations are smooth. The multiplication map

$$\mu : E(n) \times E(n) \to E(n), \quad ((v, A), (w, B)) \mapsto (v + Aw, AB)$$

is smooth, since it is the product of two smooth maps. Hence $E(n)$ is a Lie group.

A non-trivial representation of $E(n)$ that is injective can be constructed as follows. Consider

$$\Phi : E(n) \hookrightarrow \mathrm{GL}(n+1,\mathbb{R}), \quad (v,A) \mapsto \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}.$$

It is well-defined, since the block matrix is invertible if and only if A is invertible. To see that Φ is a group homomorphism, note that for any $(v,A),(w,B) \in E(n)$, we have

$$\Phi((v,A)\cdot(w,B)) = \Phi(v+Aw, AB) = \begin{pmatrix} AB & v+Aw \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B & w \\ 0 & 1 \end{pmatrix} = \Phi(v,A)\Phi(w,B).$$

Since Φ is clearly smooth, it serves as a non-trivial injective representation of $E(n)$. □

Exercise 9 Prove that the upper half-plane in $\mathbb{C}$, denoted by

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

is a homogeneous space.

Proof The Lie group $\mathrm{SL}(2,\mathbb{R})$ acts smoothly and transitively on $\mathbb{H}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. z = \frac{az+b}{cz+d}, \quad \text{where } ad-bc=1.$$

This action is clearly smooth since $cz+d \neq 0$ for all $z \in \mathbb{H}$. To see that it is transitive, let $z = x + \mathrm{i}y$ be a given point in $\mathbb{H}$. Observe that the matrix

$$\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

maps i to z. Since $z \in \mathbb{H}$ is arbitrary, the orbit of i under the action of $\mathrm{SL}(2,\mathbb{R})$ is all of $\mathbb{H}$. Therefore, the group $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}$, and $\mathbb{H}$ is a homogeneous space. □

Exercise 10 Prove that if M and N are smooth diffeomorphic, then $\dim M = \dim N$.

Proof Suppose M is a nonempty smooth m-manifold, N is a nonempty smooth n-manifold, and $f : M \to N$ is a diffeomorphism. Choose any point $p \in M$, and let (U,φ) and (V,ψ) be smooth coordinate charts containing p and $f(p)$, respectively. Then (the restriction of) $F := \psi \circ f \circ \varphi^{-1}$ is a diffeomorphism from an open subset $X \subset \mathbb{R}^m$ to an open subset $Y \subset \mathbb{R}^n$. Since $F^{-1} \circ F = \mathrm{Id}_X$, the chain rule implies that for each $x \in X$,

$$\mathrm{Id}_{\mathbb{R}^m} = \mathrm{D}(\mathrm{Id}_X)(x) = \mathrm{D}\big(F^{-1}\circ F\big)(x) = \mathrm{D}\big(F^{-1}\big)(F(x)) \circ \mathrm{D}F(x).$$

Similarly, $F \circ F^{-1} = \mathrm{Id}_Y$ implies that $\mathrm{D}F(x) \circ \mathrm{D}\big(F^{-1}\big)(F(x))$ is the identity on $\mathbb{R}^n$. This implies that $\mathrm{D}F(x)$ is invertible with inverse $\mathrm{D}\big(F^{-1}\big)(F(x))$, and therefore $n = m$. □

Homework 2

Exercise 11 Given the Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k,n)$, consider the following set

$$\gamma_{\mathbb{R}}(k,n) \coloneqq \big\{(V,v) \in \mathrm{Gr}_{\mathbb{R}}(k,n) \times \mathbb{R}^k : v \in V\big\}.$$

Prove that under the natural projection $\pi(V,v) \coloneqq V$, the structure $\pi : \gamma_{\mathbb{R}}(k,n) \to \mathrm{Gr}_{\mathbb{R}}(k,n)$ is a real vector bundle of rank-k. This vector bundle is called the tautological bundle (over $\mathrm{Gr}_{\mathbb{R}}(k,n)$).

Proof In Exercise 1 we have constructed local charts on $\mathrm{Gr}_{\mathbb{R}}(k,n)$ of the form

$$\varphi_Q : U_Q \to \mathcal{L}(P,Q) \xrightarrow{\sim} \mathbb{R}^{k(n-k)}.$$

Recall that when identifying $\mathcal{L}(P,Q)$ with $M_{(n-k)\times k}(\mathbb{R})$ and then $\mathbb{R}^{k(n-k)}$, we have chosen some bases for P and Q, which gives a natural linear isomorphism $\phi_Q : P \to \mathbb{R}^k$. Hence we can construct local trivializations of $\gamma_{\mathbb{R}}(k,n)$ as follows:

$$\begin{aligned}\Phi_Q : \pi^{-1}(U_Q) = \{(V,v) : V \in U_Q, v \in V\} &\xrightarrow{\sim} U_Q \times \mathbb{R}^k,\\ (V,v) &\longmapsto (V, \phi_Q(v)).\end{aligned}$$

It is immediate that Φ_Q preserves the fibers:

$$\Phi_Q|_{\pi^{-1}(\{V\})} : \pi^{-1}(\{V\}) = \{V\} \times V \xrightarrow{\sim} \{V\} \times \mathbb{R}^k.$$

For any two intersecting open sets U_Q and $U_{Q'}$, the map $\Phi_{Q'} \circ \Phi_Q^{-1}$ has the form

$$\begin{aligned}\Phi_{Q'} \circ \Phi_Q^{-1} : (U_Q \cap U_{Q'}) \times \mathbb{R}^k &\to (U_Q \cap U_{Q'}) \times \mathbb{R}^k,\\ (V,v) &\mapsto \Big(V, \phi_{Q'} \circ \phi_Q^{-1}(v)\Big).\end{aligned}$$

Here the transition map $\phi_{Q'} \circ \phi_Q^{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is a linear isomorphism. Therefore, the structure $\pi : \gamma_{\mathbb{R}}(k,n) \to \mathrm{Gr}_{\mathbb{R}}(k,n)$ is a real vector bundle of rank-k. □

Exercise 12 Let X, Y be vector fields on M, and locally (within some (U_α, ϕ_α)) write $X = (X_1, \cdots, X_n)$ and $Y = (Y_1, \cdots, Y_n)$ where X_i, Y_j are smooth functions on U_α for $1 \leqslant i, j \leqslant n$. Prove that the Lie bracket locally writes as follows,

$$[X,Y] = (\mathrm{D}_X Y_1 - \mathrm{D}_Y X_1, \cdots, \mathrm{D}_X Y_n - \mathrm{D}_Y X_n).$$

Use this to calculate $[X,Y]$ for $X, Y \in \Gamma\big(T\mathbb{R}^3\big)$ (in coordinate (x,y,z)) where

$$X((x,y,z)) = (-y, x, 0) \quad \text{and} \quad Y((x,y,z)) = (0, -z, y).$$

Proof By the (implicit) definition of the Lie bracket, we have

$$\mathrm{D}_{[X,Y]} f = \mathrm{D}_X \, \mathrm{D}_Y f - \mathrm{D}_Y \, \mathrm{D}_X f$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} X_i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^{n} Y_j \frac{\partial f}{\partial x^j}\right) - \sum_{j=1}^{n} Y_j \frac{\partial}{\partial x^j}\left(\sum_{i=1}^{n} X_i \frac{\partial f}{\partial x^i}\right) \\
&= \sum_{i,j=1}^{n} X_i \frac{\partial}{\partial x^i}\left(Y_j \frac{\partial f}{\partial x^j}\right) - \sum_{j,i=1}^{n} Y_j \frac{\partial}{\partial x^j}\left(X_i \frac{\partial f}{\partial x^i}\right) \\
&= \sum_{i,j=1}^{n} X_i \left(\frac{\partial Y_j}{\partial x^i}\frac{\partial f}{\partial x^j} + Y_j \frac{\partial^2 f}{\partial x^i \partial x^j}\right) - \sum_{j,i=1}^{n} Y_j \left(\frac{\partial X_i}{\partial x^j}\frac{\partial f}{\partial x^i} + X_i \frac{\partial^2 f}{\partial x^j \partial x^i}\right) \\
&= \sum_{i,j=1}^{n} X_i \frac{\partial Y_j}{\partial x^i}\frac{\partial f}{\partial x^j} - \sum_{j,i=1}^{n} Y_j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} \\
&= \sum_{i,j=1}^{n} \left(X_i \frac{\partial Y_j}{\partial x^i} - Y_i \frac{\partial X_j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} X_i \frac{\partial Y_j}{\partial x^i} - \sum_{i=1}^{n} Y_i \frac{\partial X_j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} \\
&= (\mathrm{D}_X Y_1 - \mathrm{D}_Y X_1, \cdots, \mathrm{D}_X Y_n - \mathrm{D}_Y X_n)(f)
\end{aligned}
$$

for any smooth function f. Here we have used the fact that mixed partial derivatives of a smooth function commute. Thus the local computation formula is proved. With this formula, we can calculate

$$[X, Y] = (0 - z, 0 - 0, x - 0) = (-z, 0, x).$$

□

Exercise 13 (1) Let $\mathbb{T}^2$ denote the 2-dimensional torus $\mathbb{S}^1 \times \mathbb{S}^1$. Construct a vector field $X \in \Gamma(T\mathbb{T}^2)$ that does *not* have any zero's.

(2) Construct a vector field $X \in \Gamma(T\mathbb{S}^2)$ that has only one zero.

Solution (1) Parametrize the 2-dimensional torus by

$$r : \mathbb{R}^2 \to \mathbb{R}^3, \quad (u, v) \mapsto ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u).$$

Taking partial derivatives with respect to u and v, we get

$$
\begin{aligned}
r_u(u, v) &= (-\sin u \cos v, -\sin u \sin v, \cos u), \\
r_v(u, v) &= (-(2 + \cos u)\sin v, (2 + \cos u)\cos v, 0).
\end{aligned}
$$

By construction, the vector field $X := (r_u, r_v)$ is everywhere tangential to $\mathbb{T}^2$. To see that it is nowhere vanishing, we compute

$$\|r_u\| = \sqrt{\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u} = 1$$

and

$$\|r_v\| = \sqrt{(2 + \cos u)^2(\sin^2 v + \cos^2 v)} = 2 + \cos u \geqslant 1.$$

(2) Consider the stereographic projection of $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$ onto $\mathbb{R}^2$:

$$\sigma : \mathbb{S}^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2, \quad (x, y, z) \mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right).$$

Its inverse is given by

$$\sigma^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{(0,0,1)\}, \quad (u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right).$$

The differential of σ^{-1} at $(u,v) \in \mathbb{R}^2$ is represented by its Jacobi matrix,

$$\operatorname{Jac}\bigl(\sigma^{-1}\bigr)((u,v)) = \frac{1}{\left(u^2+v^2+1\right)^2} \begin{pmatrix} 2-2u^2+2v^2 & -4uv \\ -4uv & 2+2u^2-2v^2 \\ 4u & 4v \end{pmatrix}.$$

Since $U((u,v)) = \frac{\partial}{\partial u}$ is a nowhere vanishing vector field on $\mathbb{R}^2$, the pushforward of U at (u,v) by σ^{-1} is proportional to

$$\bigl(1-u^2+v^2, -2uv, 2u\bigr),$$

and by substituting $u = \dfrac{x}{1-z}$ and $v = \dfrac{y}{1-z}$, we obtain a nowhere vanishing vector field on $\mathbb{S}^2 \setminus \{(0,0,1)\}$:

$$X_1((x,y,z)) = \frac{1}{(1-z)^2}\bigl(2-2x^2-2z, -2xy, 2x(1-z)\bigr).$$

This is proportional to the vector field

$$X((x,y,z)) = \bigl(x^2+z-1, xy, x(z-1)\bigr).$$

This expression allows us to extend X smoothly to the entire $\mathbb{S}^2$ by setting $X((0,0,1)) = (0,0,0)$. To check that X has only one zero, note that the second component xy vanishes only if $x = 0$ or $y = 0$. When $x = 0$, the vector field becomes $(z-1,0,0)$, which vanishes only at the north pole $(0,0,1)$. When $y=0$, the vector field becomes $\bigl(x^2+z-1, 0, x(z-1)\bigr)$, which again vanishes only at the north pole $(0,0,1)$. Therefore $X \in \Gamma\bigl(T\mathbb{S}^2\bigr)$ has only one zero at the north pole. □

Exercise 14 On the standard unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$, construct three smooth vector fields $X, Y, Z \in \Gamma\bigl(T\mathbb{S}^3\bigr)$ such that for every $p \in \mathbb{S}^3$, the vectors $\{X(p), Y(p), Z(p)\}$ form a basis at the fiber $T_p\mathbb{S}^3 = \pi^{-1}(p)$ of the tangent bundle $\pi : T\mathbb{S}^3 \to \mathbb{S}^3$.

Solution We use the following proposition to characterize the tangent space at each point $p \in \mathbb{S}^3$:
Proposition *Suppose M is a smooth manifold and $S \subset M$ is an embedded submanifold. If $\Phi : U \to N$ is any local defining map for S, then $T_pS = \operatorname{Ker} \mathrm{d}\Phi_p : T_pM \to T_{\Phi(p)}N$ for each $p \in S \cap U$.*

The defining map for $\mathbb{S}^3$ is given by $\Phi(x_1,x_2,x_3,x_4) = x_1^2+x_2^2+x_3^2+x_4^2-1$. The differential of Φ at $p = (x_1,x_2,x_3,x_4) \in \mathbb{S}^3$ is $\mathrm{d}\Phi_p = 2(x_1,x_2,x_3,x_4)$, hence

$$T_p\mathbb{S}^3 = \bigl\{v \in T_p\mathbb{S}^3 : p^{\mathsf{T}}v = 0\bigr\}.$$

Therefore we define for $(x,y,z,w) \in \mathbb{S}^3 \subset \mathbb{R}^4$ the following three vector fields:

$$\begin{aligned} X((x,y,z,w)) &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}, \\ Y((x,y,z,w)) &= -z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial w}, \\ Z((x,y,z,w)) &= -w\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial w}. \end{aligned}$$

By the above proposition they form a basis at each tangent space. Since $\mathbb{S}^3$ is an embedded submanifold of $\mathbb{R}^4$, X, Y, Z are all smooth vector fields on $\mathbb{S}^3$ by composition. To see linear independence, suppose $V := aX(p) + bY(p) + cZ(p) = 0$ for some $p \in \mathbb{S}^3$ and $a, b, c \in \mathbb{R}$. Since X, Y, Z are pairwise orthogonal at each point, we have

$$0 = \langle V, V \rangle = a^2 \langle X(p), X(p) \rangle + b^2 \langle Y(p), Y(p) \rangle + c^2 \langle Z(p), Z(p) \rangle.$$

This implies $a = b = c = 0$, so X, Y, Z are linearly independent at each point. □

Exercise 15 Prove that for any finite-dimensional vector spaces U, V, W, there exists a map $\varphi : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ that is an isomorphism and identifies $u \otimes (v \otimes w)$ and $(u \otimes v) \otimes w$.

Proof The map

$$f : U \times V \times W \to (U \otimes V) \otimes W, \quad (u, v, w) \mapsto (u \otimes v) \otimes w$$

is obviously multilinear, and thus by the universal property of tensor products, it descends to a linear map

$$\tilde{f} : U \otimes V \otimes W \to (U \otimes V) \otimes W, \quad u \otimes v \otimes w \mapsto (u \otimes v) \otimes w.$$

Since $(U \otimes V) \otimes W$ is spanned by elements of the form $(u \otimes v) \otimes w$, the map $\tilde{f}$ is surjective, and therefore it is an isomorphism for dimensional reasons. Similarly, there is an isomorphism

$$\tilde{g} : U \otimes V \otimes W \to U \otimes (V \otimes W), \quad u \otimes v \otimes w \mapsto u \otimes (v \otimes w).$$

Finally, the composition $\varphi := \tilde{f} \circ \tilde{g}^{-1}$ is the desired isomorphism. □

Exercise 16 Recall that an element $x \in V \otimes W$ is called *decomposable* if there exist $v \in V$ and $w \in W$ such that $x = v \otimes w$. Suppose V admits a basis $\{e_1, \cdots, e_n\}$ and W admits a basis $\{f_1, \cdots, f_m\}$. Prove that $x = \sum a_{ij}(e_i \otimes f_j) \in V \otimes W$ is decomposable if and only if the matrix $(a_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m}$ has rank 1.

Proof Denote the matrix $(a_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m}$ by A. Formally, we can write

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}(e_i \otimes f_j) = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} A \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Then

$$\begin{gathered} \operatorname{rank} A = 1 \\ \Updownarrow \\ A = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \begin{pmatrix} s_1 & \cdots & s_m \end{pmatrix} \quad \text{for some } r_i, s_j \in \mathbb{R} \\ \Updownarrow \\ x = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \begin{pmatrix} s_1 & \cdots & s_m \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \left(\sum_{i=1}^{n} r_i e_i \right) \otimes \left(\sum_{j=1}^{m} s_j f_j \right). \end{gathered}$$

□

Exercise 17 For any matrices $A \in \mathrm{GL}(k, \mathbb{R})$ and $B \in \mathrm{GL}(l, \mathbb{R})$, prove

$$\det(A \otimes B) = [\det(A)]^l [\det(B)]^k.$$

Proof (Proof 1) Let $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$ be the eigenvalues of A with associated eigenvectors $v_1, \cdots, v_k$, and let $\mu_1, \cdots, \mu_l \in \mathbb{C}$ be the eigenvalues of B with associated eigenvectors $w_1, \cdots, w_l$. Then

$$(A \otimes B)(v_i \otimes w_j) = Av_i \otimes Bw_j = \lambda_i v_i \otimes \mu_j w_j = \lambda_i \mu_j (v_i \otimes w_j).$$

Hence the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ ($1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$), counted with multiplicities. It follows that

$$\det(A \otimes B) = \prod_{i=1}^{k} \prod_{j=1}^{l} \lambda_i \mu_j = \prod_{i=1}^{k} \lambda_i^l \prod_{j=1}^{l} \mu_j^k = [\det(A)]^l [\det(B)]^k.$$

(Proof 2) Since

$$\det(A \otimes \mathbb{1}_{l \times l}) = \det(\mathbb{1}_{l \times l} \otimes A) = \det(\mathrm{diag}(\underbrace{A, \cdots, A}_{l \text{ copies}})) = [\det(A)]^l,$$

and similarly

$$\det(\mathbb{1}_{k \times k} \otimes B) = [\det(B)]^k,$$

we have

$$\det(A \otimes B) = \det((A \otimes \mathbb{1}_{l \times l})(\mathbb{1}_{k \times k} \otimes B)) = [\det(A)]^l [\det(B)]^k.$$

□

Exercise 18 Recall that on an even-dimensional manifold M, an *almost complex structure* denoted by J is a smooth family of morphisms $J_x : T_x M \to T_x M$ satisfying $J_x^2 = -\mathbb{1}$. Consider the following $(1, 2)$-tensor field

$$N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

for any $X, Y \in \Gamma(TM)$. A celebrated result from Newlander-Nirenberg says that J is integrable (induced by a complex structure) if and only if $N_J \equiv 0$. Prove that over a closed surface Σ, any almost complex structure J (if exists) is always integrable.

Proof Let Σ be a closed surface (i.e., a 2-dimensional smooth manifold), and fix a point $p \in \Sigma$. Let V be a non-vanishing local vector field defined in a neighborhood of p. Note that $\{V, JV\}$ forms a basis in this neighborhood, for if V and JV are linearly dependent, then $JV = cV$ for some $c \in \mathbb{R}$, which implies $-V = J^2 V = cJV = c^2 V$, a contradiction. Then it suffices to show that $N_J(V, V) = 0 = N_J(V, JV)$ at p since N_J is a $(1, 2)$-tensor field. In fact, using the Lie bracket properties, we have

$$\begin{aligned} N_J(V, V) &= [V, V] + J[JV, V] + J[V, JV] - [JV, JV] \\ &= J[JV, V] + J[V, JV] \\ &= J[JV, V] - J[JV, V] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} N_J(V, JV) &= [V, JV] + J[JV, JV] + J\left[V, J^2 V\right] - \left[JV, J^2 V\right] \\ &= [V, JV] + J[JV, JV] + J[V, -V] - [JV, -V] \end{aligned}$$

$$\begin{aligned} &= [V, JV] - J[V, V] + [JV, V] \\ &= [V, JV] - [V, JV] \\ &= 0. \end{aligned}$$

Since p is arbitrary, $N_J \equiv 0$ on Σ, and thus J is integrable. □

Exercise 19 Prove that on any Riemannian manifold (M, g), there exists a unique connection ∇ satisfying, for any $X, Y, Z \in \Gamma(TM)$,

(i) (compatibility) $Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.

(ii) (torsion-free) $[X, Y] = \nabla_X Y - \nabla_Y X$.

Proof We prove uniqueness first, by deriving a formula for ∇. Suppose that ∇ is a connection satisfying the above conditions (i) and (ii), and let $X, Y, Z \in \Gamma(TM)$. Writing the compatibility equation three times with X, Y, Z cyclically permuted, we obtain

$$\begin{aligned} Xg(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\ Yg(Z, X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \\ Zg(X, Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{aligned}$$

Using the torsion-free condition on the last term in each line, this can be rewritten as

$$\begin{aligned} Xg(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_Z X) + g(Y, [X, Z]), \\ Yg(Z, X) &= g(\nabla_Y Z, X) + g(Z, \nabla_X Y) + g(Z, [Y, X]), \\ Zg(X, Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Y Z) + g(X, [Z, Y]). \end{aligned}$$

Adding the first two of these equations and subtracting the third, we obtain

$$Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) = 2g(\nabla_X Y, Z) + g(Y, [X, Z]) + g(Z, [Y, X]) - g(X, [Z, Y]).$$

Finally, solving for $g(\nabla_X Y, Z)$, we get

$$g(\nabla_X Y, Z) = \frac{1}{2}[Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(Y, [X, Z]) - g(Z, [Y, X]) + g(X, [Z, Y])].$$

Now suppose ∇^1 and ∇^2 are two connections on TM that are torsion-free and compatible with g. Since the right-hand side of the above formula does not depend on the connection, it follows that

$$g\big(\nabla^1_X Y - \nabla^2_X Y, Z\big) = 0$$

for all X, Y, Z. This can happen only if $\nabla^1_X Y = \nabla^2_X Y$ for all X and Y, so $\nabla^1 = \nabla^2$.

To prove existence, one only need to check that the $\nabla_X Y$ defined by the above formula satisfies all conditions of a connection and is torsion-free and compatible with g. For any $f, h \in \mathcal{C}^\infty(M)$ and $X_1, X_2, X, Y_1, Y_2, Y, Z \in \Gamma(TM)$, with the product rule of the Lie bracket, we have

$$\begin{aligned} g(\nabla_{fX_1 + hX_2} Y, Z) =& \frac{1}{2}[(fX_1 + hX_2)g(Y, Z) + Yg(Z, fX_1 + hX_2) - Zg(fX_1 + hX_2, Y) \\ & - g(Y, [fX_1 + hX_2, Z]) - g(Z, [Y, fX_1 + hX_2]) + g(fX_1 + hX_2, [Z, Y])] \end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}[fX_1g(Y,Z)+hX_2g(Y,Z)+Yfg(Z,X_1)+Yhg(Z,X_2)\\
&-Zfg(X_1,Y)-Zhg(X_2,Y)-g(Y,[fX_1,Z])-g(Y,[hX_2,Z])\\
&-g(Z,[Y,fX_1])-g(Z,[Y,hX_2])+fg(X_1,[Z,Y])+hg(X_2,[Z,Y])]\\
=&\frac{1}{2}[fX_1g(Y,Z)+hX_2g(Y,Z)+Y(f)g(Z,X_1)+fYg(Z,X_1)\\
&Y(h)g(Z,X_2)+hYg(Z,X_2)-Z(f)g(X_1,Y)-fZg(X_1,Y)\\
&-Z(h)g(X_2,Y)-hZg(X_2,Y)-g(Y,f[X_1,Z]-Z(f)X_1)\\
&-g(Y,h[X_2,Z]-Z(h)X_2)-g(Z,f[Y,X_1]+Y(f)X_1)\\
&-g(Z,h[Y,X_2]+Y(h)X_2)+fg(X_1,[Z,Y])+hg(X_2,[Z,Y])]\\
=&fg(\nabla_{X_1}Y,Z)+hg(\nabla_{X_2}Y,Z)\\
=&g((f\nabla_{X_1}+h\nabla_{X_2})Y,Z)
\end{aligned}
$$

and

$$
\begin{aligned}
g(\nabla_X(Y_1+Y_2),Z)=&\frac{1}{2}[Xg(Y_1+Y_2,Z)+(Y_1+Y_2)g(Z,X)-Zg(X,Y_1+Y_2)\\
&-g(Y_1+Y_2,[X,Z])-g(Z,[Y_1+Y_2,X])+g(X,[Z,Y_1+Y_2])]\\
=&\frac{1}{2}[Xg(Y_1,Z)+Xg(Y_2,Z)+Y_1g(Z,X)+Y_2g(Z,X)\\
&-Zg(X,Y_1)-Zg(X,Y_2)-g(Y_1,[X,Z])-g(Y_2,[X,Z])\\
&-g(Z,[Y_1,X])-g(Z,[Y_2,X])+g(X,[Z,Y_1])+g(X,[Z,Y_2])]\\
=&g(\nabla_XY_1+\nabla_XY_2,Z).
\end{aligned}
$$

and finally

$$
\begin{aligned}
g(\nabla_X(fY),Z)=&\frac{1}{2}[Xg(fY,Z)+fYg(Z,X)-Zg(X,fY)\\
&-g(fY,[X,Z])-g(Z,[fY,X])+g(X,[Z,fY])]\\
=&\frac{1}{2}[Xfg(Y,Z)+fYg(Z,X)-Zfg(X,Y)-fg(Y,[X,Z])\\
&-g(Z,-X(f)Y-f[X,Y])+g(X,Z(f)Y+f[Z,Y])]\\
=&\frac{1}{2}[X(f)g(Y,Z)+fXg(Y,Z)+fYg(Z,X)-Z(f)g(X,Y)-fZg(X,Y)\\
&-fg(Y,[X,Z])+X(f)g(Z,Y)+fg(Z,[X,Y])+Z(f)g(X,Y)+fg(X,[Z,Y])]\\
=&\frac{1}{2}[fXg(Y,Z)+fYg(Z,X)-fZg(X,Y)\\
&-fg(Y,[X,Z])-fg(Z,[Y,X])+fg(X,[Z,Y])]+X(f)g(Y,Z)\\
=&fg(\nabla_XY,Z)+X(f)g(Y,Z)\\
=&g(X(f)Y+f\nabla_XY,Z).
\end{aligned}
$$

To check the torsion-free condition, we have

$$
\begin{aligned}
g(\nabla_XY-\nabla_YX,Z)=&\frac{1}{2}[-g(Y,[X,Z])-g(Z,[Y,X])+g(X,[Z,Y])\\
&+g(X,[Y,Z])+g(Z,[X,Y])-g(Y,[Z,X])]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{2}[-g(Z,[Y,X])+g(Z,[X,Y])]\\
&=g([X,Y],Z),
\end{aligned}
$$

which implies $[X,Y]=\nabla_X Y-\nabla_Y X$. Finally, the compatibility condition is obtained from

$$
\begin{aligned}
&g(\nabla_Z X,Y)+g(X,\nabla_Z Y)\\
=&\frac{1}{2}[Zg(X,Y)+Xg(Y,Z)-Yg(Z,X)-g(X,[Z,Y])-g(Y,[X,Z])+g(Z,[Y,X])]\\
&+\frac{1}{2}[Zg(Y,X)+Yg(X,Z)-Xg(Z,Y)-g(Y,[Z,X])-g(X,[Y,Z])+g(Z,[X,Y])]\\
=&Zg(X,Y).
\end{aligned}
$$

□

Exercise 20 Given a Riemannian manifold (M,g), prove that for any smooth function $F:M\to\mathbb{R}$, there exists a unique vector field denoted by ∇F satisfying

$$g(\nabla F,X)=\mathrm{D}_X F$$

for any $X\in\Gamma(TM)$. This vector field is called the *gradient of F on M*. Also, prove that the function F is non-decreasing along ∇F. Finally, work out (with details) the explicit formula of ∇F for $F:(\mathbb{R}^2,g)\to\mathbb{R}$ in polar coordinate (r,θ), where g is taken as the standard inner product.

Proof Since the metric tensor g is non-degenerate, it induces the musical isomorphisms $\flat : TM\to T^*M$, $X\mapsto g(X,\cdot)$ and $\sharp:=\flat^{-1}:T^*M\to TM$. In local coordinates $\{x^i\}$ we have $g=g_{ij}\,\mathrm{d}x^i\otimes\mathrm{d}x^j$ and the musicalities are given by

$$\flat\left(\frac{\partial}{\partial x^i}\right)=g_{ij}\,\mathrm{d}x^j \quad\text{and}\quad \sharp(\mathrm{d}x^i)=g^{ij}\frac{\partial}{\partial x^j},$$

where $\left[g^{ij}\right]=[g_{ij}]^{-1}$. By definition $\flat(\nabla F)=\mathrm{d}F$, so the gradient ∇F is given by

$$\nabla F=\sharp(\mathrm{d}F)=\sharp\left(\frac{\partial F}{\partial x^i}\,\mathrm{d}x^i\right)=\frac{\partial F}{\partial x^i}g^{ij}\frac{\partial}{\partial x^j}. \tag{20–1}$$

One can check that this vector field satisfies the given equation:

$$
\begin{aligned}
g(\nabla F,X)&=\frac{\partial F}{\partial x^i}g^{ij}g\left(\frac{\partial}{\partial x^j},X\right)=\frac{\partial F}{\partial x^i}g^{ij}X^k g_{jk}\\
&=\frac{\partial F}{\partial x^i}X^k\delta^i_k=\frac{\partial F}{\partial x^i}X^i=\mathrm{D}_X F.
\end{aligned}
$$

If there is another vector field $\overline{\nabla}F$ satisfying the equation, then

$$g\left(\nabla F-\overline{\nabla}F,X\right)=0,$$

which implies $\nabla F=\overline{\nabla}F$. Therefore ∇F is unique. Since $\mathrm{D}_{\nabla F}F=g(\nabla F,\nabla F)\geqslant 0$, the function F is non-decreasing along ∇F.

To get the explicit formula of ∇F for $F:\left(\mathbb{R}^2,g\right)\to\mathbb{R}$ in polar coordinates (r,θ), we need to compute the matrices $[g_{r\theta}]$ and $\left[g^{r\theta}\right]$. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},$$
$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$$

Hence we get

$$g_{rr} = g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \cos^2\theta + \sin^2\theta = 1,$$
$$g_{r\theta} = g_{\theta r} = g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = -r\cos\theta\sin\theta + r\sin\theta\cos\theta = 0,$$
$$g_{\theta\theta} = g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = r^2\sin^2\theta + r^2\cos^2\theta = r^2.$$

Therefore

$$\left[g^{r\theta}\right] = [g_{r\theta}]^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}.$$

Substituting these into (20–1) gives

$$\nabla F = \frac{\partial F}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial F}{\partial \theta}\frac{\partial}{\partial \theta}.$$

□

Homework 3

Exercise 21 Let V be a vector space with basis $\{e_1, \cdots, e_n\}$. Then for a fixed $k \in \{1, \cdots, n\}$, prove that

$$\left\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant n\right\}$$

form a basis of $\bigwedge^k V^*$. Therefore, $\dim \bigwedge^k V^* = \frac{n!}{k!(n-k)!}$.

Proof Let us introduce the multi-index notation $I = (i_1, \cdots, i_k)$ and write e_I for $(e_{i_1}, \cdots, e_{i_k})$ and α^I for $e^{i_1} \wedge \cdots \wedge e^{i_k}$. Then one has

$$\alpha^I(e_J) = \delta^I_J := \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases}$$

First, we show linear independence. Suppose $\sum_I c_I\alpha^I = 0$, $c_I \in \mathbb{R}$, and I runs over all strictly ascending multi-indices of length k. Applying both sides to e_J, $J = (j_1 < \cdots < j_k)$, we get

$$0 = \sum_I c_I\alpha^I(e_J) = \sum_I c_I\delta^I_J = c_J,$$

since among all strictly ascending multi-indices of length k, there is only one equal to J. This proves that the α^I are linearly independent.

To show that the α^I span $\bigwedge^k V^*$, let $f \in \bigwedge^k V^*$. We claim that

$$f = \sum_I f(e_I)\alpha^I,$$

where I runs over all strictly ascending multi-indices of length k. Let $g = \sum_I f(e_I)\alpha^I$. By k-linearity and the alternating property, if f and g agree on all e_J, where $J = (j_1 < \cdots < j_k)$, then they are equal. But

$$g(e_J) = \sum_I f(e_I)\alpha^I(e_J) = \sum_I f(e_I)\delta^I_J = f(e_J).$$

Therefore, $f = g = \sum_I f(e_I)\alpha^I$.

We have shown that the e_I form a basis of $\bigwedge^k V$. As a consequence, $\dim \bigwedge^k V = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. □

Exercise 22 Let V be a vector space with basis $\{e_1, \cdots, e_n\}$, equipped with an inner product $\langle \cdot, \cdot \rangle$ with signature

$$(\underbrace{-, \cdots, -}_{p}, \underbrace{+ \cdots, +}_{q}).$$

Prove that for the Hodge star operator $\star : \bigwedge^k V^* \to \bigwedge^{n-k} V^*$, it satisfies

$$\star \circ \star = (-1)^{k(n-k)+p} \cdot \mathbb{1}_{\bigwedge^k V^*}$$

for any $k \in \{1, \cdots, n\}$.

Proof By Exercise 21, it suffices to prove for a basis element $e^{i_1} \wedge \cdots \wedge e^{i_k}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$. Let $e^{i_{k+1}}, \cdots, e^{i_n}$ be the complementary basis elements with $i_{k+1} < \cdots < i_n$. Since $\star \circ \star = \pm \mathbb{1}_{\bigwedge^k V^*}$, we just need to get the sign right. We have

$$\begin{aligned}
s &:= \operatorname{sign}\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & i_{k+1} & \cdots & i_n \end{pmatrix} \operatorname{sign}\begin{pmatrix} i_{k+1} & \cdots & i_n & i_1 & \cdots & i_k \\ 1 & \cdots & n-k & n-k+1 & \cdots & n \end{pmatrix} \\
&= \operatorname{sign}\begin{pmatrix} i_1 & \cdots & i_k & i_{k+1} & \cdots & i_n \\ i_{k+1} & \cdots & i_n & i_1 & \cdots & i_k \end{pmatrix} \\
&= \operatorname{sign}\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ k+1 & \cdots & n & 1 & \cdots & k \end{pmatrix} \\
&= (-1)^{k(n-k)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\star \circ \star\big(e^{i_1} \wedge \cdots \wedge e^{i_k}\big) &= s \cdot (e_{i_1}, e_{i_1}) \cdots (e_{i_k}, e_{i_k})(e_{i_{k+1}}, e_{i_{k+1}}) \cdots (e_{i_n}, e_{i_n}) e^{i_1} \wedge \cdots \wedge e^{i_k} \\
&= s \cdot (e_1, e_1) \cdots (e_n, e_n) e^{i_1} \wedge \cdots \wedge e^{i_k} \\
&= (-1)^{k(n-k)}(-1)^p(+1)^q e^{i_1} \wedge \cdots \wedge e^{i_k} \\
&= (-1)^{k(n-k)+p} e^{i_1} \wedge \cdots \wedge e^{i_k}.
\end{aligned}$$

□

Exercise 23 Let $\{\varphi_{s,t}\}_{(s,t)\in\mathbb{R}^2}$ be a 2-parametrized group of diffeomorphisms (on a manifold M). Consider two vector fields defined via the following equations,

$$\frac{\partial \varphi_{s,t}}{\partial t} = X_s \circ \varphi_{s,t} \quad \text{and} \quad \frac{\partial \varphi_{s,t}}{\partial s} = Y_t \circ \varphi_{s,t}.$$

Then prove the following equality,

$$\frac{\partial X_s}{\partial s} - \frac{\partial Y_t}{\partial t} = [X_s, Y_t]$$

where $[\cdot,\cdot]$ denotes the Poisson bracket of vector fields.

Proof Suppose $\dim M = n$, then in local coordinates we have for each $1 \leqslant i \leqslant n$

$$\frac{\partial \varphi^i_{s,t}}{\partial t}(x) = X^i_s(\varphi_{s,t}(x)) =: X^i(s,\varphi_{s,t}(x)) \quad \text{and} \quad \frac{\partial \varphi^i_{s,t}(x)}{\partial s}(x) = Y^i_t(\varphi_{s,t}(x)) =: Y^i(s,\varphi_{s,t}(x)).$$

Differentiating both sides of the first equation with respect to s gives

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial \varphi^i_{s,t}}{\partial t}\right)(x) &= \frac{\partial X^i}{\partial s}(s,\varphi_{s,t}(x)) + \sum_{j=1}^n \frac{\partial X^i}{\partial x^j}(s,\varphi_{s,t}(x)) \cdot \frac{\partial \varphi^j_{s,t}}{\partial s}(x) \\ &= \frac{\partial X^i_s}{\partial s}(\varphi_{s,t}(x)) + \sum_{j=1}^n \frac{\partial X^i_s}{\partial x^j}(\varphi_{s,t}(x)) \cdot \frac{\partial \varphi^j_{s,t}}{\partial s}(x).\end{aligned}$$

Using $\dfrac{\partial \varphi_{s,t}}{\partial s} = Y_t \circ \varphi_{s,t}$, this becomes

$$\frac{\partial}{\partial s}\left(\frac{\partial \varphi^i_{s,t}}{\partial t}\right)(x) = \frac{\partial X^i_s}{\partial s}(\varphi_{s,t}(x)) + \sum_{j=1}^n \frac{\partial X^i_s}{\partial x^j}(\varphi_{s,t}(x)) \cdot Y^j_t(\varphi_{s,t}(x)).$$

Thus, we have

$$\frac{\partial}{\partial s}\left(\frac{\partial \varphi^i_{s,t}}{\partial t}\right) = \left(\frac{\partial X^i_s}{\partial s} + \sum_{j=1}^n Y^j_t \frac{\partial X^i_s}{\partial x^j}\right) \circ \varphi_{s,t}. \tag{23–1}$$

Similar calculations for the second equation yield

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi^i_{s,t}}{\partial s}\right) = \left(\frac{\partial Y^i_t}{\partial t} + \sum_{j=1}^n X^j_s \frac{\partial Y^i_t}{\partial x^j}\right) \circ \varphi_{s,t}. \tag{23–2}$$

Since $\varphi_{s,t}$ is smooth in both s and t, the mixed partial derivatives commute. Thus, the left-hand side of (23–1) equals the left-hand side of (23–2). And since $\varphi_{s,t}$ is a diffeomorphism, this gives

$$\frac{\partial X^i_s}{\partial s} + \sum_{j=1}^n Y^j_t \frac{\partial X^i_s}{\partial x^j} = \frac{\partial Y^i_t}{\partial t} + \sum_{j=1}^n X^j_s \frac{\partial Y^i_t}{\partial x^j}.$$

Therefore, we have

$$\frac{\partial X^i_s}{\partial s} - \frac{\partial Y^i_t}{\partial t} = \sum_{j=1}^n X^j_s \frac{\partial Y^i_t}{\partial x^j} - \sum_{j=1}^n Y^j_t \frac{\partial X^i_s}{\partial x^j} = \mathrm{D}_{X_s} Y^i_t - \mathrm{D}_{Y_t} X^i_s.$$

By Exercise 12, this implies

$$\frac{\partial X_s}{\partial s} - \frac{\partial Y_t}{\partial t} = [X_s, Y_t].$$

□

Exercise 24 Prove that, for vector fields X, Y (on a manifold M), the Lie derivative satisfies $\mathcal{L}_X Y = [X,Y]$.

Proof (Proof 1) We begin by showing

$$(\mathcal{L}_X \omega)(Y) = X(\omega(Y)) - \omega(\mathcal{L}_X Y) \tag{24–1}$$

for $\omega \in \Omega^1(M)$ and $X, Y \in \Gamma(TM)$. For any $p \in M$,

$$
\begin{aligned}
(\mathcal{L}_X \omega)(Y)_p &= \lim_{t \to 0} \frac{\left(\left(\varphi_t^X\right)^* \omega\right)_p (Y_p) - \omega_p(Y_p)}{t} \\
&= \lim_{t \to 0} \frac{\omega_{\varphi_t^X(p)}\left(\left(\mathrm{d}\varphi_t^X\right)_p (Y_p)\right) - \omega_p(Y_p)}{t} \\
&= \lim_{t \to 0} \frac{\omega_{\varphi_t^X(p)}\left(Y_{\varphi_t^X(p)}\right) - \omega_p(Y_p)}{t} + \lim_{t \to 0} \frac{\omega_{\varphi_t^X(p)}\left(\left(\mathrm{d}\varphi_t^X\right)_p (Y_p) - Y_{\varphi_t^X(p)}\right)}{t} \\
&= \lim_{t \to 0} \frac{\omega(Y)_{\varphi_t^X(p)} - \omega(Y)_p}{t} + \lim_{t \to 0} \frac{\left(\left(\varphi_t^X\right)^* \omega\right)_p \left(Y_p - \left(\mathrm{d}\varphi_{-t}^X\right)_{\varphi_t^X(p)}\left(Y_{\varphi_t^X(p)}\right)\right)}{t} \\
&= X(\omega(Y))_p + \lim_{t \to 0} \frac{\left(\left(\varphi_t^X\right)^* \omega\right)_p (-t(\mathcal{L}_X Y)_p + o(t))}{t} \\
&= X(\omega(Y))_p - \lim_{t \to 0} \left(\left(\varphi_t^X\right)^* \omega\right)_p ((\mathcal{L}_X Y)_p) \\
&= X(\omega(Y))_p - \omega_p((\mathcal{L}_X Y)_p).
\end{aligned}
$$

Thus (24–1) holds. Actually, this is a special case of (25–1). Using (24–1) and Cartan's magic formula, we get

$$
\begin{aligned}
\omega(\mathcal{L}_X Y) &= X(\omega(Y)) - (\mathcal{L}_X \omega)(Y) \\
&= X(\omega(Y)) - (\iota_X \,\mathrm{d}\omega)(Y) - (\mathrm{d}\iota_X \omega)(Y) \\
&= X(\omega(Y)) - \mathrm{d}\omega(X, Y) - \mathrm{d}(\omega(X))(Y) \\
&= X(\omega(Y)) - Y(\omega(X)) - \mathrm{d}\omega(X, Y) \\
&= \omega([X, Y]).
\end{aligned}
$$

The last equality follows from the definition of $\mathrm{d}\omega$. Since $\omega \in \Omega^1(M)$ is arbitrary, $\mathcal{L}_X Y = [X, Y]$.

(Proof 2) For any smooth function f defined near $p \in M$, we have

$$
\begin{aligned}
\left(\mathrm{d}\varphi_{-t}^X\right)_{\varphi_t^X(p)} Y_{\varphi_t^X(p)} f &= Y_{\varphi_t^X(p)}\left(f \circ \varphi_{-t}^X\right) = Y\left(f \circ \varphi_{-t}^X\right)\left(\varphi_t^X(p)\right) = \left(\varphi_t^X\right)^* Y\left(f \circ \varphi_{-t}^X\right) \\
&= \left(\varphi_t^X\right)^* Y \left(\varphi_{-t}^X\right)^* (f).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{L}_X Y f &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{d}\varphi_{-t}^X\right)_{\varphi_t^X(p)} Y_{\varphi_t^X(p)} f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\varphi_t^X\right)^* Y \left(\varphi_{-t}^X\right)^* (f) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\varphi_t^X\right)^* Y f + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Y \left(\varphi_{-t}^X\right)^* f \\
&= XYf - YXf = [X, Y] f.
\end{aligned}
$$

□

Exercise 25 Recall that given a non-degenerate 2-form ω on M, any function $H : M \to \mathbb{R}$ corresponds to a vector field X_H defined by $-\mathrm{d}H = \omega(X_H, \cdot)$. For two functions $H, G : M \to \mathbb{R}$, define

$$
\{H, G\} := \omega(X_H, X_G).
$$

Then prove that if ω is closed, i.e., $\mathrm{d}\omega = 0$, then $\{\cdot, \cdot\}$ satisfies the Jacobi identity:

$$\{\{H, G\}, F\} + \{\{G, F\}, H\} + \{\{F, H\}, G\} = 0$$

for any functions $H, G, F : M \to \mathbb{R}$.

Proof We shall apply a formula expressing the Lie derivative in terms of Lie brackets and ordinary directional derivatives of functions:

(GTM 218, Corollary 12.33) *If V is a smooth vector field and A is a smooth covariant k-tensor field, then for any smooth vector fields $X_1, \cdots, X_k$,*

$$\begin{aligned}(\mathcal{L}_V A)(X_1, \cdots, X_k) =& V(A(X_1, \cdots, X_k)) - A([V, X_1], X_2, \cdots, X_k) \\ &- \cdots - A(X_1, \cdots, X_{k-1}, [V, X_k]).\end{aligned} \tag{25–1}$$

(Proof 1) To start with, we observe that

- ⋄ $\{H, G\}$ is linear over $\mathbb{R}$ in both F and G.
- ⋄ $\{H, G\} = -\{G, H\}$.

These are obvious from the characterization $\{H, G\} = \omega(X_H, X_G)$ together with the fact that X_H depends linearly on H. Let us first prove that

$$X_{\{H,G\}} = [X_H, X_G]. \tag{25–2}$$

Because of the non-degeneracy of ω, to prove (25–2), it suffices to show that

$$\omega\big(X_{\{H,G\}}, Y\big) = \omega([X_H, X_G], Y) \tag{25–3}$$

holds for any vector field Y. On the one hand, note that

$$\omega\big(X_{\{H,G\}}, Y\big) = -\,\mathrm{d}(\{H, G\})(Y) = -Y\{H, G\} = -Y\omega(X_H, X_G) = Y\,\mathrm{d}H(X_G) = YX_GH.$$

On the other hand, by Cartan's magic formula,

$$\mathcal{L}_{X_G}\omega = \mathrm{d}\iota_{X_G}\omega + \iota_{X_G}\,\mathrm{d}\omega = \mathrm{d}(\omega(X_G, \cdot)) = -\,\mathrm{d}(\mathrm{d}G) = 0,$$

and then (25–1) yields

$$\begin{aligned}0 &= (\mathcal{L}_{X_G})\omega(X_H, Y) \\ &= X_G(\omega(X_H, Y)) - \omega([X_G, X_H], Y) - \omega(X_H, [X_G, Y]).\end{aligned} \tag{25–4}$$

The first and third terms on the right-hand side can be simplified as

$$X_G(\omega(X_H, Y)) = X_G(-\,\mathrm{d}H(Y)) = -X_GYH,$$

and

$$\begin{aligned}\omega(X_H, [X_G, Y]) &= -\,\mathrm{d}H([X_G, Y]) = -[X_G, Y]H = -X_GYH + YX_GH \\ &= -X_GYH + \omega\big(X_{\{H,G\}}, Y\big).\end{aligned}$$

Inserting these into (25–4), we obtain (25–3). Finally, by (25–2), we have

$$\begin{aligned}\{H,\{G,F\}\} &= -X_{\{G,F\}}H = -[X_G, X_F]H = -X_G X_F H + X_F X_G H \\ &= X_G\{H,F\} - X_F\{H,G\} = -\{\{H,F\},G\} + \{\{H,G\},F\} \\ &= \{\{H,G\},F\} + \{\{F,H\},G\}.\end{aligned}$$

This is the desired Jacobi identity.

(Proof 2) By (25–1), we have

$$\begin{aligned}\{\{H,G\},F\} &= \omega(X_{\{H,G\}}, X_F) = -\mathrm{d}\{H,G\}(X_F) = -X_F(\{H,G\}) = -X_F(\omega(X_H, X_G)) \\ &= -(\mathcal{L}_{X_F}\omega)(X_H, X_G) + \omega(\mathcal{L}_{X_F}X_H, X_G) + \omega(X_H, \mathcal{L}_{X_F}X_G) \\ &= \omega([X_F, X_H], X_G) + \omega(X_H, [X_F, X_G]).\end{aligned}$$

Likewise, we have

$$\begin{aligned}\{\{G,F\},H\} &= \omega([X_H, X_G], X_F) + \omega(X_G, [X_H, X_F]) \\ &= \omega([X_H, X_G], X_F) + \omega([X_F, X_H], X_G)\end{aligned}$$

and

$$\begin{aligned}\{\{F,H\},G\} &= \omega([X_G, X_F], X_H) + \omega(X_F, [X_G, X_H]) \\ &= \omega(X_H, [X_F, X_G]) + \omega([X_H, X_G], X_F).\end{aligned}$$

Hence

$$\begin{aligned}&\{\{H,G\},F\} + \{\{G,F\},H\} + \{\{F,H\},G\} \\ =&2\omega([X_F, X_H], X_G) + 2\omega(X_H, [X_F, X_G]) + 2\omega([X_H, X_G], X_F) \\ =&-2[X_F, X_H]G + 2[X_F, X_G]H - 2[X_H, X_G]F \\ =&-2X_F X_H G + 2X_H X_F G + 2X_F X_G H - 2X_G X_F H - 2X_H X_G F + 2X_G X_H F \\ =&-2X_F X_H G - 2X_H X_G F - 2X_F X_H G - 2X_G X_F H - 2X_H X_G F - 2X_G X_F H \\ =&-4(X_H X_G F + X_G X_F H + X_F X_H G).\end{aligned} \tag{25–5}$$

Since $\mathrm{d}\omega = 0$, we have

$$\begin{aligned}0 =& \mathrm{d}\omega(X_H, X_G, X_F) \\ =&X_H(\omega(X_G, X_F)) - X_G(\omega(X_H, X_F)) + X_F(\omega(X_H, X_G)) \\ &- \omega([X_H, X_G], X_F) + \omega([X_H, X_F], X_G) - \omega([X_G, X_F], X_H) \\ =&X_H(-X_F G) - X_G(-X_F H) + X_F(-X_G H) \\ &- (-[X_H, X_G]F) + (-[X_H, X_F]G) - (-[X_G, X_F]H) \\ =&-X_H X_F G + X_G X_F H - X_F X_G H + [X_H, X_G]F - [X_H, X_F]G + [X_G, X_F]H \\ =&-2X_H X_F G + 2X_G X_F H - 2X_F X_G H + X_H X_G F - X_G X_H F + X_F X_H G \\ =&2X_H X_G F + 2X_G X_F H + 2X_F X_H G + X_H X_G F + X_G X_F H + X_F X_H G \\ =&3X_H X_G F + 3X_G X_F H + 3X_F X_H G.\end{aligned}$$

Therefore, we get

$$X_H X_G F + X_G X_F H + X_F X_H G = 0. \tag{25–6}$$

Applying (25–6) to (25–5), we obtain the Jacobi identity. $\square$

Exercise 26 Consider manifold $\mathbb{R}^2_{>0}$ and $\varphi : \mathbb{R}^2_{>0} \to \mathbb{R}^2_{>0}$ defined by

$$\varphi(x, y) = \left(xy, \frac{y}{x}\right).$$

Compute the pushforward $\varphi_* X$ for a vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Do the same thing for vector field $Y = y\frac{\partial}{\partial x}$.

Solution The differential of φ at a point $(x, y) \in \mathbb{R}^2_{>0}$ is represented by its Jacobi matrix,

$$\mathrm{Jac}(\varphi)((x, y)) = \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}.$$

Hence we have

$$(\varphi_* X)_{(u,v)} = \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2xy\frac{\partial}{\partial u} = 2u\frac{\partial}{\partial u}$$

and

$$(\varphi_* Y)_{(u,v)} = \begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix} = y^2\frac{\partial}{\partial u} - \frac{y^2}{x^2}\frac{\partial}{\partial v} = uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}.$$

$\square$

Exercise 27 Consider 1-form $\alpha = x\,\mathrm{d}y$ on $\mathbb{R}^2$ and map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\varphi(x, y) = \left(xy, \mathrm{e}^{-y}\right).$$

Compute the pullback $\varphi^*\alpha$. Also, verify in this concrete case that $\varphi^*(\mathrm{d}\alpha) = \mathrm{d}(\varphi^*\alpha)$.

Solution The pullback of α is given by

$$\varphi^*\alpha = (xy)\,\mathrm{d}\left(\mathrm{e}^{-y}\right) = -xy\mathrm{e}^{-y}\,\mathrm{d}y.$$

Then

$$\mathrm{d}(\varphi^*\alpha) = \mathrm{d}\left(-xy\mathrm{e}^{-y}\,\mathrm{d}y\right) = -y\mathrm{e}^{-y}\,\mathrm{d}x \wedge \mathrm{d}y.$$

On the other hand,

$$\mathrm{d}\alpha = \mathrm{d}(x\,\mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}y,$$

so

$$\varphi^*(\mathrm{d}\alpha) = \varphi^*(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}(xy) \wedge \mathrm{d}\left(\mathrm{e}^{-y}\right) = (y\,\mathrm{d}x + x\,\mathrm{d}y) \wedge \left(-\mathrm{e}^{-y}\,\mathrm{d}y\right) = -y\mathrm{e}^{-y}\,\mathrm{d}x \wedge \mathrm{d}y.$$

Therefore $\varphi^*(\mathrm{d}\alpha) = \mathrm{d}(\varphi^*\alpha)$ in this example. $\square$

Exercise 28 Let X be a smooth vector field on M^n such that $X(p) \neq 0$ at some point $p \in M$.

(1) Prove that there exists a local chart $(U, \varphi : U \to V)$ near p, where V is an open subset of $\mathbb{R}^n$ in coordinates $(x_1, \cdots, x_n)$, such that within U, we have $\varphi_*(X) = \frac{\partial}{\partial x_1}$.

(2) Given the following three vector fields on $\mathbb{R}^3$,

$$X_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad X_2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad X_3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

Near $p = (1,0,0)$, is it possible to find a local chart as above such that X_i maps to $\frac{\partial}{\partial x_i}$ for $i = 1,2,3$ at the same time? If so, construct such a local chart; if not, please give a justifying reason.

Proof (1) Choose a local chart $\left(\widetilde{U}, y^1, \cdots, y^n\right)$ about p such that $X_p = \left.\frac{\partial}{\partial y^1}\right|_p$. Denote $X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial y^i}$ on $\widetilde{U}$, where ξ_i are smooth functions on $\widetilde{U}$. Shrinking $\widetilde{U}$ if necessary, we may assume $\xi_1 \neq 0$ on $\widetilde{U}$. Consider the system of ODEs

$$\frac{\mathrm{d}y^i}{\mathrm{d}y^1} = \frac{\xi_i\left(y^1, y^2, \cdots, y^n\right)}{\xi_1\left(y^1, y^2, \cdots, y^n\right)}, \quad 2 \leqslant i \leqslant n. \tag{28–1}$$

By basic theory of ODE, locally for any given initial data $\left(z^2, \cdots, z^n\right)$, with $|z| < \varepsilon$, the system above has a unique solution

$$y^i = y^i\left(y^1, z^2, \cdots, z^n\right), \quad \left|y^1\right| < \varepsilon$$

with initial condition

$$y^i\left(0, z^2, \cdots, z^n\right) = z^i, \quad 2 \leqslant i \leqslant n$$

and the functions y^i depend smoothly on y^1 and on z^j. Consider the coordinate transformation

$$y^1 = z^1,$$
$$y^i = y^i\left(z^1, z^2, \cdots, z^n\right), \quad 2 \leqslant i \leqslant n.$$

Since the Jacobian

$$\left.\frac{\partial\left(y^1, \cdots, y^n\right)}{\partial\left(z^1, \cdots, z^n\right)}\right|_{z^1=0} = 1,$$

we can make the change of variables from $\left(y^1, \cdots, y^n\right)$ to $\left(z^1, \cdots, z^n\right)$, i.e., there exists a neighborhood $U \subset \widetilde{U}$ of p, with $\left(z^1, \cdots, z^n\right)$ as local coordinate functions. By (28–1), in this new chart

$$X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial y^i} = \xi_1 \sum_{i=1}^n \frac{\partial y^i}{\partial z^1}\frac{\partial}{\partial y^i} = \xi_1 \frac{\partial}{\partial z^1}.$$

Finally if we let $x^1\left(z^1, \cdots, z^n\right) = \int_0^{z_1} \frac{\mathrm{d}t}{\xi_1\left(t, z^2, \cdots, z^n\right)}$ and $x_j = z_j$ for $j \geqslant 2$, then $\left\{x^1, \cdots, x^n\right\}$ are local coordinate functions on U such that $X = \frac{\partial}{\partial x^1}$ on U.

(2) Suppose there exists a local chart $(U, \varphi: U \to V)$ near $p = (1,0,0)$, where V is an open subset of $\mathbb{R}^3$ in coordinates (u, v, w), such that with U, we have

$$\varphi_*(X_1) = \frac{\partial}{\partial u}, \quad \varphi_*(X_2) = \frac{\partial}{\partial v}, \quad \varphi_*(X_3) = \frac{\partial}{\partial w}.$$

Consider the coordinate transformation

$$\begin{cases} x = x(u,v,w), \\ y = y(u,v,w), \\ z = z(u,v,w) \end{cases} \quad \text{with inverse} \quad \begin{cases} u = u(x,y,z), \\ v = v(x,y,z), \\ w = w(x,y,z). \end{cases}$$

Then

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial u}{\partial x}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial x}\dfrac{\partial}{\partial v} + \dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial w}, \\ \dfrac{\partial}{\partial y} = \dfrac{\partial u}{\partial y}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial w}, \\ \dfrac{\partial}{\partial z} = \dfrac{\partial u}{\partial z}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial z}\dfrac{\partial}{\partial v} + \dfrac{\partial w}{\partial z}\dfrac{\partial}{\partial w}. \end{cases}$$

In the new basis $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\}$, the vector fields X_1, X_2, X_3 are represented by

$$\begin{aligned} X_1 &= \left(x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}, x\frac{\partial v}{\partial y} - y\frac{\partial v}{\partial x}, x\frac{\partial w}{\partial y} - y\frac{\partial w}{\partial x}\right) = (1,0,0), \\ X_2 &= \left(y\frac{\partial u}{\partial z} - z\frac{\partial u}{\partial y}, y\frac{\partial v}{\partial z} - z\frac{\partial v}{\partial y}, y\frac{\partial w}{\partial z} - z\frac{\partial w}{\partial y}\right) = (0,1,0), \\ X_3 &= \left(z\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial z}, z\frac{\partial v}{\partial x} - x\frac{\partial v}{\partial z}, z\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial z}\right) = (0,0,1). \end{aligned}$$

However, at the point $p = (x,y,z) = (1,0,0)$, the second component of X_2 in the new basis is 0, contradicting the second equation above. Therefore, it is impossible to find a local chart such that X_i maps to $\frac{\partial}{\partial x_i}$ for $i = 1,2,3$ at the same time.

$\star$ An alternative way is to note that

$$zX_1 + xX_2 = y\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) = -yX_3.$$

Hence $\{X_1, X_2, X_3\}$ are linearly dependent near the point $(1,0,0)$. □

Exercise 29 Consider $\mathbb{R}^3$ equipped with the metric $g = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y - \mathrm{d}z \otimes \mathrm{d}z$. A *Killing vector field* on $(\mathbb{R}^3, g)$ is a complete non-trivial vector field X such that $\mathcal{L}_X g = 0$. In other words, by the definition of a Lie derivative, the flow generated by X preserves the metric g.

(1) List as many linearly independent Killing vector fields in $(\mathbb{R}^3, g)$ as possible.

(2) Verify that if X, Y are two Killing vector fields in $(\mathbb{R}^3, g)$, then $[X,Y]$ is also a Killing vector field in $(\mathbb{R}^3, g)$.

Proof (1) Let D be the Euclidean connection on $\mathbb{R}^3$, i.e.,

$$\mathrm{D}_X Y = X(Y^1)\frac{\partial}{\partial x^1} + X(Y^2)\frac{\partial}{\partial x^2} + X(Y^3)\frac{\partial}{\partial x^3}$$

for any smooth vector fields X, Y on $\mathbb{R}^3$. Suppose X is a Killing vector field in $(\mathbb{R}^3, g)$. By (25–1),

$$0 = (\mathcal{L}_X g)(Y,Z) = Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]). \tag{29–1}$$

Note that

$$\begin{aligned} Xg(Y,Z) &= Xg\left(Y^i\frac{\partial}{\partial x^i}, Z^j\frac{\partial}{\partial x^j}\right) = g_{ij}X\left(Y^iZ^j\right) = g_{ij}\left[X\left(Y^i\right)Z^j + Y^iX\left(Z^j\right)\right] \\ &= g(\mathrm{D}_X Y, Z) + g(Y, \mathrm{D}_X Z). \end{aligned}$$

Hence

$$\begin{aligned} 0 &= g(\mathrm{D}_X Y, Z) + g(Y, \mathrm{D}_X Z) - g(\mathrm{D}_X Y - \mathrm{D}_Y X, Z) - g(Y, \mathrm{D}_X Z - \mathrm{D}_Z X) \\ &= g(\mathrm{D}_Y X, Z) + g(Y, \mathrm{D}_Z X). \end{aligned}$$

This is equivalent to having

$$\begin{aligned} 0 &= g\left(\mathrm{D}_{\frac{\partial}{\partial x^i}} X, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \mathrm{D}_{\frac{\partial}{\partial x^j}} X\right) \\ &= g\left(\frac{\partial X^k}{\partial x^i}\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \frac{\partial X^k}{\partial x^j}\frac{\partial}{\partial x^k}\right) \\ &= g_{kj}\frac{\partial X^k}{\partial x^i} + g_{ik}\frac{\partial X^k}{\partial x^j} \end{aligned}$$

for all i, j. Let $G = [g_{ij}]$ and $A = [a_{ij}] = \left[\frac{\partial X^j}{\partial x^i}\right]$. Then $a_{ik}g_{kj} + g_{ik}a_{jk} = 0$, or equivalently,

$$0 = AG + GA^{\mathsf{T}} = AG + (AG)^{\mathsf{T}}.$$

Therefore the matrix AG is skew-symmetric. In this concrete case, we have

$$AG = \begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \frac{\partial X^2}{\partial x^1} & \frac{\partial X^3}{\partial x^1} \\ \frac{\partial X^1}{\partial x^2} & \frac{\partial X^2}{\partial x^2} & \frac{\partial X^3}{\partial x^2} \\ \frac{\partial X^1}{\partial x^3} & \frac{\partial X^2}{\partial x^3} & \frac{\partial X^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \frac{\partial X^2}{\partial x^1} & -\frac{\partial X^3}{\partial x^1} \\ \frac{\partial X^1}{\partial x^2} & \frac{\partial X^2}{\partial x^2} & -\frac{\partial X^3}{\partial x^2} \\ \frac{\partial X^1}{\partial x^3} & \frac{\partial X^2}{\partial x^3} & -\frac{\partial X^3}{\partial x^3} \end{pmatrix}.$$

So the skew-symmetry of AG requires that

$$\begin{cases} \frac{\partial X^1}{\partial x^1} = \frac{\partial X^2}{\partial x^2} = \frac{\partial X^3}{\partial x^3} = 0, \\ \frac{\partial X^1}{\partial x^2} + \frac{\partial X^2}{\partial x^1} = 0, \\ \frac{\partial X^1}{\partial x^3} = \frac{\partial X^3}{\partial x^1}, \\ \frac{\partial X^2}{\partial x^3} = \frac{\partial X^3}{\partial x^2}. \end{cases}$$

Thus we may set $X^1 = f\left(x^2, x^3\right)$, $X^2 = h\left(x^1, x^3\right)$, and $X^3 = k\left(x^1, x^2\right)$ for some smooth functions

f, h, k. The above equations give

$$\begin{cases} \dfrac{\partial f}{\partial x^2} + \dfrac{\partial h}{\partial x^1} = 0, \\ \dfrac{\partial f}{\partial x^3} = \dfrac{\partial k}{\partial x^1}, \\ \dfrac{\partial h}{\partial x^3} = \dfrac{\partial k}{\partial x^2}. \end{cases}$$

The first equation implies that $\dfrac{\partial^2 f}{(\partial x^2)^2} = \dfrac{\partial^2 h}{(\partial x^1)2} = 0$. Similarly, the second and third equations imply that $\dfrac{\partial^2 f}{(\partial x^3)^2} = \dfrac{\partial^2 k}{(\partial x^1)^2} = 0$ and $\dfrac{\partial^2 h}{(\partial x^3)^2} = \dfrac{\partial^2 k}{(\partial x^2)^2} = 0$. Therefore f, h, k are of the form

$$\begin{cases} f = ax^2 + bx^3 + d_1, \\ h = -ax^1 + cx^3 + d_2, \\ k = bx^1 + cx^2 + d_3, \end{cases} \qquad a, b, c, d_1, d_2, d_3 \in \mathbb{R}.$$

Hence all Killing vector fields in $(\mathbb{R}^3, g)$ are of the form

$$X\big((x^1, x^2, x^3)\big) = a\big(x^2, -x^1, 0\big) + b\big(x^3, 0, x^1\big) + c\big(0, x^3, x^2\big) + (d_1, d_2, d_3),$$

where $a, b, c, d_1, d_2, d_3 \in \mathbb{R}$.

(2) Suppose X, Y are two Killing vector fields in $(\mathbb{R}^3, g)$. The same deduction as in (29–1) gives

$$Xg(Z, W) = g([X, Z], W) + g(Z, [X, W]),$$
$$Yg(Z, W) = g([Y, Z], W) + g(Z, [Y, W]).$$

With these and the Jacobi identity, we have

$$\begin{aligned} [X, Y]g(Z, W) =& XYg(Z, W) - YXg(Z, W) \\ =& Xg([Y, Z], W) + Xg(Z, [Y, W]) - Yg([X, Z], W) - Yg(Z, [X, W]) \\ =& g([X, [Y, Z]], W) + g([Y, Z], [X, W]) + g([X, Z], [Y, W]) + g(Z, [X, [Y, W]]) \\ & - g([Y, [X, Z]], W) - g([X, Z], [Y, W]) - g([Y, Z], [X, W]) - g(Z, [Y, [X, W]]) \\ =& g([X, [Y, Z]] - [Y, [X, Z]], W) + g(Z, [X, [Y, W]] - [Y, [X, W]]) \\ =& g([[X, Y], Z], W) + g(Z, [[X, Y], W]). \end{aligned}$$

Again, applying (25–1) we find

$$\big(\mathcal{L}_{[X,Y]}g\big)(Z, W) = [X, Y]g(Z, W) - g([[X, Y], Z], W) - g(Z, [[X, Y], W]) = 0,$$

so $[X, Y]$ is also a Killing vector field in $(\mathbb{R}^3, g)$.

$\star$ In this concrete case, by (1), one can also take $X = \big(x^2, -x^1, 0\big)$, $Y = \big(x^3, 0, x^1\big)$, $Z = \big(0, x^3, x^2\big)$, and compute

$$[X, Y] = Z, \quad [X, Z] = -Y, \quad [Y, Z] = -X.$$

They are again Killing vector fields in $(\mathbb{R}^3, g)$. □

Exercise 30 Let α be a 1-form on M^3 satisfying $\alpha \wedge \mathrm{d}\alpha$ is a nowhere vanishing 3-form on M^3.

(1) Prove that there exists a vector field (called a *Reeb vector field*) denoted by R_α such that $\mathrm{d}\alpha(R_\alpha, -) = 0$ and $\alpha(R_\alpha) = 1$.

(2) Confirm that $\mathcal{L}_{R_\alpha}\alpha = 0$.

(3) In $\mathbb{R}^3$ in coordinates (x, y, z), give an example of such α and work out the associated R_α.

Proof (1) We first show that that *every smooth manifold admits a Riemannian metric*. Let M be a smooth manifold and $\{(U_\beta, \varphi_\beta) : \beta \in \Lambda\}$ a locally finite atlas so that $U_\beta \subset M$ and $\varphi_\beta : U_\beta \to \varphi_\beta(U_\beta) \subset \mathbb{R}^n$ are diffeomorphisms. Let $\{\rho_\beta : \beta \in \Lambda\}$ be a differentiable partition of unity subordinate to the given atlas, i.e. such that $\mathrm{supp}(\rho_\beta) \subset U_\beta$ for all $\beta \in \Lambda$. Define a Riemannian metric g on M by $g = \sum_{\beta\in\Lambda} \rho_\beta \tilde{g}_\beta$, where $\tilde{g}_\beta = \varphi_\beta^* g^{\mathrm{can}}$. Here g^{can} is the Euclidean metric on $\mathbb{R}^n$ and $\varphi_\beta^* g^{\mathrm{can}}$ is its pullback along φ_β. It is straightforward to check that g is a Riemannian metric.

Let $\{(U_\beta, \varphi_\beta)\}$ be an atlas on M such that $\varphi_\beta : U_\beta \to \mathbb{R}^3$ are diffeomorphisms, and g a Riemannian metric on M. Define $A : \Gamma(TM) \to \Gamma(TM)$ by

$$g(AX, Y) = \mathrm{d}\alpha(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

This is well-defined for g is non-degenerate. Sicne $g(AX, Y) = g(X, -AY)$, i.e., A is skew-symmetric, 0 is an eigenvalue of A at each point. Note that $\mathrm{tr}(A) = 0$, so the other two eigenvalues must be both zero or both non-zero. Recall that a real skew symmetric matrix is always diagonalizable over $\mathbb{C}$. If A has all eigenvalues zero, then $A = 0$, contradicting the assumption that $\mathrm{d}\alpha$ is nowhere vanishing. Therefore the eigenspace of A corresponding to the eigenvalue 0 is one-dimensional.

We can show that $\alpha(R) \neq 0$ for all eigenvectors R of A corresponding to the eigenvalue 0. Indeed, if $\alpha(R) = 0$, then

$$\iota_R(\alpha \wedge \mathrm{d}\alpha) = (\iota_R\alpha) \wedge \mathrm{d}\alpha - \alpha \wedge (\iota_R\, \mathrm{d}\alpha) = \alpha(R)\, \mathrm{d}\alpha - \alpha \wedge 0 = 0.$$

This is a contradiction to the assumption that $\alpha \wedge \mathrm{d}\alpha$ is nowhere vanishing.

By the above arguments, on each U_β, we can find eigenvector $R_\beta \in \Gamma(TU_\beta)$ of $A|_{U_\beta}$ corresponding to the eigenvalue 0 so that

$$\mathrm{d}\alpha|_{U_\beta}(R_\beta, -) = g(A|_{U_\beta} R_\beta, -) = g(0, -) = 0 \quad \text{and} \quad \alpha|_{U_\beta}(R_\beta) = 1.$$

Now we define $R_\alpha \in \Gamma(TM)$ by $R_\alpha|_{U_\beta} = R_\beta$. It is well-defined for if $U_\beta \cap U_\gamma \neq \varnothing$, then $R_\gamma = \lambda R_\beta$ for some $\lambda \in \mathbb{R}$. Then

$$1 = \alpha|_{U_\beta \cap U_\gamma}(R_\gamma) = \alpha|_{U_\beta \cap U_\gamma}(\lambda R_\beta) = \lambda \cdot \alpha|_{U_\beta \cap U_\gamma}(R_\beta) = \lambda,$$

showing $R_\gamma = R_\beta$. Therefore R_α is the desired Reeb vector field.

(2) By Cartan's magic formula,

$$\mathcal{L}_{R_\alpha}\alpha = \mathrm{d}(\iota_{R_\alpha}(\alpha)) + \iota_{R_\alpha}(\mathrm{d}\alpha) = \mathrm{d}(\alpha(R_\alpha)) + \mathrm{d}\alpha(R_\alpha, -) = \mathrm{d}(1) + 0 = 0.$$

(3) Take $\alpha = \mathrm{d}z - y\,\mathrm{d}x$, then $\mathrm{d}\alpha = \mathrm{d}x \wedge \mathrm{d}y$ and

$$\alpha \wedge \mathrm{d}\alpha = (\mathrm{d}z - y\,\mathrm{d}x) \wedge (\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

is nowhere vanishing on $\mathbb{R}^3$. The corresponding Reeb vector field is given by $R_\alpha = \dfrac{\partial}{\partial z}$ since

$$\mathrm{d}\alpha(R_\alpha, -) = (\mathrm{d}x \wedge \mathrm{d}y)\left(\frac{\partial}{\partial z}, -\right) = 0 \quad \text{and} \quad \alpha(R_\alpha) = (\mathrm{d}z - y\,\mathrm{d}x)\left(\frac{\partial}{\partial z}\right) = 1.$$

□

Homework 4

Exercise 31 Consider the unit open disk $\mathbb{B}^2$ in $\mathbb{R}^2$ defined by $\mathbb{B}^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ equipped with the following Riemannian metric

$$g((x,y)) = \frac{4}{[1-(x^2+y^2)]^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y).$$

Meanwhile, consider the open upper half plane $\mathbb{H}^2$ of $\mathbb{R}^2$, that is, $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ equipped with the following Riemannian metric

$$g'((x,y)) = \frac{1}{y^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y).$$

Prove that there exists a smooth diffeomorphism $F : \mathbb{B}^2 \to \mathbb{H}^2$ such that it preserves the metrics in the sense that for any vector fields $X, Y \in \Gamma(T\mathbb{B}^2)$, we have $g'(F_*(X), F_*(Y)) = g(X,Y)$.

Proof The Möbius transformation $z \mapsto \frac{z+\mathrm{i}}{1+\mathrm{i}z}$ is a biholomorphism from the unit disk to the upper half plane in $\mathbb{C}$. It induces the smooth diffeomorphism

$$F : \mathbb{B}^2 \to \mathbb{H}^2, \quad (x,y) \mapsto \left(\frac{2x}{x^2+(1-y)^2}, \frac{1-(x^2+y^2)}{x^2+(1-y)^2}\right).$$

The differential of F at a point $(x,y) \in \mathbb{B}^2$ is represented by its Jacobi matrix,

$$\mathrm{Jac}(F)((x,y)) = \begin{pmatrix} \dfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2} & \dfrac{4x(1-y)}{[x^2+(1-y)^2]^2} \\ \dfrac{-4x(1-y)}{[x^2+(1-y)^2]^2} & \dfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2} \end{pmatrix}.$$

For any $(x,y) \in \mathbb{B}^2$, suppose $X = X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y}$ and $Y = Y^1\frac{\partial}{\partial x} + Y^2\frac{\partial}{\partial y}$ at (x,y), then

$$g(X,Y)_{(x,y)} = \frac{4}{[1-(x^2+y^2)]^2}\left(X^1Y^1X^2Y^2\right),$$

Since

$$F_*(X((x,y))) = \text{Jac}(F)((x,y))\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} \dfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2}X^1 + \dfrac{4x(1-y)}{[x^2+(1-y)^2]^2}X^2 \\ \dfrac{-4x(1-y)}{[x^2+(1-y)^2]^2}X^1 + \dfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2}X^2 \end{pmatrix}$$

and

$$F_*(Y((x,y))) = \text{Jac}(F)((x,y))\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} \dfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2}Y^1 + \dfrac{4x(1-y)}{[x^2+(1-y)^2]^2}Y^2 \\ \dfrac{-4x(1-y)}{[x^2+(1-y)^2]^2}Y^1 + \dfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2}Y^2 \end{pmatrix},$$

we have

$$\begin{aligned}
& g'(F_*(X), F_*(Y))_{F((x,y))} \\
=& \frac{1}{\left[\frac{1-(x^2+y^2)}{x^2+(1-y)^2}\right]^2}\Bigg\{\left(\tfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2}X^1 + \tfrac{4x(1-y)}{[x^2+(1-y)^2]^2}X^2\right)\left(\tfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2}Y^1 + \tfrac{4x(1-y)}{[x^2+(1-y)^2]^2}Y^2\right) \\
&+\left(\tfrac{-4x(1-y)}{[x^2+(1-y)^2]^2}X^1 + \tfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2}X^2\right)\left(\tfrac{-4x(1-y)}{[x^2+(1-y)^2]^2}Y^1 + \tfrac{-2x^2+2(1-y)^2}{[x^2+(1-y)^2]^2}Y^2\right)\Bigg\} \\
=& \frac{4\Big\{\Big(\big[-x^2+(1-y)^2\big]^2 + [2x(1-y)]^2\Big)X^1Y^1 + \Big([2x(1-y)]^2 + \big[-x^2+(1-y)^2\big]X^2Y^2\Big)\Big\}}{[1-(x^2+y^2)]^2[x^2+(1-y)^2]^2} \\
=& \frac{4\big[x^2+(1-y)^2\big]^2\big(X^1Y^1+X^2Y^2\big)}{[1-(x^2+y^2)]^2[x^2+(1-y)^2]^2} \\
=& g(X,Y)_{(x,y)}.
\end{aligned}$$

□

Exercise 32 Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\Phi(x,y,s,t) = \big(x^2+y, x^2+y^2+s^2+t^2+y\big).$$

Show that $(0,1)$ is a regular value of Φ, and that the level set $\Phi^{-1}((0,1))$ is diffeomorphic to $\mathbb{S}^2$.

Proof The differential of Φ at $(x,y,s,t) \in \mathbb{R}^4$ is represented by its Jacobi matrix,

$$\text{Jac}(\Phi)((x,y,s,t)) = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s & 2t \end{pmatrix}.$$

The level set $\Phi^{-1}((0,1))$ is the set of points $(x,y,s,t) \in \mathbb{R}^4$ such that

$$x^2+y=0 \quad \text{and} \quad y^2+s^2+t^2=1. \tag{32–1}$$

Then for any $(x,y,s,t) \in \Phi^{-1}((0,1))$, at least one of the following subdeterminants is nonzero:

$$\begin{vmatrix} 2x & 1 \\ 2x & 2y+1 \end{vmatrix} = 4xy, \quad \begin{vmatrix} 1 & 0 \\ 2y+1 & 2s \end{vmatrix} = 2s, \quad \begin{vmatrix} 1 & 0 \\ 2y+1 & 2t \end{vmatrix} = 2t.$$

For example, if $s=t=0$, then (32–1) implies $y=-1$ and $x^2=1$, so $4xy \neq 0$. Hence rank(Jac(Φ)) = 2 at any point in $\Phi^{-1}((0,1))$, which means $(0,1)$ is a regular value of Φ. By the regular level set theorem, $\Phi^{-1}((0,1))$ is an embedded submanifold of $\mathbb{R}^4$ of dimension $4-2=2$. Consider the map

$$F:\Phi^{-1}((0,1))\to\mathbb{R}^3,\quad \big(x,-x^2,s,t\big)\mapsto(x,s,t).$$

Clearly, F is a diffeomorphism between $\Phi^{-1}((0,1))$ and its image $E:=\big\{(x,s,t)\in\mathbb{R}^3 : x^4+s^2+t^2=1\big\}$. Now consider the map

$$G:E\to\mathbb{S}^2,\quad (x,s,t)\mapsto\frac{1}{\sqrt{x^2+s^2+t^2}}(x,s,t).$$

Since E is an embedded submanifold of $\mathbb{R}^3$ and $\mathbb{S}^2$ is an immersed submanifold of $\mathbb{R}^3$, G is smooth. Likewise, the inverse of G given by

$$G^{-1}:\mathbb{S}^2\to E,\quad (u,v,w)\mapsto\left(\frac{u}{\sqrt[4]{u^4+v^2+w^2}},\frac{v}{\sqrt{u^4+v^2+w^2}},\frac{w}{\sqrt{u^4+v^2+w^2}}\right)$$

is smooth. Therefore G is a diffeomorphism between E and $\mathbb{S}^2$, and it follows that $G\circ F$ is a diffeomorphism from $\Phi^{-1}((0,1))$ to $\mathbb{S}^2$. □

Exercise 33 Let N be a nonempty smooth compact manifold. Show that there is no smooth submersion $F:N\to\mathbb{R}^k$ for any $k>0$.

Proof As a corollary of the constant rank theorem, any submersion is an open map. So if there is a smooth submersion $F:N\to\mathbb{R}^k$ for some $k>0$, then $F(N)$ is an open in $\mathbb{R}^k$. But $\mathbb{R}^k$ is Hausdorff and $F(N)$ is compact, so $F(N)$ is also closed in $\mathbb{R}^k$. Since $\mathbb{R}^k$ is connected, the only nonempty clopen set is $\mathbb{R}^k$ itself. Thus $F(N)=\mathbb{R}^k$, which is a contradiction since $\mathbb{R}^k$ is not compact. □

Exercise 34 Let $N\subset\mathbb{R}^m$ be a smooth submanifold of dimension $n\leqslant m-3$. Prove that the complement $\mathbb{R}^m\setminus N$ is connected and simply connected.

Proof We shall apply the "*Whitney Approximation Theorem*" and the "*Transversality Homotopy Theorem*":
(GTM 218, Theorem 6.26) *Suppose N is a smooth manifold with or without boundary, M is a smooth manifold (without boundary), and $f:N\to M$ is a continuous map. Then f is homotopic to a smooth map g. Moreover, if f is already smooth on a closed subset $A\subset N$, then g can be chosen so that $f|_A=g|_A$.*
(GTM 218, Theorem 6.36) *Suppose M and N are smooth manifolds and $Y\subset M$ is an embedded submanifold. Every smooth map $f:N\to M$ is homotopic to a smooth map $g:N\to M$ that is transverse to Y. Moreover, if X is an embedded submanifold of N and $f|_X$ is already transverse to Y, then g can be chosen so that $f|_X=g|_X$.*

To see that $\mathbb{R}^m\setminus N$ is path-connected, let $p,q\in\mathbb{R}^m\setminus N$ and let $\gamma(t)$ be a path in $\mathbb{R}^m$ with $\gamma(0)=p$ and $\gamma(1)=q$. By the Whitney approximation theorem, γ is homotopic to some smooth curve γ' joining p and q. Then by the transversality homotopy theorem, γ' is homotopic to some smooth map γ'' joining p and q that is transverse to N. However, since $\dim N+\dim\gamma''=n+1<\dim\mathbb{R}^m$, intersecting transversally means having empty intersection. So γ'' is a path from p to q which does not touch N, showing $\mathbb{R}^m\setminus N$ is path-connected.

To see that $\mathbb{R}^m\setminus N$ is simply connected, let $\gamma_1(t)$ and $\gamma_2(t)$ be two closed loops in $\mathbb{R}^m\setminus N$. Since $\mathbb{R}^m$ is simply connected, there is a homotopy $F(s,t)=\gamma_s(t)$ between γ_1 and γ_2. As before, we can perturb the surface $F(s,t)$ so that it intersects N transversally. However, since $\dim N+\dim F=n+2<\dim\mathbb{R}^m$, intersecting transversally means having empty intersection. So we have found a homotopy between γ_1 and γ_2 which does not touch N, showing $\mathbb{R}^m\setminus N$ is simply connected. □

Exercise 35 Let $F : M \to M$ be a smooth map. A fixed point $p \in F$ (i.e., $F(p) = p$) is called non-degenerate if 1 is *not* an eigenvalue of the pushforward $F_*(p) : T_pM \to T_pM$. The map F is called a *Lefschetz map* if all its fixed points are non-degenerate.

(1) Prove that the "horizontal" rotation $r_\theta : \mathbb{S}^2 \to \mathbb{S}^2$ by angle θ ($\neq 2k\pi$ for any $k \in \mathbb{N}$) defined by

$$r_\theta(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$$

is a Lefschetz map, where $\mathbb{S}^2$ here is viewed as a submanifold in $\mathbb{R}^3$ defined by

$$\big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\big\}.$$

(2) Let V be a vector space and $F : V \to V$ a linear map. Let $\Delta = \{(v, v) \in V \times V : v \in V\}$ be the diagonal of $V \times V$ and $\Gamma_F = \{(v, F(v)) \in V \times V : v \in V\}$ be the graph of F on V. Then deduce that if M is a compact manifold and $F : M \to M$ is a Lefschetz map, then there are only finitely many fixed points of F.

(3) When M is a compact manifold and $F : M \to M$ is a Lefschetz map, let

$$L(F) := \sum_{\text{fixed point } p \text{ of } F} \operatorname{sign}(\det(F_*(p) - \mathbb{1})).$$

Here, sign means that if $\det(F_*(p) - \mathbb{1}) > 0$, then sign $= +1$ and if $\det(F_*(p) - \mathbb{1}) < 0$, then sign $= -1$. This $L(F)$ is a well-defined number and is called the *Lefschetz number* of Lefschetz map F. Compute $L(r_\theta)$ in Question (1) above.

Proof (1) Since $\theta \neq 2k\pi$, $(0, 0, \pm 1)$ are the only two fixed points of r_θ. For the north pole $(0, 0, 1)$, take the coordinate chart

$$\varphi : \big\{(x, y, z) \in \mathbb{R}^3 : z > 0\big\} \to \mathbb{B}^2, \quad (x, y, z) \mapsto (x, y).$$

Then the pushforward of r_θ at $(0, 0, 1)$ is represented by its Jacobi matrix,

$$\operatorname{Jac}(r_\theta)((x, y, z)) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

So the eigenvalues of $(r_\theta)_*$ at $(0, 0, 1)$ are $\mathrm{e}^{\pm \mathrm{i}\theta} \neq 1$. Similarly, the south pole $(0, 0, -1)$ is also a non-degenerate fixed point of r_θ. Therefore r_θ is a Lefschetz map.

(2) Denote by $[F]$ the matrix representation of F in some basis of V. Since $\Delta \cap \Gamma_F = \{(v, v) \in V \times V : F(v) = v\}$, and for any $(v, v) \in \Delta \cap \Gamma_F$,

$$T_{(v,v)}\Delta = \{(w, w)_{(v,v)} : w \in T_vV\}, \quad T_{(v,v)}\Gamma_F = \{(w, Fw)_{(v,v)} : w \in T_vV\},$$

we have

$$\begin{aligned} \Delta \pitchfork \Gamma_F &\iff T_{(v,v)}\Delta + T_{(v,v)}\Gamma_F = T_{(v,v)}(V \times V), \ \forall (v, v) \in \Delta \cap \Gamma_F \\ &\iff 0 \neq \det\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & [F] \end{pmatrix} = \det\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ 0 & [F] - \mathbb{1} \end{pmatrix} = \det([F] - \mathbb{1}), \ \forall \text{ fixed point } v \text{ of } F \end{aligned}$$

$$\iff 1 \text{ is not an eigenvalue of } F = F_* \iff F \text{ is a Lefschetz map.}$$

Likewise, if $F : M \to M$ is a Lefschetz map, then $\Delta \pitchfork \Gamma_F$. It follows that $\Delta \cap \Gamma_F$ is an embedded submanifold of $M \times M$ of dimension $m + m - (2m) = 0$. Since a zero-dimensional manifold is a discrete set (each singleton is homeomorphic to $\mathbb{R}^0$) and $M \times M$ is compact, the set $\Delta \cap \Gamma_F$ is finite. In other words, F has only finitely many fixed points.

(3) Since the determinants of $(r_\theta)_* - \mathbb{1}$ at $(0, 0, \pm 1)$ are both equal to

$$\det\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix} = 2 - 2\cos\theta > 0,$$

we have $L(r_\theta) = 1 + 1 = 2$. □

Exercise 36 Recall that the group of $2n$-dimensional symplectic matrices is denoted by

$$\mathrm{Sp}(2n) = \left\{A \in M_{2n\times 2n}(\mathbb{R}) : AJ_0A^{\mathsf{T}} = J_0\right\}$$

where $J_0 \in M_{2n\times 2n}$ is defined by

$$J_0 = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}.$$

Prove that $\mathrm{Sp}(2n)$ is a submanifold of $M_{2n\times 2n}(\mathbb{R})$. Moreover, compute its dimension.

Proof Denote by $\mathrm{Skew}(2n) = \left\{A \in M_{2n\times 2n}(\mathbb{R}) : A^{\mathsf{T}} = -A\right\}$ the set of $2n \times 2n$ real skew-symmetric matrices. First we show that $\mathrm{Skew}(2n)$ is a smooth manifold. Consider the map

$$\Phi : \mathrm{GL}(2n, \mathbb{R}) \to M_{2n\times 2n}(\mathbb{R}), \quad A \mapsto A^{\mathsf{T}}A.$$

We want to compute the differential of Φ at $\mathbb{1}_{2n\times 2n} \in \mathrm{GL}(2n, \mathbb{R})$. For any $B \in T_{\mathbb{1}_{2n\times 2n}}\mathrm{GL}(2n, \mathbb{R}) = M_{2n\times 2n}(\mathbb{R})$, let $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{GL}(2n, \mathbb{R})$ be the curve $\gamma(t) = \mathbb{1}_{2n\times 2n} + tB$. Then

$$\mathrm{d}\Phi_{\mathbb{1}_{2n\times 2n}}(B) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi \circ \gamma(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathbb{1}_{2n\times 2n} + tB)^{\mathsf{T}}(\mathbb{1}_{2n\times 2n} + tB) = B^{\mathsf{T}} + B.$$

Note that the orthogonal group $\mathrm{O}(2n)$ is equal to the level set $\Phi^{-1}(\mathbb{1}_{2n\times 2n})$. Therefore

$$T_{\mathbb{1}_{2n\times 2n}}\mathrm{O}(2n) = \mathrm{Ker}\,\mathrm{d}\Phi_{\mathbb{1}_{2n\times 2n}} = \left\{B \in M_{2n\times 2n}(\mathbb{R}) : B^{\mathsf{T}} + B = 0\right\} = \mathrm{Skew}(2n).$$

It follows that $\mathrm{Skew}(2n)$ is a smooth manifold.

Next we consider the map

$$F : \mathrm{GL}(2n, \mathbb{R}) \to \mathrm{Skew}(2n), \quad A \mapsto AJ_0A^{\mathsf{T}}.$$

For any $B \in T_A\,\mathrm{GL}(2n, \mathbb{R}) = M_{2n\times 2n}(\mathbb{R})$, let $\beta : (-\varepsilon, \varepsilon) \to \mathrm{GL}(2n, \mathbb{R})$ be the curve $\beta(t) = A + tB$. Then

$$\mathrm{d}F_A(B) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F \circ \beta(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (A + tB)J_0(A + tB)^{\mathsf{T}}$$

$$
\begin{aligned}
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[BJ_0\left(A^{\mathsf{T}} + tB^{\mathsf{T}}\right) + (A+tB)J_0B^{\mathsf{T}}\right] \\
&= BJ_0A^{\mathsf{T}} + AJ_0B^{\mathsf{T}}.
\end{aligned}
$$

Note that $\left(BJ_0A^{\mathsf{T}}\right)^{\mathsf{T}} = AJ_0{}^{\mathsf{T}}B^{\mathsf{T}} = -AJ_0B^{\mathsf{T}}$, so the above differential can be rewritten as

$$
\mathrm{d}F_A(B) = AJ_0B^{\mathsf{T}} - \left(AJ_0B^{\mathsf{T}}\right)^{\mathsf{T}}.
$$

Since $AJ_0 \in \mathrm{GL}(2n,\mathbb{R})$, as B ranges over $M_{2n\times 2n}(\mathbb{R})$, AJ_0B^{T} also ranges over $M_{2n\times 2n}(\mathbb{R})$, and thus $\mathrm{d}F_A(B)$ ranges over $\mathrm{Skew}(2n)$. That is, $dF_A(M_{2n\times 2n}(\mathbb{R})) = \mathrm{Skew}(2n)$. Therefore dF_A is surjective, i.e., F is a submersion.

Now we are able to apply the regular level set theorem. Since F is a submersion, J_0 is a regular value of F. Thus $\mathrm{Sp}(2n) = F^{-1}(J_0)$ is an embedded submanifold of $\mathrm{GL}(2n,\mathbb{R})$, and

$$
\dim \mathrm{Sp}(2n) = \dim \mathrm{GL}(2n,\mathbb{R}) - \dim \mathrm{Skew}(2n) = (2n)^2 - n(2n-1) = 2n^2 + n. \qquad \square
$$

Exercise 37 Prove by definition that if $N_1 \subset \mathbb{R}^{m_1}$ and $N_2 \subset \mathbb{R}^{m_2}$ are submanifolds of dimensions n_1 and n_2 respectively, then $N_1 \times N_2$ is a submanifold (of $\mathbb{R}^{m_1+m_2}$) of dimension $n_1 + n_2$.

Proof For any $(p,q) \in_1 \times N_2$, we can find local charts $\left(U_1, \varphi : U_1 \xrightarrow{\sim} V_1 \subset \mathbb{R}^{m_1}\right)$ of $\mathbb{R}^{m_1}$ near p and $\left(U_2, \psi : U_2 \xrightarrow{\sim} V_2 \subset \mathbb{R}^{m_2}\right)$ of $\mathbb{R}^{m_2}$ near q such that

$$
\begin{aligned}
\varphi(U_1 \cap N_1) &= \{x \in V_1 \subset \mathbb{R}^{m_1} : x_{n_1+1} = \cdots = x_{m_1} = 0\}, \\
\psi(U_2 \cap N_2) &= \{x \in V_2 \subset \mathbb{R}^{m_2} : x_{n_2+1} = \cdots = x_{m_2} = 0\}.
\end{aligned}
$$

Then $\left(U_1 \times U_2, \varphi \times \psi : U_1 \times U_2 \xrightarrow{\sim} V_1 \times V_2\right)$ is a local chart of $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \simeq \mathbb{R}^{m_1+m_2}$ near (p,q) such that

$$
\varphi \times \psi((U_1 \times U_2) \cap (N_1 \times N_2)) = \left\{x \in V_1 \times V_2 \subset \mathbb{R}^{m_1+m_2} : {\scriptstyle x_{n_1+1} = \cdots = x_{m_1} = 0, \atop x_{m_1+n_2+1} = \cdots = x_{m_1+m_2} = 0}\right\}.
$$

Thus $N_1 \times N_2$ is a submanifold of $\mathbb{R}^{m_1+m_2}$ of dimension $n_1 + n_2$. $\square$

Exercise 38 (1) Prove the Inverse Mapping Theorem: Let $F : N \to M$ be a smooth map such that $F_*(p) : T_pN \to T_{F(p)}M$ is an isomorphism, then F is a diffeomorphism locally near p.

(2) Deduce from (1) that there is no immersion from $\mathbb{S}^n$ to $\mathbb{R}^n$.

Proof (1) The fact that $F_*(p) : T_pN \to T_{F(p)}M$ is bijective implies that N and M have the same dimension, say n. Choose smooth charts (U,φ) centered at p and (V,ψ) centered at $F(p)$, with $F(U) \subset V$. Then $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is a smooth map from the open subset $\widehat{U} = \varphi(U) \subset \mathbb{R}^n$ into $\widehat{V} = \psi(V) \subset \mathbb{R}^n$, with $\widehat{F}(0) = 0$. Since φ and ψ are diffeomorphisms, the differential $\mathrm{d}\widehat{F}_0 = \mathrm{d}\psi_{F(p)} \circ \mathrm{d}F_p \circ \mathrm{d}\left(\varphi^{-1}\right)_0$ is nonsingular. The ordinary inverse function theorem shows that there connected open subsets $\widehat{U}_0 \subset \widehat{U}$ and $\widehat{V}_0 \subset \widehat{V}$ containing 0 such that $\widehat{F}$ restricts to a diffeomorphism from $\widehat{U}_0$ to $\widehat{V}_0$. Then $U_0 = \varphi^{-1}\left(\widehat{U}_0\right)$ and $V_0 = \psi^{-1}\left(\widehat{V}_0\right)$ are connected open neighborhoods of p and $F(p)$, respectively, and it follows by composition that $F|_{U_0}$ is a diffeomorphism from U_0 to V_0.

(2) Suppose there is an immersion $F : \mathbb{S}^n \to \mathbb{R}^n$. Since $\mathbb{S}^n$ and $\mathbb{R}^n$ have the same dimension, F is also a submersion. Hence F is an open map, $F(\mathbb{S}^n)$ is open in $\mathbb{R}^n$. But since $\mathbb{S}^n$ is compact and F

is continuous, $F(\mathbb{S}^n)$ is compact, i.e., $F(\mathbb{S}^n)$ is closed and bounded in $\mathbb{R}^n$. This is a contradiction since $\mathbb{R}^n$ is connected and the only nonempty clopen set is $\mathbb{R}^n$ itself, which is unbouded. Therefore there is no immersion from $\mathbb{S}^n$ to $\mathbb{R}^n$. $\square$

Exercise 39 Let $F : N \to M$ be a smooth map. Recall that the pullback of F is a functor $F^* : TM \to TN$. In particular, F^* defines a map for sections (forms) from $\Omega^k(M)$ to $\Omega^k(N)$ for any $k \in \mathbb{N}$, defined explicitly as follows,

$$(F^*\alpha)(X_1, \cdots, X_k) \coloneqq \alpha(F_*(X_1), \cdots, F_*(X_k))$$

or even more explicitly when the positions are specified,

$$(F^*\alpha)(p)(X_1(p), \cdots, X_k(p)) \coloneqq \alpha(F(p))(F_*(p)(X_1(p)), \cdots, F_*(p)(X_k(p))).$$

Now, consider map $F : \mathbb{R}^2 \to \mathbb{R}^3$, where $\mathbb{R}^2$ is in coordinate (x, y) and $\mathbb{R}^3$ in coordinate (u, v, w), by $F(x, y) = \big(xy, x^2, 3x + y\big)$. For $\alpha = uv\,\mathrm{d}u + 2w\,\mathrm{d}v - v\,\mathrm{d}w \in \Omega^1(\mathbb{R}^3)$, compute $F^*\alpha$ and express it in terms of $\mathrm{d}x$ and $\mathrm{d}y$.

Solution The pullback $F^*\alpha$ is computed as follows:

$$\begin{aligned} F^*(uv\,\mathrm{d}u + 2w\,\mathrm{d}v - v\,\mathrm{d}w) &= (xy)x^2\,\mathrm{d}(xy) + 2(3x+y)\,\mathrm{d}\big(x^2\big) - x^2\,\mathrm{d}(3x+y) \\ &= x^3y(y\,\mathrm{d}x + x\,\mathrm{d}y) + (6x+2y)(2x\,\mathrm{d}x) - x^2(3\,\mathrm{d}x + \mathrm{d}y) \\ &= \big(x^3y^2 + 9x^2 + 4xy\big)\,\mathrm{d}x + \big(x^4y - x^2\big)\,\mathrm{d}y. \end{aligned}$$

We can also compute $F^*\alpha$ from its definition. First we find the pushforward F_* in its Jacobi matrix representation:

$$\mathrm{Jac}(F)((x,y)) = \begin{pmatrix} y & x \\ 2x & 0 \\ 3 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} (F^*\alpha)((x,y))\left(\frac{\partial}{\partial x}\right) &= \alpha(F(x,y))\left(F_*((x,y))\left(\frac{\partial}{\partial x}\right)\right) \\ &= \alpha\big(\big(xy, x^2, 3x+y\big)\big)\left(y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v} + 3\frac{\partial}{\partial w}\right) \\ &= (xy)x^2\,\mathrm{d}u\left(y\frac{\partial}{\partial u}\right) + 2(3x+y)\,\mathrm{d}v\left(2x\frac{\partial}{\partial v}\right) - x^2\,\mathrm{d}w\left(3\frac{\partial}{\partial w}\right) \\ &= x^3y^2 + 9x^2 + 4xy. \end{aligned}$$

Similarly, we have

$$(F^*\alpha)((x,y))\left(\frac{\partial}{\partial y}\right) = x^4y - x^2.$$

These lead to the same result as before. $\square$

Exercise 40 Define the map $F : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$F(x,y) = \big(\mathrm{e}^y \cos x, \mathrm{e}^y \sin x, \mathrm{e}^{-y}\big).$$

Denote by $S_r(0) \subset \mathbb{R}^3$ the standard 2-sphere centered at 0 with radius r. Recall/Define that a map

$F : N \to M$ is transverse to a submanifold $S \subset M$ means for any $x \in F^{-1}(S)$, the linear spaces $T_{F(x)}S$ and $F_*(x)(T_xN)$ span $T_{F(x)}M$.

(1) For which positive numbers r is F transverse to $S_r(0)$ in $\mathbb{R}^3$?

(2) For which positive numbers r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$?

Solution (1) The map F will not be transverse to $S_r(0)$ if and only if there is a point $(x, y) \in \mathbb{R}^2$ such that $|F(x, y)| = \sqrt{\mathrm{e}^{2y} + \mathrm{e}^{-2y}} = r$ and the vectors

$$\partial_x F(x, y) = (-\mathrm{e}^y \sin x, \mathrm{e}^y \cos x, 0) \quad \text{and} \quad \partial_y F(x, y) = \left(\mathrm{e}^y \cos x, \mathrm{e}^y \sin x, -\mathrm{e}^{-y}\right)$$

are parallel to $T_{F(x,y)}S_r(0)$. This last condition is equivalent to

$$\partial_x F(x, y) \cdot F(x, y) = 0 \quad \text{and} \quad \partial_y F(x, y) \cdot F(x, y) = 0.$$

The first equation holds everywhere, and the second equation gives $\mathrm{e}^{2y} - \mathrm{e}^{-2y} = 0$, which has solution $y = 0$ and therefore $r = \sqrt{2}$. So F is transverse to $S_r(0)$ unless $r = \sqrt{2}$.

(2) By (1), for positive numbers $r \neq \sqrt{2}$, $F^{-1}(S_r(0))$ is an embedded submanifold of $\mathbb{R}^2$. In the case $r = \sqrt{2}$, we have

$$F^{-1}(S_r(0)) = \left\{(x, y) \in \mathbb{R}^2 : \mathrm{e}^{2y} + \mathrm{e}^{-2y} = 2\right\} = \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\},$$

which is just the x-axis and is clearly an embedded submanifold of $\mathbb{R}^2$. Therefore $F^{-1}(S_r(0))$ is an embedded submanifold of $\mathbb{R}^2$ for all positive numbers r. □

Homework 5

Exercise 41 Let M be a smooth manifold and $F : M \to \mathbb{R}^k$ be a *continuous* map. Prove that for any positive continuous function $\varepsilon : M \to \mathbb{R}$, there exists a smooth map $G : M \to \mathbb{R}^k$ such that $\|G(x) - F(x)\| \leqslant \varepsilon(x)$ for any $x \in M$.

Proof We shall show that there are countably many points $\{x_i\}_{i=1}^\infty$ in M and open neighborhoods U_i of x_i in M such that $\{U_i\}_{i=1}^\infty$ is an open cover of M and

$$\|F(y) - F(x_i)\| < \varepsilon(y), \quad \forall y \in U_i. \tag{41–1}$$

To see this, for any $x \in M$, let U_x be an open neighborhood of x small enough such that

$$\varepsilon(y) > \tfrac{1}{2}\varepsilon(x) \quad \text{and} \quad \|F(y) - F(x)\| < \tfrac{1}{2}\varepsilon(x)$$

for all $y \in U_x$. (Such a neighborhood exists by continuity of ε and F.) Then if $y \in U_x$, we have

$$\|F(y) - F(x)\| < \tfrac{1}{2}\varepsilon(x) < \varepsilon(x).$$

The collection $\{U_x : x \in M\}$ is an open cover of M. Choosing a countable subcover $\{U_{x_i}\}_{i=1}^\infty$ and setting $U_i = U_{x_i}$, we have (41–1). Let $\{\rho_i\}$ be a smooth partition of unity subordinate to the cover $\{U_i\}$ of M, and define $G : M \to \mathbb{R}^k$ by

$$G(y) = \sum_{i=1}^{\infty} \rho_i(y) F(x_i).$$

Then clearly G is smooth. For any $y \in M$, the fact that $\sum_{i=1}^{\infty} \rho_i \equiv 1$ implies that

$$\begin{aligned} \|G(y) - F(y)\| &= \left\| \sum_{i=1}^{\infty} \rho_i(y)[F(x_i) - F(y)] \right\| \\ &\leqslant \sum_{i=1}^{\infty} \rho_i(y) \|F(x_i) - F(y)\| \\ &< \sum_{i=1}^{\infty} \rho_i(y) \varepsilon(y) \\ &= \varepsilon(y). \end{aligned}$$

□

Exercise 42 Consider $\theta \in \Omega^2(\mathbb{R}^3)$ defined by

$$\theta = x^2 \, \mathrm{d}y \wedge \mathrm{d}z + y \, \mathrm{d}z \wedge \mathrm{d}x + z \, \mathrm{d}x \wedge \mathrm{d}y.$$

Denote by $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Compute the integration $\int_{\mathbb{S}^2} i^*\theta$ where $i : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion.

Solution Let $\mathbb{D}^3 \subset \mathbb{R}^3$ be the closed unit ball. By Stokes' theorem, we have

$$\begin{aligned} \int_{\mathbb{S}^2} i^*\theta = \int_{\mathbb{D}^3} \mathrm{d}\theta = \int_{\mathbb{D}^3} (2x + 2) \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \left(\int_{\mathbb{D}^3} 2x \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \right) + 2 \cdot \frac{4\pi}{3} \\ = \left(\int_{\mathbb{S}^2} x^2 \, \mathrm{d}y \wedge \mathrm{d}z \right) + \frac{8\pi}{3}. \end{aligned}$$

Consider the map

$$F : (0, 2\pi) \times (0, \pi) \to \mathbb{S}^2, \quad (\psi, \varphi) \mapsto (\sin\varphi \cos\psi, \sin\varphi \sin\psi, \cos\varphi).$$

Since

$$\mathrm{Jac}(F)(\psi, \varphi) = \begin{pmatrix} -\sin\varphi \sin\psi & \cos\varphi \cos\psi \\ \sin\varphi \cos\psi & \cos\varphi \sin\psi \\ 0 & -\sin\varphi \end{pmatrix},$$

at the point $(\psi, \varphi) = \left(\pi, \frac{\pi}{2}\right)$, we have

$$F_*\left(\frac{\partial}{\partial\psi}\right) = \begin{pmatrix} -\sin\varphi \sin\psi \\ \sin\varphi \cos\psi \\ 0 \end{pmatrix}_{\left(\pi, \frac{\pi}{2}\right)} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\frac{\partial}{\partial y}\bigg|_{T_p\mathbb{S}^2}$$

and

$$F_*\left(\frac{\partial}{\partial\varphi}\right) = \begin{pmatrix} \cos\varphi \cos\psi \\ \cos\varphi \sin\psi \\ -\sin\varphi \end{pmatrix}_{\left(\pi, \frac{\pi}{2}\right)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\frac{\partial}{\partial z}\bigg|_{T_p\mathbb{S}^2}.$$

At the point $p := F(\pi, \frac{\pi}{2}) = (-1,0,0)$, the three tangent vectors $\left\{-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right\}$ are of opposite orientation to the standard orientation $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ of $\mathbb{R}^3$, i.e., $\left\{F_*\left(\frac{\partial}{\partial \psi}\right), F_*\left(\frac{\partial}{\partial \varphi}\right)\right\}$ is an orientation-reversing basis of $T_p\mathbb{S}^2$. Thus F is an orientation-reversing diffeomorphism, and

$$\begin{aligned}\int_{\mathbb{S}^2} x^2\,\mathrm{d}y\wedge\mathrm{d}z &= \int_{F((0,2\pi)\times(0,\pi))} x^2\,\mathrm{d}y\wedge\mathrm{d}z\\ &= -\int_{(0,2\pi)\times(0,\pi)} F^*\big(x^2\,\mathrm{d}y\wedge\mathrm{d}z\big)\\ &= -\int_{(0,2\pi)\times(0,\pi)} \sin^2\varphi\cos^2\psi\,\mathrm{d}(\sin\varphi\sin\psi)\wedge\mathrm{d}(\cos\varphi)\\ &= \int_{(0,2\pi)\times(0,\pi)} \sin^4\varphi\cos^3\psi\,\mathrm{d}\psi\wedge\mathrm{d}\varphi\\ &= \int_0^{2\pi}\cos^3\psi\,\mathrm{d}\psi\int_0^{\pi}\sin^4\varphi\,\mathrm{d}\varphi\\ &= 0.\end{aligned}$$

Therefore, we have $\displaystyle\int_{\mathbb{S}^2} i^*\theta = \frac{8\pi}{3}$. □

Exercise 43

(1) Given a manifold M and two 1-forms $\alpha,\beta\in\Omega^1(M)$, prove the following identity

$$\alpha\wedge(\mathrm{d}\alpha)^n - \beta\wedge(\mathrm{d}\beta)^n = (\alpha-\beta)\wedge\sum_{j=0}^{n}(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-j} + \mathrm{d}\left(\alpha\wedge\beta\wedge\sum_{j=0}^{n-1}(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-1-j}\right)$$

for any $n\in\mathbb{N}$. Here $(\mathrm{d}\alpha)^n := \mathrm{d}\alpha\wedge\cdots\wedge\mathrm{d}\alpha$, wedged n times, similarly to others.

(2) Deduce the following proposition from (1): given a closed (i.e., compact without boundary) orientable manifold M of dimension $2n+1$ and a smooth vector field $X\in\Gamma(TM)$, if two 1-forms $\alpha,\beta\in\Omega^1(M)$ satisfy $\left(\phi_X^t\right)^*\alpha=\alpha$ and $\left(\phi_X^t\right)^*\beta=\beta$ for any $t\in\mathbb{R}$ (invariant condition), moreover $\alpha(X)=\beta(X)=1$, then

$$\int_M \alpha\wedge(\mathrm{d}\alpha)^n = \int_M \beta\wedge(\mathrm{d}\beta)^n.$$

Proof (1) Direct computation gives

$$\begin{aligned}&\mathrm{d}\left(\alpha\wedge\beta\wedge\sum_{j=0}^{n-1}(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-1-j}\right)\\ =&\,\mathrm{d}(\alpha\wedge\beta)\wedge\sum_{j=0}^{n-1}(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-1-j} + \alpha\wedge\beta\wedge\sum_{j=0}^{n-1}\underbrace{\mathrm{d}\big((\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-1-j}\big)}_{=0}\\ =&(\mathrm{d}\alpha\wedge\beta-\alpha\wedge\mathrm{d}\beta)\wedge\sum_{j=0}^{n-1}(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-1-j}\\ =&\sum_{j=0}^{n-1}\mathrm{d}\alpha\wedge\beta\wedge(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-1-j} - \sum_{j=0}^{n-1}\alpha\wedge\mathrm{d}\beta\wedge(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-1-j}\\ =&\,\beta\wedge\sum_{j=0}^{n-1}(\mathrm{d}\alpha)^{j+1}\wedge(\mathrm{d}\beta)^{n-1-j} - \alpha\wedge\sum_{j=0}^{n-1}(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-j}\end{aligned}$$

$$
\begin{aligned}
&=\alpha \wedge \left((\mathrm{d}\alpha)^n - \sum_{j=0}^{n}(\mathrm{d}\alpha)^j \wedge (\mathrm{d}\beta)^{n-j}\right) - \beta \wedge \left((\mathrm{d}\beta)^n - \sum_{j=0}^{n}(\mathrm{d}\alpha)^j \wedge (\mathrm{d}\beta)^{n-j}\right)\\
&=\alpha \wedge (\mathrm{d}\alpha)^n - \beta \wedge (\mathrm{d}\beta)^n - (\alpha-\beta)\sum_{j=0}^{n}(\mathrm{d}\alpha)^j \wedge (\mathrm{d}\beta)^{n-j}.
\end{aligned}
$$

(2) Note that $\alpha \wedge \beta \wedge \sum_{j=0}^{n-1}(\mathrm{d}\alpha)^j \wedge (\mathrm{d}\beta)^{n-1-j} \in \Omega^{2n}(M)$. By (1) and Stokes' theorem, since M is closed, we have

$$
\int_M [\alpha \wedge (\mathrm{d}\alpha)^n - \beta \wedge (\mathrm{d}\beta)^n] = \int_M (\alpha-\beta) \wedge \sum_{j=0}^{n}(\mathrm{d}\alpha)^j \wedge (\mathrm{d}\beta)^{n-j}.
$$

The invariant condition implies that

$$
\mathcal{L}_X\alpha = \lim_{\to 0}\frac{(\phi_X^t)^*\alpha - \alpha}{t} = 0, \quad \mathcal{L}_X\beta = \lim_{t\to 0}\frac{(\phi_X^t)^*\beta - \beta}{t} = 0.
$$

So by Cartan's magic formula,

$$
0 = \mathcal{L}_X\alpha = \mathrm{d}(\iota_X\alpha) + \iota_X(\mathrm{d}\alpha) = \underbrace{\mathrm{d}(\alpha(X))}_{=\mathrm{d}(1)=0} + \iota_X(\mathrm{d}\alpha) = \iota_X(\mathrm{d}\alpha),
$$

and similarly $\iota_X(\mathrm{d}\beta) = 0$.

We claim that $\theta := (\alpha-\beta) \wedge \sum_{j=0}^{n}(\mathrm{d}\alpha)^j \wedge (\mathrm{d}\beta)^{n-j}$ is in fact identically zero. Since $\alpha(X) = 1$, the vector field X is nowhere vanishing. At any point $p \in M$, we can extend X_p to an oriented basis for T_pM, say $X_p, v_1, \cdots, v_{2n}$. Now

$$
\begin{aligned}
&\theta_p(X, v_1, \cdots, v_{2n})\\
=&\frac{1}{1!(2n)!}\sum_{\sigma\in\mathfrak{S}_{2n+1}}\mathrm{sign}(\sigma)\sigma \cdot \left((\alpha-\beta)\otimes\sum_{j=0}^{n}(\mathrm{d}\alpha)^j \wedge (\mathrm{d}\beta)^{n-j}\right)_p (X_p, v_1, \cdots, v_{2n})\\
\overset{\iota_X(\mathrm{d}\alpha)=0}{\underset{\iota_X(\mathrm{d}\beta)=0}{=\!=}}&\frac{1}{(2n)!}\sum_{\tau\in\mathfrak{S}_{2n}}\mathrm{sign}(\tau)\left((\alpha-\beta)_p(X_p)\sum_{j=0}^{n}(\mathrm{d}\alpha)_p^j \wedge (\mathrm{d}\beta)_p^{n-j}\big(v_{\tau(1)}, \cdots, v_{\tau(2n)}\big)\right)\\
\overset{\alpha(X)=1}{\underset{\beta(X)=1}{=\!=}}&0.
\end{aligned}
$$

Since p is arbitrary, we have $\theta \equiv 0$. Therefore,

$$
\int_M \alpha \wedge (\mathrm{d}\alpha)^n - \int_M \beta \wedge (\mathrm{d}\beta)^n = \int_M \theta = 0.
$$

□

Exercise 44 Let M be a closed manifold of dimension $2n$.

(1) Let $\omega \in \Omega^2(M)$ be a 2-form, then ω is non-degenerate (in the sense that at any point $x \in M$, if $v \in T_xM$ is not zero, then there exists some $w \in T_xM$ such that $\omega_x(v, w) \neq 0$) if and only if ω^n is a volume form of M.

(2) From Exercise 25, we have seen the (Poisson) bracket of two functions $H, G : M \to \mathbb{R}$ defined by

$$\{H, G\} \coloneqq \omega(X_H, X_G), \quad \text{where } -\mathrm{d}H = \omega(X_H, \cdot), \text{ similarly to } X_G.$$

Suppose further that ω is closed, then prove that

$$\int_M \{F, G\}\omega^n = 0.$$

Proof (1) Suppose first that ω is non-degenerate. For any $p \in M$, we show that T_pM admits a basis $u_1, \cdots, u_n, v_1, \cdots, v_n$ such that

$$\omega_p(u_j, u_k) = \omega_p(v_j, v_k) = 0, \quad \omega_p(u_j, v_k) = \delta_{jk}. \tag{44–1}$$

The proof is by induction over n. Since ω is non-degenerate, there exist $u_1, v_1 \in T_pM$ such that $\omega_p(u_1, v_1) = 1$. Also, $\omega_p(u_1, u_1) = \omega_p(v_1, v_1) = 0$ always holds. Let

$$W = \{v \in T_pM : \omega_p(v, w) = 0, \forall w \in \text{Span}\{u_1, v_1\}\}.$$

Define a linear map $\Phi : T_pM \to T_p^*M$ by $\Phi(v)(w) = \omega(v, w)$. Since ω is non-degenerate, Φ is an isomorphism. It identifies W with the annihilator of Span$\{u_1, v_1\}$ in T_p^*M. Thus W is a vector space of dimension $2n - 2$. By the induction hypothesis, there exists a basis $u_2, \cdots, u_n, v_2, \cdots, v_n$ of W satisfying (44–1). Hence $u_1, \cdots, u_n, v_1, \cdots, v_n$ forms a basis of T_pM satisfying (44–1).

By (44–1), there is a vector space isomorphism $\Psi : T_p^*M \to T_0^*\mathbb{R}^{2n}$ sending ω_p to

$$\omega_0 = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j,$$

where $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}$ is the standard basis of $T_0\mathbb{R}^{2n}$.

Since $\mathrm{d}x_i \wedge \mathrm{d}x_i = \mathrm{d}y_i \wedge \mathrm{d}y_i = 0$, we have

$$\omega_0^n = \left(\sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j\right)^n = n!\, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n,$$

which is nonzero. Thus ω^n is non-vanishing at p. Since p is arbitrary, ω^n is a nowhere vanishing $2n$-form on M.

Conversely, suppose ω is degenerate. Choose $p \in M$ and nonzero $v_1 \in T_pM$ such that $\omega_p(v_1, w) = 0$ for all $w \in T_pM$, and extend it to a basis $v_1, \cdots, v_{2n}$ of T_pM. Then $\omega_p^n(v_1, \cdots, v_{2n}) = 0$. Hence ω^n is not a volume form of M.

(2) First, observe that

$$\begin{aligned}
\iota_{X_G}\omega^n &= \iota_{X_G}\big(\omega\wedge\omega^{n-1}\big)\\
&= (\iota_{X_G}\omega)\wedge\omega^{n-1}+\omega\wedge\big(\iota_{X_G}\omega^{n-1}\big)\\
&= -\,\mathrm{d}G\wedge\omega^{n-1}+\omega\wedge\big(\iota_{X_G}\omega^{n-1}\big)\\
&= -\,\mathrm{d}G\wedge\omega^{n-1}+\omega\wedge\big(-\,\mathrm{d}G\wedge\omega^{n-2}+\omega\wedge\big(\iota_{X_G}\omega^{n-2}\big)\big)\\
&= -2\,\mathrm{d}G\wedge\omega^{n-1}+\omega^2\wedge\big(\iota_{X_G}\omega^{n-2}\big)\\
&= \cdots\\
&= -(n-1)\,\mathrm{d}G\wedge\omega^{n-1}+\omega^{n-1}\wedge(\iota_{X_G}\omega)\\
&= -(n-1)\,\mathrm{d}G\wedge\omega^{n-1}-\omega^{n-1}\wedge\mathrm{d}G\\
&= -n\,\mathrm{d}G\wedge\omega^{n-1},
\end{aligned}\tag{44–2}$$

and similarly

$$\iota_{X_F}\omega^{n+1}=-(n+1)\,\mathrm{d}F\wedge\omega^n.\tag{44–3}$$

Since $\dim M=2n$, we have $\omega^{n+1}=0$, and then

$$\begin{aligned}
0 &= \iota_{X_G}\iota_{X_F}\omega^{n+1}\\
&\overset{(44–3)}{=\!=} -(n+1)\iota_{X_G}(\mathrm{d}F\wedge\omega^n)\\
&= -(n+1)(\iota_{X_G}\,\mathrm{d}F)\wedge\omega^n+(n+1)\,\mathrm{d}F\wedge(\iota_{X_G}\omega^n)\\
&= -(n+1)\,\mathrm{d}F(X_G)\wedge\omega^n+(n+1)\,\mathrm{d}F\wedge(\iota_{X_G}\omega^n)\\
&= (n+1)\omega(X_F,X_G)\wedge\omega^n+(n+1)\,\mathrm{d}F\wedge(\iota_{X_G}\omega^n)\\
&\overset{(44–2)}{=\!=} (n+1)\{F,G\}\omega^n+(n+1)\,\mathrm{d}F\wedge\big(-n\,\mathrm{d}G\wedge\omega^{n-1}\big)\\
&= (n+1)\big(\{F,G\}\omega^n+n\,\mathrm{d}G\wedge\mathrm{d}F\wedge\omega^{n-1}\big).
\end{aligned}$$

Rearranging the terms, we obtain

$$\{F,G\}\omega^n=-n\,\mathrm{d}G\wedge\mathrm{d}F\wedge\omega^{n-1}.$$

Let $\theta=F\,\mathrm{d}G\wedge\omega^{n-1}\in\Omega^{2n-1}(M)$. Since ω is closed, so is ω^{n-1}. Hence

$$\begin{aligned}
\mathrm{d}\theta &= \mathrm{d}F\wedge\mathrm{d}G\wedge\omega^{n-1}+F\,\mathrm{d}\big(\mathrm{d}G\wedge\omega^{n-1}\big)\\
&= \mathrm{d}F\wedge\mathrm{d}G\wedge\omega^{n-1}+F\big(\mathrm{d}^2G\wedge\omega^{n-1}-\mathrm{d}G\wedge\mathrm{d}\big(\omega^{n-1}\big)\big)\\
&= \mathrm{d}F\wedge\mathrm{d}G\wedge\omega^{n-1}.
\end{aligned}$$

Since M is closed, by Stokes' theorem, we have

$$\int_M\{F,G\}\omega^n=n\int_M\mathrm{d}F\wedge\mathrm{d}G\wedge\omega^{n-1}=n\int_M\mathrm{d}\theta=0.$$

□

Exercise 45 Let M^m,N^n be orientable manifolds. Let $\pi_M:M\times N\to M$ and $\pi_N:M\times N\to N$ be the projections. Then for forms $\alpha \in \Omega^m(M)$ and $\beta \in \Omega^n(N)$, consider their "product" defined by

$$\alpha \times \beta \coloneqq \pi_M^* \alpha \wedge \pi_N^* \beta \in \Omega^{m+n}(M \times N).$$

Prove from definition (of integration on manifolds) that

$$\int_{M \times N} \alpha \times \beta = \left(\int_M \alpha\right) \cdot \left(\int_N \beta\right).$$

Proof (1) Choose oriented atlases $\{(U_i, \varphi_i) : i \in I\}$ and $\{(V_j, \psi_j) : j \in J\}$ for M and N, respectively. Then the atlas $\{(U_i \times V_j, \varphi_i \times \psi_j) : i \in I, j \in J\}$ is an oriented atlas for $M \times N$, because

$$\begin{aligned}
\det \operatorname{Jac}\Big((\psi_{\beta_1} \times \psi_{\beta_2}) \circ (\varphi_{\alpha_1} \times \varphi_{\alpha_2})^{-1}\Big) &= \det \operatorname{Jac}\big((\psi_{\beta_1} \circ \varphi_{\alpha_1}^{-1}) \times (\psi_{\beta_2} \circ \varphi_{\alpha_2}^{-1})\big) \\
&= \det \begin{pmatrix} \operatorname{Jac}(\psi_{\beta_1} \circ \varphi_{\alpha_1}^{-1}) & 0 \\ 0 & \operatorname{Jac}(\psi_{\beta_2} \circ \varphi_{\alpha_2}^{-1}) \end{pmatrix} \\
&= \det \operatorname{Jac}(\psi_{\beta_1} \circ \varphi_{\alpha_1}^{-1}) \det \operatorname{Jac}(\psi_{\beta_2} \circ \varphi_{\alpha_2}^{-1}) \\
&> 0.
\end{aligned}$$

So $M \times N$ is orientable.

(2) Assume first that α is compactly supported in a local chart (U, φ) and β is compactly supported in a local chart (V, ψ). Suppose

$$(\varphi^{-1})^* \alpha = f \, dx_1 \cdots dx_m, \quad (\psi^{-1})^* \beta = g \, dy_1 \cdots dy_n.$$

Then $\alpha \times \beta$ is compactly supported in the local chart $(U \times V, \varphi \times \psi)$, and

$$\begin{aligned}
\big((\varphi \times \psi)^{-1}\big)^* (\alpha \times \beta) &= \big((\varphi \times \psi)^{-1}\big)^* (\pi_M^* \alpha \wedge \pi_N^* \beta) \\
&= \big((\varphi \times \psi)^{-1}\big)^* (\pi_M^* \alpha) \wedge \big((\varphi \times \psi)^{-1}\big)^* (\pi_N^* \beta) \\
&= \big(\pi_M \circ (\varphi \times \psi)^{-1}\big)^* \alpha \wedge \big(\pi_N \circ (\varphi \times \psi)^{-1}\big)^* \beta \\
&= (\varphi^{-1})^* \alpha \wedge (\psi^{-1})^* \beta \\
&= f g \, dx_1 \cdots dx_m \, dy_1 \cdots dy_n.
\end{aligned}$$

So by Fubini's theorem on $\mathbb{R}^m \times \mathbb{R}^n$, we have

$$\begin{aligned}
\int_{U \times V} \alpha \times \beta &= \int_{\varphi(U) \times \psi(V)} f g \, dx_1 \cdots dx_m \, dy_1 \cdots dy_n \\
&= \left(\int_{\varphi(U)} f \, dx_1 \cdots dx_m\right) \cdot \left(\int_{\psi(V)} g \, dy_1 \cdots dy_n\right) \\
&= \left(\int_U \alpha\right) \cdot \left(\int_V \beta\right).
\end{aligned}$$

(3) Let $\{U_i\}$ be a finite open cover of $\operatorname{supp} \alpha$ by domains of oriented smooth charts, and let $\{\rho_i\}$ be a subordinate smooth partition of unity. Likewise, choose open cover $\{V_j\}$ for $\operatorname{supp} \beta$ and a subordinate partition of unity $\{\sigma_j\}$. Then $\{\rho_i \sigma_j\}$ is a partition of unity subordinate to the open cover

$\{U_i \times V_j\}$, since for $(p,q) \in M \times N$,

$$\sum_i \sum_j (\rho_i\sigma_j)(p,q) = \left(\sum_i \rho_i(p)\right) \cdot \left(\sum_j \sigma_j(q)\right) = 1,$$

and other requirements of a partition of unity are easily checked. Hence

$$\begin{aligned}\int_{M\times N} \alpha \times \beta &= \sum_{i,j} \int_M (\rho_i\sigma_j)\alpha \times \beta \\ &= \sum_{i,j} \int_M (\rho_i\alpha) \times (\sigma_j\beta) \\ &\overset{(2)}{=\!=} \sum_{i,j} \left(\int_M \rho_i\alpha\right) \cdot \left(\int_M \sigma_j\beta\right) \\ &= \left(\sum_i \int_M \rho_i\alpha\right) \cdot \left(\sum_j \int_M \sigma_j\beta\right) \\ &= \left(\int_M \alpha\right) \cdot \left(\int_N \beta\right).\end{aligned}$$

□

Homework 6

Exercise 46 For the following matrix groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{U}(n)$, and $\mathrm{Sp}(2n)$, compute/ confirm their induced Lie algebras as follows.

(1) $\mathfrak{sl}(n,\mathbb{R}) \coloneqq \{A \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr}(A) = 0\}$.

(2) $\mathfrak{sl}(n,\mathbb{C}) \coloneqq \{A \in \mathfrak{gl}(n,\mathbb{C}) : \mathrm{tr}(A) = 0\}$.

(3) $\mathfrak{o}(n) \coloneqq \{A \in \mathfrak{gl}(n,\mathbb{R}) : A^{\mathsf{T}} + A = 0\}$.

(4) $\mathfrak{u}(n) \coloneqq \{A \in \mathfrak{gl}(n,\mathbb{C}) : A^{\mathsf{H}} + A = 0\}$.

(5) $\mathfrak{sp}(2n) \coloneqq \{A \in \mathfrak{gl}(2n,\mathbb{R}) : A^{\mathsf{T}}J + JA = 0\}$, where $J = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}$.

Proof We shall apply the following theorem.

(GTM 94, Theorem 3.34) *Let A be an abstract subgroup of a Lie group G, and let $\mathfrak{a}$ be a subspace of $\mathfrak{g}$. Let U be an open neighborhood of 0 in $\mathfrak{g}$ diffeomorphic under the exponential map to an open neighborhood V of e in G. Suppose that*

$$\exp(U \cap \mathfrak{a}) = V \cap A.$$

Then A with the subspace topology is a Lie subgroup of G, $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, and $\mathfrak{a}$ is the Lie algebra of A.

(1) Clearly $\mathfrak{sl}(n,\mathbb{R})$ is a subspace of $\mathfrak{gl}(n,\mathbb{R})$. Let U be an open neighborhood of 0 in $\mathfrak{gl}(n,\mathbb{R})$, diffeomorphic under the exponential map to an open neighborhood V of $\mathbb{1}_{n\times n}$ in $\mathrm{GL}(n,\mathbb{R})$. If $A \in \mathfrak{sl}(n,\mathbb{R})$, then $\det(\exp(A)) = \det(\mathrm{e}^A) = \mathrm{e}^{\mathrm{tr}(A)} = 1$, so $\exp(A) \in \mathrm{SL}(n,\mathbb{R})$. Conversely, if $\det(\exp(A)) = 1$, since $\mathrm{tr}(A) \in \mathbb{R}$, we get $\mathrm{tr}(A) = 0$. Thus the above theorem implies that $\mathfrak{g}_{\mathrm{SL}(n,\mathbb{R})} = \mathfrak{sl}(n,\mathbb{R})$.

(2) Clearly $\mathfrak{sl}(n,\mathbb{C})$ is a subspace of $\mathfrak{gl}(n,\mathbb{C})$. Let U be an open neighborhood of 0 in $\mathfrak{gl}(n,\mathbb{C})$, diffeomorphic under the exponential map to an open neighborhood V of $\mathbb{1}_{n\times n}$ in $\mathrm{GL}(n,\mathbb{C})$. Since the trace function is continuous, we can assume that $|\mathrm{tr}(A)| < 2\pi$ for all $A \in U$. If $A \in \mathfrak{sl}(n,\mathbb{C})$, then $\det(\exp(A)) = \det\left(\mathrm{e}^A\right) = \mathrm{e}^{\mathrm{tr}(A)} = 1$, so $\exp(A) \in \mathrm{SL}(n,\mathbb{C})$. Conversely, if $\det(\exp(A)) = 1$, then $\mathrm{tr}(A) = 2\pi k\mathrm{i}$ for some $k \in \mathbb{Z}$. If in addition $A \in U$, then $\mathrm{tr}(A) = 0$. Thus the above theorem implies that $\mathfrak{g}_{\mathrm{SL}(n,\mathbb{C})} = \mathfrak{sl}(n,\mathbb{C})$.

(3) Clearly $\mathfrak{o}(n)$ is a subspace of $\mathfrak{gl}(n,\mathbb{R})$. Let U be an open neighborhood of 0 in $\mathfrak{gl}(n,\mathbb{R})$, diffeomorphic under the exponential map to an open neighborhood V of $\mathbb{1}_{n\times n}$ in $\mathrm{GL}(n,\mathbb{R})$. We can assume, in addition, that if $A \in U$, then A^{T} and $-A$ belong to U. For let W be an open neighborhood of 0 in $\mathfrak{gl}(n,\mathbb{R})$ that is small enough for the exponential map to be a diffeomorphism, and then let $U = W \cap W^{\mathsf{T}} \cap (-W)$. If $A \in U \cap \mathfrak{o}(n)$, then

$$(\exp(A))^{\mathsf{T}} = \left(\mathrm{e}^A\right)^{\mathsf{T}} = \mathrm{e}^{A^{\mathsf{T}}} = \mathrm{e}^{-A} = (\exp(A))^{-1},$$

so $\exp(A) \in \mathrm{O}(n)$. Conversely, suppose that $A \in U$ and that $\exp(A) \in \mathrm{O}(n) \cap V$. Then

$$\exp(-A) = (\exp(A))^{-1} = (\exp(A))^{\mathsf{T}} = \exp\left(A^{\mathsf{T}}\right),$$

which implies that $-A = A^{\mathsf{T}}$ since $-A$ and A^{T} also belong to U and since the exponential map is bijective on U. Thus $A \in U \cap \mathfrak{o}(n)$. It follows from the theorem above that $\mathfrak{g}_{\mathrm{O}(n)} = \mathfrak{o}(n)$.

(4) Clearly $\mathfrak{u}(n)$ is a subspace of $\mathfrak{gl}(n,\mathbb{C})$. Let U be an open neighborhood of 0 in $\mathfrak{gl}(n,\mathbb{C})$, diffeomorphic under the exponential map to an open neighborhood V of $\mathbb{1}_{n\times n}$ in $\mathrm{GL}(n,\mathbb{C})$. We can assume, in addition, that if $A \in U$, then $\overline{A}$, A^{T}, and $-A$ belong to U. For let W be an open neighborhood of 0 in $\mathfrak{gl}(n,\mathbb{C})$ that is small enough for the exponential map to be a diffeomorphism, and then let $U = W \cap \overline{W} \cap W^{\mathsf{T}} \cap (-W)$. If $A \in U \cap \mathfrak{u}(n)$, then

$$(\exp(A))^{\mathsf{H}} = \left(\overline{\mathrm{e}^A}\right)^{\mathsf{T}} = \left(\mathrm{e}^{\overline{A}}\right)^{\mathsf{T}} = \mathrm{e}^{(\overline{A})^{\mathsf{T}}} = \mathrm{e}^{A^{\mathsf{H}}} = \mathrm{e}^{-A} = \exp(-A) = (\exp(A))^{-1},$$

so $\exp(A) \in \mathrm{U}(n)$. Conversely, suppose that $A \in U$ and that $\exp(A) \in \mathrm{U}(n) \cap V$. Then

$$\exp(-A) = (\exp(A))^{-1} = (\exp(A))^{\mathsf{H}} = \left(\overline{\mathrm{e}^A}\right)^{\mathsf{T}} = \mathrm{e}^{(\overline{A})^{\mathsf{T}}} = \exp\left(\overline{A}^{\mathsf{T}}\right),$$

which implies that $-A = \left(\overline{A}\right)^{\mathsf{T}}$ since $-A$ and $\left(\overline{A}\right)^{\mathsf{T}}$ also belong to U and since the exponential map is bijective on U. Thus $A \in U \cap \mathfrak{u}(n)$. It follows from the above theorem that $\mathfrak{g}_{\mathrm{U}(n)} = \mathfrak{u}(n)$.

(5) Clearly $\mathfrak{sp}(2n)$ is a subspace of $\mathfrak{gl}(2n,\mathbb{R})$. Let U be an open neighborhood of 0 in $\mathfrak{gl}(2n,\mathbb{R})$, diffeomorphic under the exponential map to an open neighborhood V of $\mathbb{1}_{2n\times 2n}$ in $\mathrm{GL}(2n,\mathbb{R})$. We can assume, in addition, that if $A \in U$, then A^{T} and $J(-A)J^{-1}$ belong to U. For let W be an open neighborhood of 0 in $\mathfrak{gl}(2n,\mathbb{R})$ that is small enough for the exponential map to be a diffeomorphism, and then let $U = W \cap W^{\mathsf{T}} \cap J(-W)J^{-1}$. If $A \in \mathfrak{sp}(2n)$, then

$$A^{\mathsf{T}}J = -JA \implies A^{\mathsf{T}} = J(-A)J^{-1} \implies \mathrm{e}^{A^{\mathsf{T}}} = \mathrm{e}^{J(-A)J^{-1}} = J\mathrm{e}^{-A}J^{-1}.$$

It follows that

$$(\exp(A))^{\mathsf{T}}J\exp(A) = \mathrm{e}^{A^{\mathsf{T}}}J\mathrm{e}^{A} = J\mathrm{e}^{-A}J^{-1}J\mathrm{e}^{A} = J,$$

so $\exp(A) \in \mathrm{Sp}(2n)$. Conversely, suppose that $A \in U$ and that $\exp(A) \in \mathrm{Sp}(2n) \cap V$. Then

$$\mathrm{e}^{A^{\mathsf{T}}} J \mathrm{e}^{A} = J \implies \mathrm{e}^{A^{\mathsf{T}}} = J \mathrm{e}^{-A} J^{-1} = \mathrm{e}^{J(-A)J^{-1}},$$

which implies that $A^{\mathsf{T}} = J(-A)J^{-1}$ since A^{T} and $J(-A)J^{-1}$ also belong to U and since the exponential map is bijective on U. Thus $A^{\mathsf{T}} J = -JA$ and $A \in U \cap \mathrm{Sp}(2n)$. It follows from the above theorem that $\mathfrak{g}_{\mathrm{Sp}(2n)} = \mathfrak{sp}(2n)$. ☐

Exercise 47 Given a Lie group G, prove the following equality

$$\exp(-tX)\exp(-tY)\exp(tX)\exp(tY) = \exp\big(t^2[X,Y] + O(t^3)\big)$$

for any $X, Y \in \mathfrak{g}_G$, when parameter t is sufficiently small.

Proof For any $X \in \mathfrak{g}_G$, $g \in G$ and $t \in \mathbb{R}$, we have

$$(Xf)(g\exp(tX)) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f(g\exp(tX)\exp(sX)) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f(g\exp((t+s)X)) = \frac{\mathrm{d}}{\mathrm{d}t} f(g\exp(tX)).$$

Using this, one can show by induction that

$$(X^n f)(g\exp(tX)) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(g\exp(tX)).$$

In particular, we have

$$(X^n f)(g) = \left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right|_{t=0} f(g\exp(tX)).$$

Using this formula twice, we get

$$(X^n Y^m f)(e) = \left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right|_{t=0} (Y^m f)(\exp(tX)) = \left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right|_{t=0} \left.\frac{\mathrm{d}^m}{\mathrm{d}s^m}\right|_{s=0} f(\exp(tX)\exp(sY)).$$

Therefore, the Taylor series for $f(\exp(tX)\exp(sY))$ is

$$f(\exp(tX)\exp(sY)) = \sum_{m,n=0}^{\infty} \frac{t^n}{n!}\frac{s^m}{m!}(X^n Y^m f)(e)$$

for sufficiently small t and s. When $s = t$, we obtain

$$f(\exp(tX)\exp(tY)) = f(e) + t[(Xf)(e) + (Yf)(e)] + \frac{t^2}{2}\big[\big(X^2 f\big)(e) + 2(XYf)(e) + \big(Y^2 f\big)(e)\big] + O\big(t^3\big).$$

Now apply this formula to the inverse of the exponential map near e, i.e., the map f defined by

$$f(\exp(tX)) = tX$$

for t sufficiently small. Then $f(e) = 0$,

$$(Xf)(e) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\exp(tX)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (tX) = X,$$

and for any $n > 1$,

$$(X^n f)(e) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\bigg|_{t=0} f(\exp(tX)) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\bigg|_{t=0} (tX) = 0.$$

Note that

$$X^2 + 2XY + Y^2 = (X+Y)^2 + [X,Y],$$

it follows that

$$f(\exp(tX)\exp(tY)) = t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3).$$

Thus

$$\exp(tX)\exp(tY) = \exp\left\{t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\right\}.$$

Using this formula twice, we get

$$\begin{aligned}
&\exp(-tX)\exp(-tY)\exp(tX)\exp(tY) \\
=&\exp\left\{t\left(-(X+Y) + \frac{t}{2}[X,Y] + O(t^2)\right)\right\}\exp\left\{t\left((X+Y) + \frac{t}{2}[X,Y] + O(t^2)\right)\right\} \\
=&\exp\left\{t\big(t[X,Y] + O(t^2)\big) + \frac{t^2}{2}\left[-(X+Y) + \frac{t}{2}[X,Y], (X+Y) + \frac{t}{2}[X,Y]\right] + O(t^3)\right\} \\
=&\exp\big(t^2[X,Y] + O(t^3)\big).
\end{aligned}$$

□

Exercise 48 Prove that the matrix exponential map on elements in $M_{n\times n}(\mathbb{R})$ satisfies

$$\det(\mathrm{e}^A) = \mathrm{e}^{\mathrm{tr}(A)}.$$

Here, $\mathrm{e}^A = \mathbb{1} + A + \frac{A^2}{2} + \cdots$. Please provide all necessary details in your argument. Use this conclusion to confirm that the following matrix

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

can *not* be written as e^A for any $A \in M_{2\times 2}(\mathbb{R})$.

Proof (1) Let $\|\cdot\|$ be a matrix norm on $M_{n\times n}(\mathbb{C})$. Then

$$\left\|\sum_{k=0}^{\infty}\frac{A^k}{k!}\right\| \leqslant \sum_{k=0}^{\infty}\frac{\|A^k\|}{k!} \leqslant \sum_{k=0}^{\infty}\frac{\|A\|^k}{k!} = \mathrm{e}^{\|A\|} < \infty,$$

so the series $\displaystyle\sum_{k=0}^{\infty}\frac{A^k}{k!}$ converges for any $A \in M_{n\times n}(\mathbb{C})$.

Since any complex square matrix is triangularizable, one can find $P \in \mathrm{GL}(n,\mathbb{C})$ such that

$$A = P\begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_n \end{pmatrix}P^{-1}, \quad \text{where } \lambda_1, \cdots, \lambda_n \in \mathbb{C}.$$

Then

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} P \begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_n \end{pmatrix}^k P^{-1} = P \begin{pmatrix} e^{\lambda_1} & * & * \\ & \ddots & * \\ & & e^{\lambda_n} \end{pmatrix} P^{-1}. \tag{48–1}$$

It follows that $\det(e^A) = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\mathrm{tr}(A)}$.

(2) Suppose $e^A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ for $A \in M_{2\times 2}(\mathbb{R})$ and the eigenvalues of A are α and β. By (48–1), we can assume $e^\alpha = -2$ and $e^\beta = -1$. Hence $\alpha, \beta \notin \mathbb{R}$ and they must be complex conjugates of each other. However, $|e^A| \neq |e^\beta|$, which is a contradiction. □

Exercise 49 Given a Riemannian metric g, recall that the associated curvature tensor (as a $(0,4)$-tensor) is defined by

$$R(X, Y, Z, W) := g(R(X, Y)Z, W)$$

for vector fields X, Y, Z, W. Prove the following equalities.

(1) $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$.

(2) $R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z)$.

(3) $R(X, Y, Z, W) = R(Z, W, X, Y)$.

Proof (1) Since

$$\begin{aligned} &R(X, Y)Z + R(Y, Z)X + R(Z, X)Y \\ =&(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) + (\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X) \\ &+ (\nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y) \\ =&\nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) \\ &- \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\ =&\nabla_X([Y, Z]) + \nabla_Y([Z, X]) + \nabla_Z([X, Y]) - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\ =&[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\ =&0, \end{aligned}$$

we have $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$.

(2) Since $R(X, Y)Z = -R(Y, X)Z$, we have $R(X, Y, Z, W) = -R(Y, X, Z, W)$. Using compatibility with the metric, we have

$$\begin{aligned} XY|Z|^2 &= X(2\langle \nabla_Y Z, Z\rangle) = 2\langle \nabla_X \nabla_Y Z, Z\rangle + 2\langle \nabla_Y Z, \nabla_X Z\rangle, \\ YX|Z|^2 &= Y(2\langle \nabla_X Z, Z\rangle) = 2\langle \nabla_Y \nabla_X Z, Z\rangle + 2\langle \nabla_X Z, \nabla_Y Z\rangle, \\ [X, Y]|Z|^2 &= 2\langle \nabla_{[X,Y]} Z, Z\rangle. \end{aligned}$$

Subtracting the second and third equations from the first, we get

$$0 = 2\langle \nabla_X \nabla_Y Z, Z\rangle - 2\langle \nabla_Y \nabla_X Z, Z\rangle - 2\langle \nabla_{[X,Y]} Z, Z\rangle$$

$$
\begin{aligned}
&= 2\langle R(X,Y)Z, Z\rangle \\
&= 2R(X,Y,Z,Z).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
0 &= R(X,Y,Z+W,Z+W) \\
&= R(X,Y,Z,Z) + R(X,Y,W,W) + R(X,Y,Z,W) + R(X,Y,W,Z) \\
&= R(X,Y,Z,W) + R(X,Y,W,Z).
\end{aligned}
$$

(3) Writing the identity in (1) four times with indices cyclically permuted gives

$$
\begin{aligned}
R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) &= 0, \\
R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) &= 0, \\
R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) &= 0, \\
R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) &= 0.
\end{aligned}
$$

Now add up all four equations. Applying (2) makes all the terms in the first columns cancel, and in the last column it yields

$$
2R(Z,X,Y,W) + 2R(W,Y,Z,X) = 0,
$$

which is equivalent to $R(X,Y,Z,W) = R(Z,W,X,Y)$. □

Exercise 50 Consider the following (real) 2-dimensional Lie group

$$
G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbb{R} \right\}.
$$

Complete the following questions.

(1) Verify that its Lie algebra is

$$
\mathfrak{g}_G = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}.
$$

(2) Take the following basis of $\mathfrak{g}_G$ in (1),

$$
X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
$$

Construct a left-invariant metric g on G such that $\{X_1, X_2\}$ forms an orthonormal basis.

(3) Verify that the Levi-Civita connection ∇ of g in (2) satisfies the following relations,

$$
\nabla_{X_1} X_1 = \nabla_{X_1} X_2 = 0, \quad \nabla_{X_2} X_1 = -X_2, \quad \nabla_{X_2} X_2 = X_1.
$$

(4) Compute sectional curvatures of (G, g, ∇) for g and ∇ in (2) and (3).

Proof (1) Let $A=\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}$, where $a,b\in\mathbb{R}$.

- If $a=0$, then $A^n=0$ for all $n\geqslant 2$ and $\mathrm{e}^A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$.
- If $a\neq 0$, then $A^n=a^{n-1}A$ for all $n\geqslant 1$ and

$$\mathrm{e}^A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\sum_{n=1}^{\infty}\frac{A^n}{n!}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{1}{a}\sum_{n=1}^{\infty}\frac{a^n}{n!}A=\begin{pmatrix}\mathrm{e}^a & \frac{b(\mathrm{e}^a-1)}{a}\\ 0 & 1\end{pmatrix}.$$

Clearly $\mathfrak{a}:=\left\{\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}: a,b\in\mathbb{R}\right\}$ is a subspace of $\mathfrak{gl}(2,\mathbb{R})$. Let U be an open neighborhood of 0 in $\mathfrak{gl}(2,\mathbb{R})$, diffeomorphic under the exponential map to an open neighborhood V of $\mathbb{1}_{2\times 2}$ in $\mathrm{GL}(2,\mathbb{R})$. If $B\in\mathfrak{a}$, then the above calculation implies that $\exp(B)\in G$. Conversely, suppose that $B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in U$ and that $\mathrm{e}^B=\begin{pmatrix}x & y\\ 0 & 1\end{pmatrix}\in G\cap V$. Note that $\mathrm{e}^B B=B\mathrm{e}^B$, i.e.,

$$\begin{pmatrix}ax+cy & bx+dy\\ c & d\end{pmatrix}=\begin{pmatrix}ax & ay+b\\ cx & cy+d\end{pmatrix}.$$

This implies $c=0$, and then $B^n=\begin{pmatrix}a^n & *\\ 0 & d^n\end{pmatrix}$ for all $n\geqslant 0$. Hence $\begin{pmatrix}x & y\\ 0 & 1\end{pmatrix}=\mathrm{e}^B=\begin{pmatrix}\mathrm{e}^a & *\\ 0 & \mathrm{e}^d\end{pmatrix}$ and then $d=0$. Thus $B=\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}\in\mathfrak{a}$. The theorem in the proof of Exercise 46 implies that $\mathfrak{g}_G=\mathfrak{a}$.

(2) Consider the inner product $\langle\,,\rangle$ on $\mathfrak{g}_G$ given by

$$\left\langle\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix},\begin{pmatrix}c & d\\ 0 & 0\end{pmatrix}\right\rangle=ac+bd.$$

Then define the metric g on G by

$$g_x(X,Y)=\langle (L_{x^{-1}})_*X,(L_{x^{-1}})_*Y\rangle.$$

Now for any $h\in G$, we have

$$\begin{aligned}(L_h)^*g_x(X,Y)&:=g_{hx}((L_h)_*X,(L_h)_*Y)\\ &=\left\langle\left(L_{(hx)^{-1}}\right)_*(L_h)_*X,\left(L_{(hx)^{-1}}\right)_*(L_h)_*Y\right\rangle\\ &=\langle (L_{x^{-1}})_*X,(L_{x^{-1}})_*Y\rangle\\ &=g_x(X,Y).\end{aligned}$$

Therefore g is left-invariant, and $\{X_1,X_2\}$ is an orthonormal basis (easily seen at the point $\mathbb{1}$). Let $\dfrac{\partial}{\partial x}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $\dfrac{\partial}{\partial y}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ be the standard coordinate vector fields on G, and denote by $\{\mathrm{d}x,\mathrm{d}y\}$ the dual basis of $\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right\}$. Then $X_1=x\frac{\partial}{\partial x}$ and $X_2=x\frac{\partial}{\partial y}$. Since G is a Lie subgroup of

$\mathrm{GL}(2,\mathbb{R})$, we get

$$X_1\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = x\frac{\partial}{\partial x},$$
$$X_2\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = x\frac{\partial}{\partial y}.$$

Then we have $g = \frac{1}{x^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$.

(3) The Lie bracket of X_1 and X_2 is

$$[X_1, X_2] = \begin{pmatrix} 0-0 & x-0 \\ 0-0 & 0-0 \end{pmatrix} = X_2.$$

For left-invariant vector fields X, Y, Z, the Koszul formula simplifies to

$$2g(\nabla_X Y, Z) = g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X). \tag{50–1}$$

Since $\{X_1, X_2\}$ is an orthonormal basis, we have

$$\nabla_{X_1}X_1 = \theta_1^2(X_1)X_2, \quad \nabla_{X_1}X_2 = \theta_2^1(X_1)X_1, \quad \nabla_{X_2}X_1 = \theta_1^2(X_2)X_2, \quad \nabla_{X_2}X_2 = \theta_2^1(X_2)X_1.$$

Using (50–1), we obtain

$$\begin{aligned} 2g(\nabla_{X_1}X_1, X_2) &= -g([X_1,X_2],X_1) - g([X_1,X_2],X_1) = -2g(X_2,X_1) = 0, \\ 2g(\nabla_{X_1}X_2, X_1) &= g([X_1,X_2],X_1) - g([X_2,X_1],X_1) = 2g(X_2,X_1) = 0, \\ 2g(\nabla_{X_2}X_1, X_2) &= g([X_2,X_1],X_2) - g([X_1,X_2],X_2) = -2g(X_2,X_2) = -2, \\ 2g(\nabla_{X_2}X_2, X_1) &= -g([X_2,X_1],X_2) - g([X_2,X_1],X_2) = 2g(X_2,X_2) = 2. \end{aligned}$$

It follows that

$$\nabla_{X_1}X_1 = 0, \quad \nabla_{X_1}X_2 = 0, \quad \nabla_{X_2}X_1 = -X_2, \quad \nabla_{X_2}X_2 = X_1.$$

(4) From (3) we see that

$$\theta_2^1(X_1) = 0 \quad \text{and} \quad \theta_2^1(X_2) = 1,$$

which implies

$$\theta_2^1 = \frac{1}{x}\,\mathrm{d}y.$$

Thus

$$\Omega_2^1 = \mathrm{d}\theta_2^1 + \sum_{k=1}^{2}\theta_2^k \wedge \theta_k^1 = \mathrm{d}\theta_2^1 = -\frac{1}{x^2}\,\mathrm{d}x \wedge \mathrm{d}y.$$

It follows that

$$R(X_1,X_2)X_2 = \sum_{j=1}^{2}\Omega_2^j(X_1,X_2)X_j = \Omega_2^1(X_1,X_2)X_1 = -X_1$$

and the sectional curvature at the point $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ is

$$R(X_1, X_2, X_2, X_1) = g(R(X_1, X_2)X_2, X_1) = g(-X_1, X_1) = -1. \qquad \square$$

Homework 7

Exercise 51 Prove that any short exact sequence of cochian complexes (of $\mathbf{k}$-modules)

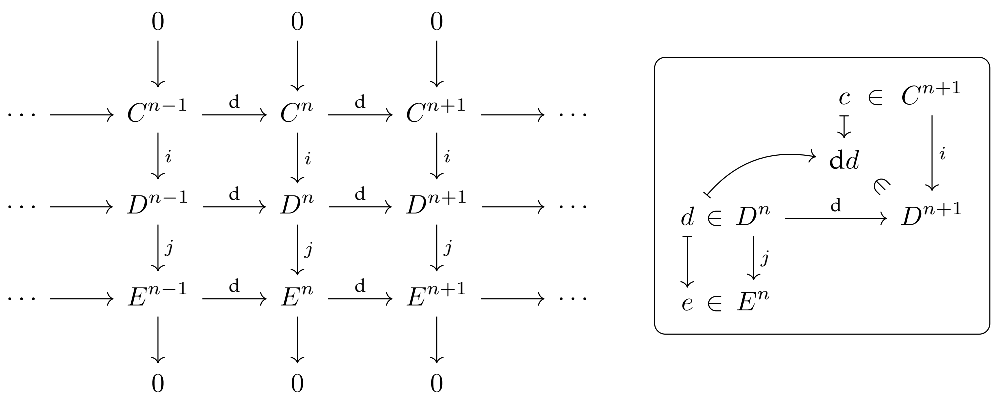
$$0 \longrightarrow (C^\bullet, d_C^\bullet) \xrightarrow{\ i\ } (D^\bullet, d_D^\bullet) \xrightarrow{\ j\ } (E^\bullet, d_E^\bullet) \longrightarrow 0$$

induces a long exact sequence on cohomoogy groups,

$$\cdots \longrightarrow H^*(C^\bullet; \mathbf{k}) \xrightarrow{\ i_*\ } H^*(D^\bullet; \mathbf{k}) \xrightarrow{\ j_*\ } H^*(E^\bullet; \mathbf{k}) \xrightarrow{\ \delta\ } H^{*+1}(C^\bullet; \mathbf{k}) \longrightarrow \cdots$$

Please provide all necessary details.

Proof By the definition of short exact sequence of cochain complexes, we have the following commutative diagram with exact columns,



The commutativity of the squares means that i and j are chain maps. These therefore induce maps i_* and j_* on cohomology. To define the boundary map $\delta : H^n(E^\bullet; \mathbf{k}) \to H^{n+1}(C^\bullet; \mathbf{k})$, let $e \in E^n$ be a cycle. Since j is surjective, $e = j(d)$ for some $d \in D^n$. The element $\mathrm{d}d \in D^{n+1}$ is in $\operatorname{Ker} j$ since $j(\mathrm{d}d) = \mathrm{d}j(d) = \mathrm{d}e = 0$. So $\mathrm{d}d = i(c)$ for some $c \in C^{n+1}$ since $\operatorname{Ker} j = \operatorname{Im} i$. Note that $\mathrm{d}c = 0$ since $i(\mathrm{d}c) = \mathrm{d}i(c) = \mathrm{d}^2 d = 0$ and i is injective. We define $\delta : H^n(E^\bullet; \mathbf{k}) \to H^{n+1}(C^\bullet; \mathbf{k})$ by sending the cohomology class of e to the cohomology class of c, $\delta[e] = [c]$. This is well-defined since:

- The element c is uniquely determined by $\mathrm{d}d$ since i is injective.
- A different choice of d' for d would have $j(d') = j(d)$, so $d' - d$ is in $\operatorname{Ker} j = \operatorname{Im} i$. Thus $d' - d = i(c')$ for some c', hence $d' = d + i(c')$. The effect of replacing d by $d + i(c')$ is to change c to the cohomologous element $c + \mathrm{d}c'$ since $i(c + \mathrm{d}c') = i(c) + i(\mathrm{d}c') = \mathrm{d}d + \mathrm{d}i(c') = \mathrm{d}(d + i(c'))$.
- A different choice of e within its cohomology class would have the form $e + \mathrm{d}e'$. Since $e' = j(d')$ for some d', we then have $e + \mathrm{d}e' = e + \mathrm{d}j(d') = e + j(\mathrm{d}d') = j(d + \mathrm{d}d')$, so d is replaced by $d + \mathrm{d}d'$, which leaves $\mathrm{d}d$ and therefore also c unchanged.

The map $\delta : H^n(E^\bullet;\mathbf{k}) \to H^{n+1}(C^\bullet;\mathbf{k})$ is a homomorphism since if $\mathrm{d}[e_1] = [c_1]$ and $\mathrm{d}[e_2] = [c_2]$ via elements d_1 and d_2 as above, then $j(d_1 + d_2) = j(d_1) + j(d_2) = e_1 + e_2$ and $i(c_1 + c_2) = i(c_1) + i(c_2) = \mathrm{d}d_1 + \mathrm{d}d_2 = \mathrm{d}(d_1 + d_2)$, so $\mathrm{d}([e_1] + [e_2]) = [c_1] + [c_2]$.

To show that the sequence is exact, there are six things to verify:

$\boxed{\operatorname{Im} i_* \subset \operatorname{Ker} j_*}$ This is immediate since $ji = 0$ implies $j_* i_* = 0$.

$\boxed{\operatorname{Im} j_* \subset \operatorname{Ker} \delta}$ We have $\delta j_* = 0$ since in this case $\mathrm{d}d = 0$ in the definition of δ.

$\boxed{\operatorname{Im} \delta \subset \operatorname{Ker} i_*}$ Here $i_*\delta = 0$ since $i_*\delta$ takes $[e]$ to $[\mathrm{d}d] = 0$.

$\boxed{\operatorname{Ker} j_* \subset \operatorname{Im} i_*}$ A cohomology class in $\operatorname{Ker} j_*$ is represented by a cycle $d \in D^n$ with $j(d)$ a boundary, so $j(d) = \mathrm{d}e'$ for some $e' \in E^{n+1}$. Since j is surjective, $e' = j(d')$ for some $d' \in D^{n-1}$. We have $j(d - \mathrm{d}d') = j(d) - j(\mathrm{d}d') = j(d) - \mathrm{d}j(d') = j(d) - \mathrm{d}e' = 0$. So $d - \mathrm{d}d' = i(c)$ for some $c \in C^n$. This c is a cycle since $i(\mathrm{d}a) = \mathrm{d}i(a) = \mathrm{d}(d - \mathrm{d}d') = \mathrm{d}d = 0$ and i is injective. Thus $i_*[c] = [d - \mathrm{d}d'] = [d]$, showing that i_* maps onto $\operatorname{Ker} j_*$.

$\boxed{\operatorname{Ker} \delta \subset \operatorname{Im} j_*}$ In the notation used in the definition of δ, if e represents a cohomology class in $\operatorname{Ker} \delta$, then $c = \mathrm{d}c'$ for some $c' \in C^n$. The element $d - i(c')$ is a cycle since $\mathrm{d}(d - i(c')) = \mathrm{d}d - \mathrm{d}i(c') = \mathrm{d}d - i(\mathrm{d}c') = \mathrm{d}d - i(c) = 0$. And $j(d - i(c')) = j(d) - ji(c') = j(d) = c$, so j_* maps $[d - i(c')]$ to $[e]$.

$\boxed{\operatorname{Ker} i_* \subset \operatorname{Im} \delta}$ Given a cycle $c \in C^{n+1}$ such that $i(c) = \mathrm{d}d$ for some $d \in D^n$, then $j(d)$ is a cycle since $\mathrm{d}j(d) = j(\mathrm{d}d) = ji(c) = 0$, and δ takes $[j(d)]$ to $[c]$. □

Exercise 52 Prove the Künneth formula of de Rham cohomology groups. Explicitly, for manifolds M and N with finite good covers, one has

$$H^k_{\mathrm{dR}}(M \times N;\mathbb{R}) \simeq \bigoplus_{0 \leqslant p,q \leqslant k,\, p+q=k} H^p_{\mathrm{dR}}(M;\mathbb{R}) \otimes_{\mathbb{R}} H^q_{\mathrm{dR}}(N;\mathbb{R})$$

for any $0 \leqslant k \leqslant \dim M + \dim N$.

Proof Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ be the standard projections. Then we get a map

$$\Psi : \Omega^*(M) \otimes \Omega^*(N) \to \Omega^*(M \times N), \quad \omega_1 \otimes \omega_2 \mapsto \pi_M^*\omega_1 \wedge \pi_N^*\omega_2.$$

One can check that this map induces a map on cohomologies:

$$\Psi : H^*_{\mathrm{dR}}(M;\mathbb{R}) \otimes_{\mathbb{R}} H^*_{\mathrm{dR}}(N;\mathbb{R}) \to H^*_{\mathrm{dR}}(M \times N;\mathbb{R}), \quad [\omega_1] \otimes [\omega_2] \mapsto [\pi_M^*\omega_1 \wedge \pi_N^*\omega_2].$$

To prove that this map is in fact a linear isomorphism, we work by induction on the number l of elements in a good cover of M.

If $l = 1$, i.e., M is diffeomorphic to $\mathbb{R}^n$, then the Künneth formula follows from the fact that $\mathbb{R}^n \times N$ is homotopy equivalent to N, and $H^k_{\mathrm{dR}}(\mathbb{R}^n)$ equals $\mathbb{R}$ for $k = 0$ and 0 otherwise.

Now suppose that the Künneth formula holds for manifolds admitting a good cover with no more than $l - 1$ open sets, and suppose that $M = U_1 \cup \cdots \cup U_l$ is a good cover. Let $U = U_1 \cup \cdots \cup U_{l-1}$ and $V = U_l$. For simplicity, we denote

$$\Sigma^k(M,N) \coloneqq \bigoplus_{0 \leqslant p,q \leqslant k,\, p+q=k} H^p_{\mathrm{dR}}(M;\mathbb{R}) \otimes_{\mathbb{R}} H^q_{\mathrm{dR}}(N;\mathbb{R}).$$

Consider the following diagram with exact rows given by the Mayer–Vietoris sequences (note that tensoring with the vector space $H^q_{\mathrm{dR}}(N;\mathbb{R})$ preserves exactness):

$$\begin{array}{ccccccc}
\Sigma^k(M,N) & \xrightarrow{\alpha} & \Sigma^k(U,N)\oplus\Sigma^k(V,N) & \xrightarrow{\beta} & \Sigma^k(U\cap V,N) & \xrightarrow{\delta} & \Sigma^{k+1}(M,N) \\
\downarrow\Psi & & \downarrow\Psi & & \downarrow\Psi & & \downarrow\Psi \\
H^k_{\mathrm{dR}}(M\times N;\mathbb{R}) & \xrightarrow{\alpha} & H^k_{\mathrm{dR}}(U\times N;\mathbb{R})\oplus H^k_{\mathrm{dR}}(V\times N;\mathbb{R}) & \xrightarrow{\beta} & H^k_{\mathrm{dR}}((U\cap V)\times N;\mathbb{R}) & \xrightarrow{\delta} & H^{k+1}_{\mathrm{dR}}(M\times N;\mathbb{R})
\end{array}$$

We must prove that this diagram commutes. The only question is in the square at extreme right because it involves the δ operator used to define the long exact sequence for the Mayer–Vietoris sequence. We start with an element of $\Sigma^k(U\cap V,N)$ in the upper left corner of this square. We can deal with each element of this sum separately, so ignore the "$\oplus$" sign. Let $[\omega_1]\otimes[\omega_2]$ be in $H^p_{\mathrm{dR}}(U\cap V;\mathbb{R})\otimes_{\mathbb{R}} H^{k-p}_{\mathrm{dR}}(N;\mathbb{R})$. Then

$$\Psi\delta([\omega_1]\otimes[\omega_2]) = \pi_M^*(\delta[\omega_1])\wedge\pi_N^*[\omega_2],$$
$$\delta\Psi([\omega_1]\otimes[\omega_2]) = \delta[\pi_M^*\omega_1\wedge\pi_N^*\omega_2].$$

Let $\{\rho_U,\rho_V\}$ be a partition of unity subordinate to $\{U,V\}$ such that $\mathrm{supp}(\rho_U)\Subset U$ and $\mathrm{supp}(\rho_V)\Subset V$. To find out δ, we let $[\omega]\in H^p_{\mathrm{dR}}(U\cap V;\mathbb{R})$ represented by ω and $\tau=(\rho_U\omega,-\rho_V\omega)\in\Omega^p(U)\oplus\Omega^p(V)$, so that $\beta[\tau]=[\rho_U\omega-(-\rho_V\omega)]=[\omega]$. By diagram chasing, one has

$$\delta[\omega]=\begin{cases}[\mathrm{d}(\rho_U\omega)], & \text{on } U,\\ -[\mathrm{d}(\rho_V\omega)], & \text{on } V.\end{cases} \tag{52–1}$$

Since the pullback functions $\{\pi_M^*\rho_U,\pi^*\rho_V\}$ form a partition of unity on $M\times N$ subordinate to the cover $\{U\times N,V\times N\}$, by (52–1), on $(U\cap V)\times N$ we have

$$\delta[\pi_M^*\omega_1\wedge\pi_N^*\omega_2]=[\mathrm{d}((\pi_M^*\rho_U)\pi_M^*\omega_1\wedge\pi_N^*\omega_2)]=[\mathrm{d}\pi_M^*(\rho_U\omega_1)]\wedge\pi_N^*[\omega_2]=\pi_M^*(\delta[\omega_1])\wedge\pi_N^*[\omega_2].$$

So the diagram is commutative.

Now the second and the third Ψ in this commutative diagram are linear isomorphisms by the induction hypothesis. Thus the other Ψ are also linear isomorphisms by the five lemma. ☐

Exercise 53 Compute the de Rham cohomology groups (over $\mathbb{R}$) of the real projective space $\mathbb{R}P^n$ using Mayer–Vietoris sequence.

Solution We work by induction on n to show that

$$H^k_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R})=\begin{cases}\mathbb{R}, & \text{if } k=0,\\ \mathbb{R}, & \text{if } k=n \text{ is odd},\\ 0, & \text{otherwise}.\end{cases} \tag{53–1}$$

For $n=1$, (53–1) is true since $\mathbb{R}P^1\simeq\mathbb{S}^1$. Now suppose that (53–1) holds for $1,\cdots,n-1$ ($n\geqslant 2$). Let

$$U=\mathbb{R}P^n\setminus\{[0:\cdots:0:1]\}\simeq\mathbb{R}P^{n-1},$$
$$V=\mathbb{R}P^n\setminus\mathbb{R}P^{n-1}=\{[x_0:\cdots:x_n]\in\mathbb{R}P^n : x_n\neq 0\}\simeq\mathbb{R}^n.$$

Then

$$U \cap V \simeq \mathbb{R}^n \setminus \{0\} \sim \mathbb{S}^{n-1}.$$

Define the inclusion map

$$i : \mathbb{R}P^{n-1} \to U, \quad [x_0 : \cdots : x_{n-1}] \mapsto [x_0 : \cdots : x_{n-1} : 0]$$

and the projection map

$$\pi : U \to \mathbb{R}P^{n-1}, \quad [x_0 : \cdots : x_{n-1} : x_n] \mapsto [x_0 : \cdots : x_{n-1}].$$

Then we have $\pi \circ i = \mathrm{Id}_{\mathbb{R}P^{n-1}}$ and $i \circ \pi \sim \mathrm{Id}_U$. So U is homotopy equivalent to $\mathbb{R}P^{n-1}$. The Mayer–Vietoris sequence for $\mathbb{R}P^n$ is

$$\begin{aligned}
0 \to H^0_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R}) &\longrightarrow H^0_{\mathrm{dR}}(\mathbb{R}P^{n-1};\mathbb{R}) \oplus H^0_{\mathrm{dR}}(\mathbb{R}^n;\mathbb{R}) \longrightarrow H^0_{\mathrm{dR}}(\mathbb{S}^{n-1};\mathbb{R}) \dashrightarrow \\
\dashrightarrow H^1_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R}) &\longrightarrow H^1_{\mathrm{dR}}(\mathbb{R}P^{n-1};\mathbb{R}) \oplus H^1_{\mathrm{dR}}(\mathbb{R}^n;\mathbb{R}) \longrightarrow H^1_{\mathrm{dR}}(\mathbb{S}^{n-1};\mathbb{R}) \dashrightarrow \\
\dashrightarrow \cdots\cdots\cdots\cdots &\longrightarrow H^{k-1}_{\mathrm{dR}}(\mathbb{R}P^{n-1};\mathbb{R}) \oplus H^{k-1}_{\mathrm{dR}}(\mathbb{R}^n;\mathbb{R}) \longrightarrow H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{n-1};\mathbb{R}) \dashrightarrow \\
\dashrightarrow H^k_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R}) &\longrightarrow H^k_{\mathrm{dR}}(\mathbb{R}P^{n-1};\mathbb{R}) \oplus H^k_{\mathrm{dR}}(\mathbb{R}^n;\mathbb{R}) \longrightarrow H^k_{\mathrm{dR}}(\mathbb{S}^{n-1};\mathbb{R}) \to \cdots
\end{aligned}$$

The first two cases in (53–1) are immediate from the facts that $\mathbb{R}P^n$ is connected and is orientable if and only if n is odd. So we are left to show that $H^k_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R}) = 0$ for $1 \leqslant k \leqslant n-1$.

- If n is odd and $n \geqslant 3$, then $H^k_{\mathrm{dR}}(\mathbb{R}P^{n-1};\mathbb{R}) = 0$ for $k \geqslant 1$ by the induction hypothesis. From the above Mayer–Vietoris sequence, we have

  $$\begin{aligned}
  \cdots &\longrightarrow 0 \dashrightarrow \\
  \dashrightarrow H^k_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R}) &\longrightarrow 0 \oplus 0 \longrightarrow \cdots
  \end{aligned}$$

  which implies $H^k_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R}) = 0$.

- If n is even and $n \geqslant 2$, then $H^k_{\mathrm{dR}}(\mathbb{R}P^{n-1};\mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } k = n-1, \\ 0, & \text{otherwise.} \end{cases}$ for $1 \leqslant k \leqslant n-1$ by the induction hypothesis. When $k < n-1$, the same argument as above shows that $H^k_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R}) = 0$. When $k = n-1$, the Mayer–Vietoris sequence gives

  $$\begin{aligned}
  \cdots &\longrightarrow 0 \dashrightarrow \\
  \dashrightarrow H^{n-1}_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R}) &\longrightarrow \mathbb{R} \oplus 0 \longrightarrow \mathbb{R} \dashrightarrow \\
  \dashrightarrow \underbrace{H^n_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R})}_{=0 \text{ since } n \text{ is even}} &\longrightarrow \cdots
  \end{aligned}$$

which implies $H^{n-1}_{\mathrm{dR}}(\mathbb{R}P^n;\mathbb{R})=0$.

Therefore (53–1) holds for all $n\geqslant 1$. □

Exercise 54 Let M be a compact oriented manifold. Prove that if $\dim M=4n+2$, then its Euler characteristic $\chi(M)$ is even.

Proof Without loss of generality, assume M is connected. Since M is compact, $H^*_{\mathrm{dR}}(M;\mathbb{R})$ is finite-dimensional over $\mathbb{R}$ by the de Rham theorem. Moreover, by Poincaré duality,

$$H^*_{\mathrm{dR}}(M;\mathbb{R})\simeq\left(H^{4n+2-*}_{\mathrm{c}}(M;\mathbb{R})\right)^*=\left(H^{4n+2-*}_{\mathrm{dR}}(M;\mathbb{R})\right)^*\simeq H^{4n+2-*}_{\mathrm{dR}}(M;\mathbb{R}).$$

Thus

$$\begin{aligned}\chi(M)&=\sum_{k=0}^{4n+2}(-1)^k\dim_{\mathbb{R}}H^k_{\mathrm{dR}}(M;\mathbb{R})\\&=\sum_{k=0}^{2n}\left[(-1)^k+(-1)^{4n+2-k}\right]\dim_{\mathbb{R}}H^k_{\mathrm{dR}}(M;\mathbb{R})+(-1)^{2n+1}\dim_{\mathbb{R}}H^{2n+1}_{\mathrm{dR}}(M;\mathbb{R})\\&=2\sum_{k=0}^{2n}(-1)^k\dim_{\mathbb{R}}H^k_{\mathrm{dR}}(M;\mathbb{R})-\dim_{\mathbb{R}}H^{2n+1}_{\mathrm{dR}}(M;\mathbb{R}).\end{aligned}$$

So the parity of $\chi(M)$ is determined by the parity of $\dim_{\mathbb{R}}H^{2n+1}_{\mathrm{dR}}(M;\mathbb{R})$. Now consider the pairing

$$P:H^{2n+1}_{\mathrm{dR}}(M;\mathbb{R})\times H^{2n+1}_{\mathrm{dR}}(M;\mathbb{R})\to H^{4n+2}_{\mathrm{dR}}(M;\mathbb{R}),\quad([\alpha],[\beta])\mapsto[\alpha\wedge\beta].$$

Since $2n+1$ is odd, we have

$$P([\alpha],[\beta])=(-1)^{(2n+1)(2n+1)}P([\beta],[\alpha])=-P([\beta],[\alpha]).$$

Assume $H^{2n+1}_{\mathrm{dR}}(M;\mathbb{R})\simeq\mathbb{R}^m$ for some m, and note that $H^{4n+2}_{\mathrm{dR}}(M;\mathbb{R})\simeq H^0_{\mathrm{dR}}(M;\mathbb{R})\simeq\mathbb{R}$ by Poincaré duality and the connectedness of M. Then, P simply defines an antisymmetric bilinear form $\phi:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$. Hence we can represent ϕ by a non-singular skew-symmetric matrix $A\in M_{m\times m}(\mathbb{R})$. Then

$$\det(A)=\det\left(-A^{\mathsf{T}}\right)=(-1)^m\det(A)$$

implies that m is even since $\det(A)\neq 0$, and we conclude that $\chi(M)$ is even. □

Exercise 55 Complete the following two questions on mapping degree.

(1) Let $f:\mathbb{T}^n\to\mathbb{T}^n$ be the map $f\left(\mathrm{e}^{\mathrm{i}\theta_1},\cdots,\mathrm{e}^{\mathrm{i}\theta_n}\right)=\left(\mathrm{e}^{\mathrm{i}k_1\theta_1},\cdots,\mathrm{e}^{\mathrm{i}k_n\theta_n}\right)$. Compute $\deg(f)$.

(2) Prove that there does *not* exist a map $\mathbb{S}^2\times\mathbb{S}^2\to\mathbb{C}P^2$ with odd degree.

Proof (1) The wedge product

$$[\omega]:=[\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_n],$$

where $\theta_1,\cdots,\theta_n$ are angular coordinates on $\mathbb{T}^n$, is a generator of $H^n_{\mathrm{dR}}(\mathbb{T}^n;\mathbb{R})=H^n_{\mathrm{c}}(\mathbb{T}^n;\mathbb{R})$. The map f induces a pullback f^* on differential forms:

$$f^*(\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_n)=\mathrm{d}(k_1\theta_1)\wedge\cdots\mathrm{d}(k_n\theta_n)=(k_1\cdots k_n)\,\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_n.$$

So $f^*[\omega] = k_1 \cdots k_n[\omega]$ and $\deg(f) = k_1 \cdots k_n$.

(2) Recall the cohomologies of $\mathbb{C}P^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$:

$$H^2\big(\mathbb{C}P^2;\mathbb{Z}\big) = \mathbb{Z}[\alpha], \quad H^4\big(\mathbb{C}P^2;\mathbb{Z}\big) = \mathbb{Z}[[\alpha] \cup [\alpha]],$$
$$H^2\big(\mathbb{S}^2 \times \mathbb{S}^2;\mathbb{Z}\big) = \mathbb{Z}[\pi_1^*\alpha_1] \oplus \mathbb{Z}[\pi_2^*\alpha_2], \quad H^4\big(\mathbb{S}^2 \times \mathbb{S}^2;\mathbb{Z}\big) = \mathbb{Z}[[\pi_1^*\alpha_1] \cup [\pi_2^*\alpha_2]],$$

where $[\alpha_1]$ and $[\alpha_2]$ are both generators of $H^2\big(\mathbb{S}^2;\mathbb{Z}\big)$, and π_1 and π_2 are the standard projections. Let $f : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{C}P^2$ and suppose

$$f^*[\alpha] = k_1[\pi_1^*\alpha_1] + k_2[\pi_2^*]\alpha_2, \quad k_1, k_2 \in \mathbb{Z}.$$

Then

$$\begin{aligned} f^*([\alpha] \cup [\alpha]) &= (f^*[\alpha]) \cup (f^*[\alpha]) \\ &= (k_1[\pi_1^*\alpha_1] + k_2[\pi_2^*\alpha_2]) \cup (k_1[\pi_1^*\alpha_1] + k_2[\pi_2^*\alpha_2]) \\ &= k_1^2\pi_1^*(\underbrace{[\alpha_1] \cup [\alpha_1]}_{=0}) + k_2^2\pi_2^*(\underbrace{[\alpha_2] \cup [\alpha_2]}_{=0}) + 2k_1k_2[\pi_1^*\alpha_1] \cup [\pi_2^*\alpha_2] \\ &= 2k_1k_2[\pi_1^*\alpha_1] \cup [\pi_2^*\alpha_2]. \end{aligned}$$

So $\deg(f) = 2k_1k_2$ is even. □

Homework 8

Exercise 56 Complete the following questions on Hodge–Laplace operator.

(1) Let M be a connected closed manifold and $f : M \to \mathbb{R}$ be a smooth function. Fix a volume form Ω on M. Prove that $\Delta f = 0$ or $\Delta(f\Omega) = 0$ if and only if f is a constant function.

(2) Under the same hypothesis of (1) above. Prove that $\displaystyle\int_M f\Omega = 0$ if and only if there exists a smooth function $g : M \to \mathbb{R}$ such that $\Delta g = f$.

Proof (1) The "if" part in either case is trivial. For the "only if" part, since M is connected, f is constant if and only it is locally constant. Let us pick for every $p \in M$ a local coordinate chart (U, φ) around p such that $\varphi(U) = \mathbb{R}^n$, where $n = \dim M$, and compute Δ in terms of the local coordinates $x_1, \cdots, x_n$. For a differential k-form of the shape $F\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$. Beginning with the action of $\mathrm{d}\delta$, we obtain

$$\begin{aligned} &\mathrm{d}\delta(F\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k) \\ =&(-1)^{n(k-1)+1}\,\mathrm{d} \star \mathrm{d} \star (F\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k) \\ =&(-1)^{n(k-1)+1}\,\mathrm{d} \star \mathrm{d}(F\,\mathrm{d}x_{k+1} \wedge \cdots \wedge \mathrm{d}x_n) \\ =&(-1)^{n(k-1)+1}\sum_{i=1}^{k}\mathrm{d} \star \left(\frac{\partial F}{\partial x_i}\,\mathrm{d}x_i \wedge \mathrm{d}x_{k+1} \wedge \cdots \wedge \mathrm{d}x_n\right) \\ =&(-1)^{n(k-1)+1}\sum_{i=1}^{k}(-1)^{(k-1)(n-k)+i-1}\,\mathrm{d}\left(\frac{\partial F}{\partial x_i}\,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k\right) \end{aligned}$$

$$= \sum_{i=1}^{k} (-1)^i \left(\frac{\partial^2 F}{\partial x_i^2} \, dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \right.$$

$$\left. + \sum_{j=k+1}^{n} (-1)^{k-1} \frac{\partial^2 F}{\partial x_i \partial x_j} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \wedge dx_j \right)$$

$$= -\sum_{i=1}^{k} \frac{\partial^2 F}{\partial x_i^2} \, dx_1 \wedge \cdots \wedge dx_k + \sum_{i=1}^{k} \sum_{j=k+1}^{n} (-1)^{i+k-1} \frac{\partial^2 F}{\partial x_i \partial x_j} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \wedge dx_j.$$

The simplification of the sign in the second to last equality uses that $n(k-1) + (k-1)(n-k) = (k-1)(2n-k)$ which is even since $k(k-1)$ is always even. Meanwhile,

$$\begin{aligned}
&\delta \, d(F \, dx_1 \wedge \cdots \wedge dx_k) \\
=&(-1)^{nk+1} \star d \star (dF \wedge dx_1 \wedge \cdots \wedge dx_k) \\
=&(-1)^{nk+k+1} \star d \star \left(\sum_{j=k+1}^{n} \frac{\partial F}{\partial x_j} \, dx_1 \wedge \cdots \wedge dx_k \wedge dx_j \right) \\
=&(-1)^{nk+k+1} \star d \left(\sum_{j=k+1}^{n} (-1)^{j-k-1} \frac{\partial F}{\partial x_j} \, dx_{k+1} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \right) \\
=& \sum_{j=k+1}^{n} (-1)^{nk+j} \star \left((-1)^{j-k-1} \frac{\partial^2 F}{\partial x_j^2} \, dx_{k+1} \wedge \cdots \wedge dx_n \right. \\
&\left. + \sum_{i=1}^{k} \frac{\partial^2 F}{\partial x_i \partial x_j} \, dx_i \wedge dx_{k+1} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \right) \\
=&(-1)^{nk+k-1+k(n-k)} \sum_{j=k+1}^{n} \frac{\partial^2 F}{\partial x_j^2} \, dx_1 \wedge \cdots \wedge dx_k \\
&+ \sum_{i=1}^{k} \sum_{j=k+1}^{n} (-1)^{nk+j+(i-1)(n-k)+(k-i)(n-k-1)+n-j} \frac{\partial^2 F}{\partial x_i \partial x_j} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \wedge dx_j.
\end{aligned}$$

Now $nk + k - 1 + k(n-k) = -1 + k(2n+1-k)$ is always odd. Meanwhile $nk + j + (i-1)(n-k) + (k-i)(n-k-1) + n - j = n(k+1) + (n-k)(k-1) - (k-i) = 2kn - k(k-1) - k - i$ has the same parity as $i + k$. So we obtain

$$\begin{aligned}
&\delta \, d(F \, dx_1 \wedge \cdots \wedge dx_k) \\
=& -\sum_{j=k+1}^{n} \frac{\partial^2 F}{\partial x_j^2} \, dx_1 \wedge \cdots \wedge dx_k + \sum_{i=1}^{k} \sum_{j=k+1}^{n} (-1)^{i+k} \frac{\partial^2 F}{\partial x_i \partial x_j} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \wedge dx_j.
\end{aligned}$$

Therefore, we have

$$\Delta(F \, dx_1 \wedge \cdots \wedge dx_k) = -\sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i^2} \, dx_1 \wedge \cdots \wedge dx_k. \tag{56–1}$$

◇ Take $k = 0$. Then

$$\Delta F = 0 \iff \sum_{j=1}^{n} \frac{\partial^2 F}{\partial x_j^2} = 0.$$

By Liouville's theorem, any bounded harmonic function on $\mathbb{R}^n$ is constant. So $\Delta f = 0$ implies that f is (locally) constant.

◇ Take $k = n$. Then

$$\Delta(F\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = 0 \iff \sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i^2}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k = 0 \iff \sum_{j=1}^{n} \frac{\partial^2 F}{\partial x_j^2} = 0.$$

Thus again f is (locally) constant.

(2) By the Hodge decomposition theorem, we can write $f = \Delta g + h$ for some $g \in \Omega^0(M)$ and $h \in \mathcal{H}^0(M)$. By (1) we know that $\mathcal{H}^0(M)$ consists of constant functions on M, so h is in fact a constant. Since Δg is orthogonal to $\mathcal{H}^0(M)$, we have

$$\int_M \Delta g \Omega = 0.$$

Therefore,

$$\int_M f\Omega = 0 \iff \int_M (\Delta g + h)\Omega = 0 \iff h\,\mathrm{Vol}(M) = 0 \iff h = 0 \iff f = \Delta g.$$

□

Exercise 57 A *contact 1-form* on M^3 is a 1-form $\alpha \in \Omega^1(M)$ such that $\mathrm{d}\alpha \wedge \alpha$ is nowhere vanishing (i.e., a volume form). Complete the following questions.

(1) Prove that the hyperplane field $\mathcal{D}^2$ defined by

$$\mathcal{D}^2(p) := \mathrm{Ker}\,\alpha(p) = \{v \in T_pM : \alpha_p(v) = 0\}$$

for any $p \in M$ is *not* integrable anywhere (called *completely non-integrable*). Such a completely non-integrable $\mathcal{D}^2$ is called a *contact structure* on M^3.

(2) Following the terminology in (1) right above, for $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ in coordinates (x, y, z), prove that $\mathcal{D}^2$ defined as follows,

$$\mathcal{D}^2 = \mathrm{Span}_{\mathbb{R}}\left\langle \frac{\partial}{\partial z}, \cos(2\pi z)\frac{\partial}{\partial x} - \sin(2\pi z)\frac{\partial}{\partial y} \right\rangle$$

is a contact structure on $\mathbb{T}^3$.

(3) Draw a closed curve γ in $\mathbb{T}^3$ such that everywhere its tangent vector lies in $\mathcal{D}^2$. Note that this does *not* contradict the Frobenius integrability theorem!

Proof (1) For any $X, Y \in \mathcal{D}^2$, we have

$$\mathrm{d}\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) = 0 - 0 - \alpha([X, Y]).$$

So if $\mathcal{D}^2$ is integrable at some point $p \in M$, then by the Frobenius integrability theorem, $[X, Y]_p \in \mathcal{D}^2$ and then $\mathrm{d}\alpha(X, Y)_p = 0$. This implies that $\mathrm{d}\alpha \wedge \alpha$ vanishes at p, which is a contradiction. Therefore, $\mathcal{D}^2$ is completely non-integrable.

(2) Let $\alpha = \sin(2\pi z)\,\mathrm{d}x + \cos(2\pi z)\,\mathrm{d}y \in \Omega^1(\mathbb{T}^3)$ (it is invariant under the action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ by

translations, so it descends to a 1-form on $\mathbb{T}^3$). Then

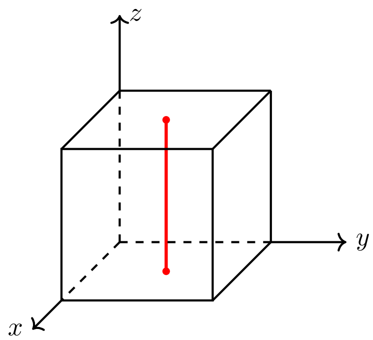
$$\mathrm{d}\alpha = -2\pi\cos(2\pi z)\,\mathrm{d}x\wedge\mathrm{d}z + 2\pi\sin(2\pi z)\,\mathrm{d}y\wedge\mathrm{d}z,$$

and

$$\mathrm{d}\alpha\wedge\alpha = \left[2\pi\cos^2(2\pi z) + 2\pi\sin^2(2\pi z)\right]\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = 2\pi\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z,$$

which is a volume form on $\mathbb{T}^3$. So α is a contact 1-form on $\mathbb{T}^3$. Obviously $\operatorname{Ker}\alpha(p)\supset\mathcal{D}^2(p)$ for any $p=(x,y,z)\in\mathbb{T}^3$. And since $\dim\operatorname{Ker}\alpha(p) = 3-1 = \dim\mathcal{D}^2(p)$, they must be equal. Hence by (1), $\mathcal{D}^2$ is completely non-integrable and thus a contact structure on $\mathbb{T}^3$.

(3) The red "line" $\gamma(t) = \left(\frac{1}{2},\frac{1}{2},t\right)$ for $t\in[0,1]$ is a closed curve in $\mathbb{T}^3$ whose tangent vector at each point $\left(\frac{1}{2},\frac{1}{2},z\right)\in\gamma([0,1])$ is $\frac{\partial}{\partial z}$.



□

Exercise 58 Use Sard's theorem and stereographic projection to prove that the n-sphere $\mathbb{S}^n$ (for $n\geqslant 2$) is simply connected. (Recall that a smooth manifold X is *simply connected* if it is connected and any smooth map $\mathbb{S}^1\to X$ can be continuously deformed to a constant map.)

Proof Let $f:\mathbb{S}^1\to\mathbb{S}^n$ be a smooth map. Sard's theorem implies that there is a point $p\in\mathbb{S}^n$ such that p is a regular value of f. Let $\sigma:\mathbb{S}^n\setminus\{p\}\to\mathbb{R}^n$ be the stereographic projection from p. If there is an $x\in\mathbb{S}^1$ such that $p=f(x)$, then $\mathrm{d}f_x: T_x\mathbb{S}^1\to T_p\mathbb{S}^n$ is a map from a 1-dimensional vector space to an n-dimensional vector space. This cannot be surjective for dimension reasons. Hence $p\notin\operatorname{Im} f$. Then $\sigma\circ f:\mathbb{S}^1\to\mathbb{R}^n$ is null-homotopic since $\mathbb{R}^n$ is contractible. That is, $\operatorname{Im} f$ is contractible and f is null-homotopic. Therefore, $\mathbb{S}^n$ $(n\geqslant 2)$ is simply connected. □

Optional Exercises

Exercise 59 Prove that if n is odd, then $\mathbb{R}P^n$ is orientable.

Proof Recall the atlas $\{(U_i, \varphi_i) : 0 \leqslant i \leqslant n\}$, where $U_i = \{[x_0, x_1, \cdots, x_n] \in \mathbb{R}P^n : x_i \neq 0\}$ and

$$\varphi_i : U_i \to \mathbb{R}^n, \quad [x_0, x_1, \cdots, x_n] \mapsto \left(\frac{x_0}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i} \cdots, \frac{x_n}{x_i}\right).$$

To show the transition maps have positive Jacobian determinant, it suffices to consider $0 \leqslant i < j \leqslant n$, since if $i = j$, the transition map is the identity map which has determinant 1, and if $i > j$, the transition is the inverse (so the determinant will still have the same sign). Now the transition maps are given by

$$\left(\varphi_j \circ \varphi_i^{-1}\right)(t_1, \cdots, t_n) = \left(\frac{t_1}{t_j}, \cdots, \frac{t_i}{t_j}, \frac{1}{t_j}, \frac{t_{i+1}}{t_j}, \cdots, \frac{t_{j-1}}{t_j}, \frac{t_{j+1}}{t_j}, \cdots, \frac{t_n}{t_j}\right).$$

Clearly, this ordering is not pretty; the factor $\frac{1}{t_j}$ seems out of place, and we have a jump in $\frac{t_{j-1}}{t_j}, \frac{t_{j+1}}{t_j}$. So, it would be nice to permute the columns $(j-1)-(i+1)+1 = j-i-1$ times so that we get the mapping

$$f_{ij}(t_1, \cdots, t_n) = \left(\frac{t_1}{t_j}, \cdots, \frac{t_{j-1}}{t_j}, \frac{1}{t_j}, \frac{t_{j+1}}{t_j}, \cdots, \frac{t_n}{t_j}\right).$$

In other words, $\left(\varphi_j \circ \varphi_i^{-1}\right) = \sigma \circ f_{ij}$, where σ is a permutation that makes $j-i-1$ many column swaps. Thus,

$$\det\!\left(\mathrm{Jac}\left(\varphi_j \circ \varphi_i^{-1}\right)(t)\right) = (-1)^{j-i-1} \det(\mathrm{Jac}(f_{ij})(t)).$$

We start by calculating the Jacobian matrix for f_{ij}:

$$\mathrm{Jac}(f_{ij})(t) = \left(\begin{array}{c|c|c} \frac{1}{t_j}\mathbb{1}_{(j-1)\times(j-1)} & \begin{matrix} -\frac{t_1}{t_j^2} \\ \vdots \\ -\frac{t_{j-1}}{t_j^2} \end{matrix} & \\ \hline & -\frac{1}{t_j^2} & \\ \hline & \begin{matrix} -\frac{t_{j+1}}{t_j^2} \\ \vdots \\ -\frac{t_n}{t_j^2} \end{matrix} & \frac{1}{t_j}\mathbb{1}_{(n-j)\times(n-j)} \end{array}\right).$$

We compute

$$\begin{aligned} &\det\!\left(\mathrm{Jac}\left(\varphi_j \circ \varphi_i^{-1}\right)(t)\right) \\ =&(-1)^{j-i-1} \det(\mathrm{Jac}(f_{ij})(t)) \end{aligned}$$

$$
\begin{aligned}
&=(-1)^{j-i-1}\left(\frac{1}{t_j}\right)^{j-1}\left(-\frac{1}{t_j^2}\right)\left(\frac{1}{t_j}\right)^{n-j}\det\left(\begin{array}{c|c|c} \mathbb{1}_{(j-1)\times(j-1)} & \begin{matrix} t_1 \\ \vdots \\ t_{j-1}\end{matrix} & \\ \hline & 1 & \\ \hline & \begin{matrix} t_{j+1} \\ \vdots \\ t_n\end{matrix} & \mathbb{1}_{(n-j)\times(n-j)} \end{array}\right) \\
&=\frac{(-1)^{j-i}}{t_j^{n+1}}\det\mathbb{1}_{n\times n} \\
&=\frac{(-1)^{j-i}}{t_j^{n+1}}.
\end{aligned}
$$

Unfortunately, these charts are not the oriented ones. Consider $\psi_i = (-1)^i\varphi_i$. Then, the transition map is

$$
(\psi_j \circ \psi_i^{-1})(t_1, \cdots, t_n) = (-1)^j\left(\frac{t_1}{t_j}, \cdots, \frac{t_i}{t_j}, \frac{(-1)^i}{t_j}, \frac{t_{i+1}}{t_j}, \cdots, \frac{t_{j-1}}{t_j}, \frac{t_{j+1}}{t_j}, \cdots, \frac{t_n}{t_j}\right).
$$

Hence,

$$
\begin{aligned}
\det\big(\mathrm{Jac}\big(\psi_j \circ \psi_i^{-1}\big)(t)\big) &= (-1)^{nj}(-1)^i\det\big(\mathrm{Jac}\big(\varphi_j \circ \varphi_i^{-1}\big)(t)\big) \\
&= (-1)^{nj}(-1)^i\frac{(-1)^{j-i}}{t_j^{n+1}} \\
&= \frac{(-1)^{(n+1)j}}{t_j^{n+1}}.
\end{aligned}
$$

Thus, for odd values of n, this determinant is positive, and hence for odd n, $\mathbb{R}P^n$ is orientable, and the ψ_i's provide an oriented atlas. □

Exercise 60 Let M^m, N^n be smooth manifolds. Prove that $M \times N$ is orientable if and only if M and N are orientable.

Proof ($\Leftarrow$) Suppose M, N are both orientable, and let $\{(U_i, \varphi_i) : i \in I\}$ and $\{(V_j, \psi_j) : j \in J\}$ be oriented atlases for M and N, respectively. Then the atlas $\{(U_i \times V_j, \varphi_i \times \psi_j) : i \in I, j \in J\}$ is an oriented atlas for $M \times N$, because

$$
\begin{aligned}
\det\mathrm{Jac}\Big((\psi_{\beta_1} \times \psi_{\beta_2}) \circ (\varphi_{\alpha_1} \times \varphi_{\alpha_2})^{-1}\Big) &= \det\mathrm{Jac}\big(\big(\psi_{\beta_1} \circ \varphi_{\alpha_1}^{-1}\big) \times \big(\psi_{\beta_2} \circ \varphi_{\alpha_2}^{-1}\big)\big) \\
&= \det\begin{pmatrix} \mathrm{Jac}\big(\psi_{\beta_1} \circ \varphi_{\alpha_1}^{-1}\big) & 0 \\ 0 & \mathrm{Jac}\big(\psi_{\beta_2} \circ \varphi_{\alpha_2}^{-1}\big) \end{pmatrix} \\
&= \det\mathrm{Jac}\big(\psi_{\beta_1} \circ \varphi_{\alpha_1}^{-1}\big)\det\mathrm{Jac}\big(\psi_{\beta_2} \circ \varphi_{\alpha_2}^{-1}\big) \\
&> 0.
\end{aligned}
$$

($\Rightarrow$) Note that any open submanifold of an orientable manifold is orientable. So if we pick an open subset $V \subset N$ homeomorphic to $\mathbb{R}^n$, then $M \times V \simeq M \times \mathbb{R}^n$ is orientable. By induction, it suffices to show that if $M \times \mathbb{R}$ is orientable, then M is orientable. Choose an open cover $\{U_\alpha : \alpha \in \Lambda\}$ of M such that there are homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$. Then $\big\{U_\alpha \times \mathbb{R}, \psi_\alpha := \varphi_\alpha \times \mathrm{Id} : U_\alpha \times \mathbb{R} \to \mathbb{R}^{m+1}\big\}$ is an atlas for $M \times \mathbb{R}$. If needed, we can modify each ψ_α by changing the sign of the first component into $\mathbb{R}^{m+1}$ to make it compatible with a fixed orientation in $M \times \mathbb{R}$. This changes correspondingly the φ_α. Thus

$$\det \operatorname{Jac}\left(\psi_\beta \circ \psi_\alpha^{-1}\right) = \det \begin{pmatrix} \operatorname{Jac}\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right) & 0 \\ 0 & 1 \end{pmatrix} = \det \operatorname{Jac}\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right) > 0.$$

Therefore $\{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ is a positive atlas of M, and M is orientable. □

Exercise 61 Prove that the inversion condition is redundant in the definition of a Lie group. That is, if G is a group with the property that the multiplication map $\mu : G \times G \to G$ is smooth, then the inverse map $i : G \to G$ is smooth.

Proof Consider the map

$$F : G \times G \to G \times G, \quad (g, h) \mapsto (g, gh).$$

Then F is smooth, since μ is smooth. It is straightforward to check that the differential of F at a point $(g, h) \in G \times G$ is given by

$$(\mathrm{d}F)_{(g,h)} : T_gG \times T_hG \to T_gG \times T_{gh}G, \quad (X, Y) \mapsto (X, (\mathrm{d}R_h)_g(X) + (\mathrm{d}L_g)_h(Y)),$$

where $R_h : G \to G$ is right multiplication by h and $L_g : G \to G$ is left multiplication by g.

The map $L_g : G \to G$ has a smooth inverse $L_{g^{-1}}$, so it is a diffeomorphism. Thus, $(\mathrm{d}L_g)_h$ is an isomorphism and hence $(\mathrm{d}F)_{(g,h)}$ is surjective. Since the domain and range have the same dimension, $(\mathrm{d}F)_{(g,h)}$ is an isomorphism. This shows that F is a local diffeomorphism. But F is bijective, so F is a diffeomorphism. In particular, its inverse

$$F^{-1} : G \times G \to G \times G, \quad (g, h) \mapsto \left(g, g^{-1}h\right)$$

is smooth, and hence the following composition is smooth:

$$g \mapsto (g, e) \overset{F^{-1}}{\mapsto} \left(g, g^{-1}\right) \mapsto g^{-1}.$$ □

Exercise 62 The *dual bundle* of a vector bundle $\pi : E \to M$ is the vector bundle $\pi^* : E^* \to M$ whose fibers are the dual spaces to the fibers of E. Prove that if $g_{\alpha\beta}(x) \in \mathrm{GL}(n, \mathbb{R})$ are the transition maps for E, then the transition maps for E^* are $\left(g_{\alpha\beta}(x)^{-1}\right)^{\mathsf{T}}$.

Proof Fix $x \in U_\alpha \cap U_\beta$ and let $\ell \in (\mathbb{R}^n)^*$. Then for any $u \in \mathbb{R}^n$,

$$\left\langle g^*_{\alpha\beta}(x)\ell, g_{\alpha\beta}(x)u \right\rangle = \left\langle (\Phi^*_\alpha)^{-1}(x, \ell), \left(\Phi_\alpha^{-1}\right)(x, u) \right\rangle = \langle \ell, u \rangle.$$

Thus for every $v \in \mathbb{R}^n$, we have

$$\left\langle g^*_{\alpha\beta}(x)\ell, v \right\rangle = \left\langle \ell, g_{\alpha\beta}(x)^{-1}v \right\rangle = \left\langle \left(g_{\alpha\beta}(x)^{-1}\right)^{\mathsf{T}}\ell, v \right\rangle.$$

Therefore, $g^*_{\alpha\beta}(x) = \left(g_{\alpha\beta}(x)^{-1}\right)^{\mathsf{T}}$. □

Exercise 63 Every vector bundle admits a connection.

Proof Assume $\pi : E \to M$ is a vector bundle and $\{(U_\alpha, \Phi_\alpha)\}$ is a system of local trivializations. Since

M is paracompact, we can replace $\{U_\alpha\}$ with a locally finite refinement and choose a smooth partition of unity $\{\rho_\alpha\}$. With the trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, any section s over U_α can be identified with a smooth map $s_\alpha : U_\alpha \to \mathbb{R}^k$. Then define ∇^α by $\nabla^\alpha_X s = \mathrm{D}_X s_\alpha$ (the directional derivative of s_α along X) for any $X \in \Gamma(TM)$. Now we can define a connection ∇ in E by

$$\nabla = \sum_\alpha \rho_\alpha \nabla^\alpha.$$

Because the set of supports of the ρ_α's is locally finite, the sum on the right-hand side has only finitely many nonzero terms in a neighborhood of each point, so it defines a smooth vector field on M. It is immediate from this definition that $\nabla_X Y$ is linear over $\mathbb{R}$ in Y and linear over $\mathcal{C}^\infty(M)$ in X. For the product rule, by direct computation,

$$\begin{aligned}
\nabla_X(fY) &= \sum_\alpha \rho_\alpha \nabla^\alpha_X(fY) \\
&= \sum_\alpha \rho_\alpha[(Xf)Y + f\nabla^\alpha_X Y] \\
&= (Xf)Y\sum_\alpha \rho_\alpha + f\sum_\alpha \rho_\alpha \nabla^\alpha_X Y \\
&= (Xf)Y + f\nabla_X Y.
\end{aligned}$$

□

Exercise 64 Prove that if $\varphi : G \to H$ is a Lie group homomorphism, then $(\mathrm{d}\varphi)(e) : \mathfrak{g}_G \to \mathfrak{g}_H$ is a Lie algebra homomorphism.

Proof Since $(\mathrm{d}\varphi)(e)$ is a linear map, it suffices to show that $(\mathrm{d}\varphi)(e)$ preserves the Lie bracket. This follows from the naturality of Lie brackets (see the proposition below) that

$$[(\mathrm{d}\varphi)(e)(v), (\mathrm{d}\varphi)(e)(w)] = (\mathrm{d}\varphi)(e)([v,w]), \quad \forall v, w \in \mathfrak{g}_G.$$

(GTM 218, Proposition 8.30) *Let $F : M \to N$ be a smooth map between manifolds with or without boundary, and let $X_1, X_2 \in \Gamma(TM)$ and $Y_1, Y_2 \in \Gamma(TN)$ be vector fields such that X_i is F-related to Y_i for $i = 1, 2$. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.* □

Exercise 65 If $\pi_1(M)$ is a finite group, then $H^1_{\mathrm{dR}}(M;\mathbb{R}) = 0$.

Proof Choose $\omega \in \Omega^1(M)$ with $\mathrm{d}\omega = 0$ and fix any base point x_0 in M. For any loop γ in M based at x_0, we have $[\gamma]_p^n = e$ in $\pi_1(M, x_0)$ for some $n \in \mathbb{Z} \setminus \{0\}$, since $|\pi_1(M)| < \infty$. Hence, there exists a path homotopy $F : [0,1] \times [0,1] \to M$ such that

$$F(0,t) = \underbrace{\gamma * \cdots * \gamma}_{n}(t) \quad \text{and} \quad F(1,t) = \gamma_{x_0}(t) \equiv x_0, \text{ the constant loop at } x_0.$$

By Stokes' theorem (for manifolds with corners), we have

$$0 = \int_{[0,1]\times[0,1]} F^*\,\mathrm{d}\omega = \int_{[0,1]\times[0,1]} \mathrm{d}(F^*\omega) = \int_{[0,1]} (\gamma_{x_0})^*\omega - \int_{[0,1]} (\underbrace{\gamma * \cdots * \gamma}_{n})^*\omega = 0 - n\int_\gamma \omega.$$

Hence $\displaystyle\int_\gamma \omega = 0$ holds for any loop γ based at x_0, and so ω is exact. Therefore $H^1_{\mathrm{dR}}(M;\mathbb{R}) = 0$. □