

微分几何 (H) 作业

林晓烁

<https://xiaoshuo-lin.github.io>

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1. 微分几何 (H) 课程主页: http://staff.ustc.edu.cn/~spliu/Teach_DG2022.html.
2. 刘世平教授对本课程给出的参考书如下:
 - 微分几何, 彭家贵, 陈卿, 高等教育出版社, 2021.
 - A Comprehensive Introduction to Differential Geometry, Michael Spivak, Vol. 2, Publish or Perish, 1999.
 - Differential Geometry of Curves and Surfaces, Manfredo P. do Carmo, Dover Publications, 2016.
 - Elementary Differential Geometry, Barrett O'Neill, Academic Press, 2006.
 - 整体微分几何初步, 沈一兵, 高等教育出版社, 2009.
3. 杨振宁先生写过一首《赞陈氏级》:

天衣岂无缝, 匠心剪接成。
浑然归一体, 广邃妙绝伦。
造化爱几何, 四力纤维能。
千古存心事, 欧高黎嘉陈。



Portrait of Carl Friedrich Gauß (1777-1855)

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第一章 欧氏空间

§1.1 向量运算、欧氏变换、向量分析、向量场和微分 1-形式

题 1 设 $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$. 证明:

$$\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3) = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \mathbf{v}_2 - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \mathbf{v}_3.$$

证明 ① 若 \mathbf{v}_2 与 \mathbf{v}_3 共线, 则易知两边均为 $\mathbf{0}$, 等式成立.

② 若 \mathbf{v}_2 与 \mathbf{v}_3 不共线, 由 $\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3)$ 与 $\mathbf{v}_2 \wedge \mathbf{v}_3$ 垂直知 $\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3)$ 可由 \mathbf{v}_2 与 \mathbf{v}_3 线性表出, 设 $\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3) = \lambda \mathbf{v}_2 + \mu \mathbf{v}_3$. 由 $\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3)$ 与 \mathbf{v}_1 垂直知 $\langle \mathbf{v}_1, \lambda \mathbf{v}_2 + \mu \mathbf{v}_3 \rangle = \lambda \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \mu \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0$. 因此可设 $\lambda = \omega \langle \mathbf{v}_1, \mathbf{v}_3 \rangle$, $\mu = -\omega \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$, 其中 $\omega \in \mathbb{R}$. 取 \mathbb{R}^3 的一组标准正交基 $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 使得

$$\mathbf{v}_1 = a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3, \quad \mathbf{v}_2 = a_{22}\mathbf{e}_2 + a_{23}\mathbf{e}_3, \quad \mathbf{v}_3 = a_{33}\mathbf{e}_3.$$

将其代入

$$\mathbf{v}_1 \wedge (\mathbf{v}_2 \wedge \mathbf{v}_3) = \omega (\langle \mathbf{v}_1, \mathbf{v}_3 \rangle \mathbf{v}_2 - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \mathbf{v}_3)$$

得

$$(a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3) \wedge (a_{22}a_{33}\mathbf{e}_1) = \omega (a_{13}a_{22}a_{33}\mathbf{e}_2 - a_{12}a_{22}a_{33}\mathbf{e}_3),$$

即

$$a_{13}a_{22}a_{33}\mathbf{e}_2 - a_{12}a_{22}a_{33}\mathbf{e}_3 = \omega (a_{13}a_{22}a_{33}\mathbf{e}_2 - a_{12}a_{22}a_{33}\mathbf{e}_3),$$

因此 $\omega = 1$. 这就证明了原式. □

题 2 设 $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^3$. 证明 Lagrange 恒等式:

$$\langle \mathbf{v}_1 \wedge \mathbf{v}_2, \mathbf{v}_3 \wedge \mathbf{v}_4 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \langle \mathbf{v}_2, \mathbf{v}_4 \rangle - \langle \mathbf{v}_1, \mathbf{v}_4 \rangle \langle \mathbf{v}_2, \mathbf{v}_3 \rangle.$$

证明 由混合积的轮换对称性,

$$\langle \mathbf{v}_1 \wedge \mathbf{v}_2, \mathbf{v}_3 \wedge \mathbf{v}_4 \rangle = \langle \mathbf{v}_1 \wedge \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle = \langle \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1 \wedge \mathbf{v}_2 \rangle = \langle \mathbf{v}_3, \mathbf{v}_4 \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2) \rangle.$$

再利用题 1 结论可进一步化为

$$\langle \mathbf{v}_3, \mathbf{v}_4 \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2) \rangle = \langle \mathbf{v}_3, \langle \mathbf{v}_4, \mathbf{v}_2 \rangle \mathbf{v}_1 - \langle \mathbf{v}_4, \mathbf{v}_1 \rangle \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle \langle \mathbf{v}_2, \mathbf{v}_4 \rangle - \langle \mathbf{v}_1, \mathbf{v}_4 \rangle \langle \mathbf{v}_2, \mathbf{v}_3 \rangle.$$

□

题 3 证明: \mathbb{R}^3 中任一微分 1-形式 ϕ 都可表成 $\phi = \sum_{i=1}^3 \phi(\mathbf{u}_i) dx^i$.

证明 任取 $p \in \mathbb{R}^3$, 对任意 $\mathbf{v}_p \in T_p\mathbb{R}^3$, 设 $\mathbf{v}_p = \sum_{i=1}^3 v_p^i \mathbf{u}_i(p)$, 则

$$\phi(\mathbf{v}_p) = \phi\left(\sum_{i=1}^3 v_p^i \mathbf{u}_i(p)\right) = \sum_{i=1}^3 v_p^i \phi(\mathbf{u}_i(p)) = \sum_{i=1}^3 \phi(\mathbf{u}_i)(p) dx^i(\mathbf{v}_p) = \sum_{i=1}^3 \phi(\mathbf{u}_i) dx^i(\mathbf{v}_p).$$

□

题 4 设 $\mathbf{a}(t)$ 是向量值函数, 证明:

- (1) $|\mathbf{a}| = \text{常数}$ 当且仅当 $\langle \mathbf{a}(t), \mathbf{a}'(t) \rangle = 0$;
 (2) $\mathbf{a}(t)$ 的方向不变当且仅当 $\mathbf{a}(t) \wedge \mathbf{a}'(t) = \mathbf{0}$.

证明 (1) $|\mathbf{a}| = \text{常数} \iff \langle \mathbf{a}, \mathbf{a} \rangle = \text{常数} \iff \frac{d}{dt} \langle \mathbf{a}, \mathbf{a} \rangle = 0 \iff \langle \mathbf{a}(t), \mathbf{a}'(t) \rangle = 0$.

(2) \Rightarrow : 若 $\mathbf{a}(t)$ 方向不变, 则存在常向量 \mathbf{v} 与可微函数 f , 使得 $\mathbf{a}(t) = f(t)\mathbf{v}$. 由 $\mathbf{a}'(t) = f'(t)\mathbf{v}$ 得

$$\mathbf{a}(t) \wedge \mathbf{a}'(t) = f(t)f'(t)\mathbf{v} \wedge \mathbf{v} = \mathbf{0}.$$

\Leftarrow : 若 $\mathbf{a} = \mathbf{0}$ 则结论平凡, 下设 $\mathbf{a} \neq \mathbf{0}$. 设 $\mathbf{b}(t) = \frac{\mathbf{a}(t)}{|\mathbf{a}(t)|}$, 则由 $|\mathbf{b}(t)| \equiv 1$ 得 $\langle \mathbf{b}(t), \mathbf{b}'(t) \rangle = 0$. 设 $f(t) = |\mathbf{a}(t)|$, 则 $\mathbf{a}(t) = f(t)\mathbf{b}(t)$, $\mathbf{a}'(t) = f'(t)\mathbf{b}(t) + f(t)\mathbf{b}'(t)$. 由

$$\mathbf{0} = \mathbf{a}(t) \wedge \mathbf{a}'(t) = (f(t))^2 \mathbf{b}(t) \wedge \mathbf{b}'(t)$$

及 $(f(t))^2 = |\mathbf{a}(t)|^2 > 0$ 得 $\mathbf{b}(t)$ 与 $\mathbf{b}'(t)$ 共线, 即存在 $\lambda \in \mathbb{R}$, 使得 $\mathbf{b}'(t) = \lambda \mathbf{b}(t)$. 于是

$$0 = \langle \mathbf{b}(t), \mathbf{b}'(t) \rangle = \lambda \langle \mathbf{b}(t), \mathbf{b}(t) \rangle = \lambda,$$

故 $\mathbf{b}'(t) \equiv 0$, 即 \mathbf{b} 是常向量, $\mathbf{a}(t)$ 方向不变. □

题 5 设 \mathcal{T} 是 E^3 的一个合同变换, \mathbf{v} 与 \mathbf{w} 是 E^3 的两个向量, 求 $(\mathcal{T}\mathbf{v}) \wedge (\mathcal{T}\mathbf{w})$ 与 $\mathcal{T}(\mathbf{v} \wedge \mathbf{w})$ 的关系.

解 设 $\mathcal{T}: X \mapsto XT + P$, 其中 T 是正交方阵, 则 $(\mathcal{T}\mathbf{v}) \wedge (\mathcal{T}\mathbf{w}) = \det(T)\mathcal{T}(\mathbf{v} \wedge \mathbf{w})$.

不妨设 \mathbf{v} 与 \mathbf{w} 不共线. 于是存在 $\mathbf{u} \in E^3$, 使得方阵 $A = \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \\ \mathbf{u} \end{pmatrix}$ 可逆, 则

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} * & * & (\mathbf{v} \wedge \mathbf{w})^\top \end{pmatrix},$$

$$T^{-1}A^{-1} = (AT)^{-1} = \frac{1}{\det(A)\det(T)} \begin{pmatrix} * & * & ((\mathbf{v}T) \wedge (\mathbf{w}T))^\top \end{pmatrix}.$$

将第一式两边左乘 T^{-1} 并与第二式比较就得到

$$T^{-1}(\mathbf{v} \wedge \mathbf{w})^\top = \frac{1}{\det(T)} ((\mathbf{v}T) \wedge (\mathbf{w}T))^\top,$$

两边取转置即

$$(\mathbf{v} \wedge \mathbf{w})T = \frac{1}{\det(T)} (\mathbf{v}T) \wedge (\mathbf{w}T).$$

故

$$(\mathcal{T}\mathbf{v}) \wedge (\mathcal{T}\mathbf{w}) = \det(T)\mathcal{T}(\mathbf{v} \wedge \mathbf{w}).$$

□

§1.2 欧氏空间上的微分形式和外微分运算

题 6 设 $f, g \in C^\infty(\mathbb{R}^3)$, $h \in C^\infty(\mathbb{R})$. 验证如下性质:

- (1) $d(f + g) = df + dg$.
 (2) $d(fg) = f dg + g df$.
 (3) $d(h(f)) = h'(f) df$.

证明 (1) $d(f+g) = \sum_{i=1}^3 \frac{\partial(f+g)}{\partial x^i} dx^i = \sum_{i=1}^3 \frac{\partial f}{\partial x^i} dx^i + \sum_{i=1}^3 \frac{\partial g}{\partial x^i} dx^i = df + dg.$

(2) 我们有

$$d(fg) = \sum_{i=1}^3 \frac{\partial(fg)}{\partial x^i} dx^i = \sum_{i=1}^3 \left(f \frac{\partial g}{\partial x^i} + g \frac{\partial f}{\partial x^i} \right) dx^i = f \sum_{i=1}^3 \frac{\partial g}{\partial x^i} dx^i + g \sum_{i=1}^3 \frac{\partial f}{\partial x^i} dx^i = f dg + g df.$$

(3) $d(h(f)) = \sum_{i=1}^3 \frac{\partial h(f)}{\partial x^i} dx^i = \sum_{i=1}^3 h'(f) \frac{\partial f}{\partial x^i} dx^i = h'(f) \sum_{i=1}^3 \frac{\partial f}{\partial x^i} dx^i = h'(f) df. \quad \square$

题 7 证明: \mathbb{R}^3 中任一微分 3-形式 η 都可表成 $\eta = \eta(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) dx^1 \wedge dx^2 \wedge dx^3.$

证明 任取 $p \in \mathbb{R}^3$, 对任意 $(\mathbf{r}_p, \mathbf{s}_p, \mathbf{t}_p) \in \mathbb{R}_p^3 \times \mathbb{R}_p^3 \times \mathbb{R}_p^3$, 设 $\mathbf{r}_p = \sum_{i=1}^3 r_p^i \mathbf{u}_i(p)$, $\mathbf{s}_p = \sum_{i=1}^3 s_p^i \mathbf{u}_i(p)$, $\mathbf{t}_p = \sum_{i=1}^3 t_p^i \mathbf{u}_i(p)$, 则

$$\begin{aligned} \eta(\mathbf{r}_p, \mathbf{s}_p, \mathbf{t}_p) &= \eta\left(\sum_{i=1}^3 r_p^i \mathbf{u}_i(p), \sum_{j=1}^3 s_p^j \mathbf{u}_j(p), \sum_{k=1}^3 t_p^k \mathbf{u}_k(p)\right) = \sum_{i,j,k} r_p^i s_p^j t_p^k \eta(\mathbf{u}_i(p), \mathbf{u}_j(p), \mathbf{u}_k(p)) \\ &= \sum_{i,j,k} \eta(\mathbf{u}_i(p), \mathbf{u}_j(p), \mathbf{u}_k(p)) dx^i(\mathbf{r}_p) dx^j(\mathbf{s}_p) dx^k(\mathbf{t}_p) \\ &= \sum_{i < j < k} \eta(\mathbf{u}_i(p), \mathbf{u}_j(p), \mathbf{u}_k(p)) dx^i \wedge dx^j \wedge dx^k(\mathbf{r}_p, \mathbf{s}_p, \mathbf{t}_p) \\ &= \eta(\mathbf{u}_1(p), \mathbf{u}_2(p), \mathbf{u}_3(p)) dx^1 \wedge dx^2 \wedge dx^3(\mathbf{r}_p, \mathbf{s}_p, \mathbf{t}_p). \end{aligned}$$

\square

题 8 证明: 对任一 $\phi \in \Omega^k(\mathbb{R}^3)$ 都有 $d^2\phi = 0$, 其中 $\Omega^k(\mathbb{R}^3) := \{\mathbb{R}^3 \text{ 上的所有光滑 } k\text{-形式}\}.$

证明 只需证 $k = 0, 1$ 的情形.

① 若 $\phi \in \Omega^0(\mathbb{R}^3)$, 即 $\phi \in C^\infty(\mathbb{R}^3)$, 则

$$d(d\phi) = d\left(\sum_{i=1}^3 \frac{\partial \phi}{\partial x^i} dx^i\right) = \sum_{i=1}^3 d\left(\frac{\partial \phi}{\partial x^i}\right) \wedge dx^i = \sum_{i=1}^3 \left(\sum_{j=1}^3 \frac{\partial^2 \phi}{\partial x^j \partial x^i} dx^j \wedge dx^i\right).$$

因为 $\frac{\partial^2 \phi}{\partial x^j \partial x^i} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$ 且 $dx^j \wedge dx^i$, 所以

$$d(d\phi) = \sum_{i < j} \left(\frac{\partial^2 \phi}{\partial x^i \partial x^j} - \frac{\partial^2 \phi}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0.$$

② 若 $\phi \in \Omega^1(\mathbb{R}^3)$. 先考虑 $\phi = \phi(u_1) dx^1$, 有

$$\begin{aligned} d(d\phi) &= d(d\phi(u_1) \wedge dx^1) = d\left(\sum_{i=1}^3 \frac{\partial \phi(u_1)}{\partial x^i} dx^i \wedge dx^1\right) \\ &= d\left(\frac{\partial \phi(u_1)}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \phi(u_1)}{\partial x^3} dx^3 \wedge dx^1\right) \\ &= \frac{\partial^2 \phi(u_1)}{\partial x^3 \partial x^2} dx^3 \wedge dx^2 \wedge dx^1 + \frac{\partial^2 \phi(u_1)}{\partial x^2 \partial x^3} dx^2 \wedge dx^3 \wedge dx^1 \\ &= 0. \end{aligned}$$

同理, 对 $\phi = \phi(u_i) dx^i$ ($i = 2, 3$) 也有 $d^2\phi = 0$. 因此由线性性即得 $d^2\phi = 0, \forall \phi \in \Omega^1(\mathbb{R}^3).$

\square

题 9 设 $f, g \in \Omega^0(\mathbb{R}^3)$, $\phi, \psi \in \Omega^1(\mathbb{R}^3)$. 证明:

- (1) $d(fg) = f dg + g df$.
- (2) $d(f\phi) = df \wedge \phi + f d\phi$.
- (3) $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$.

证明 (1) 此即题 6 (2).

(2) 设 $\phi = \sum_{i=1}^3 \phi(\mathbf{u}_i) dx^i$, 则

$$\begin{aligned} d(f\phi) &= \sum_{i=1}^3 d(f\phi(\mathbf{u}_i)) \wedge dx^i = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial(f\phi(\mathbf{u}_i))}{\partial x^j} dx^j \wedge dx^i = \sum_{i=1}^3 \sum_{j=1}^3 \left(\frac{\partial f}{\partial x^j} \phi(\mathbf{u}_i) + f \frac{\partial \phi(\mathbf{u}_i)}{\partial x^j} \right) dx^j \wedge dx^i \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \phi(\mathbf{u}_i) \frac{\partial f}{\partial x^j} dx^j \wedge dx^i + f \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial \phi(\mathbf{u}_i)}{\partial x^j} dx^j \wedge dx^i = df \wedge \phi + f d\phi. \end{aligned}$$

(3) 设 $\phi = \sum_{i=1}^3 \phi(\mathbf{u}_i) dx^i$, $\psi = \sum_{j=1}^3 \psi(\mathbf{u}_j) dx^j$, 则

$$\begin{aligned} d(\phi \wedge \psi) &= d \left(\sum_{i=1}^3 \sum_{j=1}^3 \phi(\mathbf{u}_i) \psi(\mathbf{u}_j) dx^i \wedge dx^j \right) = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial(\phi(\mathbf{u}_i) \psi(\mathbf{u}_j))}{\partial x^k} dx^k \wedge dx^i \wedge dx^j \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \left(\frac{\partial \phi(\mathbf{u}_i)}{\partial x^k} \psi(\mathbf{u}_j) + \phi(\mathbf{u}_i) \frac{\partial \psi(\mathbf{u}_j)}{\partial x^k} \right) dx^k \wedge dx^i \wedge dx^j \\ &= \sum_{i=1}^3 \sum_{k=1}^3 \sum_{j=1}^3 \frac{\partial \phi(\mathbf{u}_i)}{\partial x^k} \psi(\mathbf{u}_j) dx^k \wedge dx^i \wedge dx^j - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \phi(\mathbf{u}_i) \frac{\partial \psi(\mathbf{u}_j)}{\partial x^k} dx^i \wedge dx^k \wedge dx^j \\ &= d\phi \wedge \psi - \phi \wedge d\psi. \end{aligned}$$

□

题 10 设 $\phi_i = \sum_{j=1}^3 f_{ij} dx^j$ ($i = 1, 2, 3$) 是 3 个 1-形式. 证明:

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \det \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} dx^1 \wedge dx^2 \wedge dx^3.$$

证明 用 $\tau(j_1, j_2, j_3)$ 表示 $(1, 2, 3)$ 的排列 (j_1, j_2, j_3) 的逆序数, 则

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \sum_{j_1, j_2, j_3} f_{1j_1} f_{2j_2} f_{3j_3} dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} = \sum_{(j_1, j_2, j_3) \in S_3} (-1)^{\tau(j_1, j_2, j_3)} f_{1j_1} f_{2j_2} f_{3j_3} dx^1 \wedge dx^2 \wedge dx^3.$$

由行列式定义,

$$\sum_{(j_1, j_2, j_3) \in S_3} (-1)^{\tau(j_1, j_2, j_3)} f_{1j_1} f_{2j_2} f_{3j_3} = \det \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}.$$

故结论得证.

□

第二章 曲线的局部理论

§2.1 平面曲线

题 11 设曲线 C 在极坐标 (r, θ) 下的表示为 $r = f(\theta)$, 证明: 曲线 C 的曲率表达式为

$$\kappa(\theta) = \frac{f^2(\theta) + 2\left(\frac{df}{d\theta}\right)^2 - f(\theta)\frac{d^2f}{d\theta^2}}{\left[f^2(\theta) + \left(\frac{df}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

证明 曲线 C 在 E^2 的正交标架下以 θ 为参数的表示为 $\mathbf{r}(\theta) = (x(\theta), y(\theta)) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. 由

$$\mathbf{r}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta),$$

$$\mathbf{r}''(\theta) = (f''(\theta)\cos\theta - 2f'(\theta)\sin\theta - f(\theta)\cos\theta, f''(\theta)\sin\theta + 2f'(\theta)\cos\theta - f(\theta)\sin\theta)$$

即得曲线 C 的曲率表达式

$$\kappa(\theta) = \frac{x'(\theta)y''(\theta) - x''(\theta)y'(\theta)}{\left[(x')^2 + (y')^2\right]^{\frac{3}{2}}} = \frac{f^2(\theta) + 2\left(\frac{df}{d\theta}\right)^2 - f(\theta)\frac{d^2f}{d\theta^2}}{\left[f^2(\theta) + \left(\frac{df}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

□

§2.2 空间曲线

题 12 证明: E^3 的正则曲线 $\mathbf{r}(t)$ 的曲率和挠率分别为

$$\kappa(t) = \frac{|\mathbf{r}' \wedge \mathbf{r}''|}{|\mathbf{r}'|^3}, \quad \tau(t) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{|\mathbf{r}' \wedge \mathbf{r}''|^2}.$$

证明 对弧长参数 s 与一般参数 t , 有

$$\begin{cases} \mathbf{r}'(t) = \dot{\mathbf{r}}(s)\frac{ds}{dt}, \\ \mathbf{r}''(t) = \dot{\mathbf{r}}(s)\frac{d^2s}{dt^2} + \ddot{\mathbf{r}}(s)\left(\frac{ds}{dt}\right)^2, \\ \mathbf{r}'''(t) = \dot{\mathbf{r}}(s)\frac{d^3s}{dt^3} + 3\ddot{\mathbf{r}}(s)\frac{ds}{dt}\frac{d^2s}{dt^2} + \ddot{\mathbf{r}}(s)\left(\frac{ds}{dt}\right)^3. \end{cases}$$

由 $\dot{\mathbf{r}}(s) \wedge \ddot{\mathbf{r}}(s) = \mathbf{t}(s) \wedge \kappa\mathbf{n}(s) = \kappa\mathbf{b}(s)$ 知 $|\dot{\mathbf{r}}(s) \wedge \ddot{\mathbf{r}}(s)| = \kappa$. 又有

$$\begin{aligned} |\mathbf{r}'(t)| &= |\dot{\mathbf{r}}(s)| \cdot \left|\frac{ds}{dt}\right| = \frac{ds}{dt}, \\ |\mathbf{r}'(t) \wedge \mathbf{r}''(t)| &= \left(\frac{ds}{dt}\right)^3 |\dot{\mathbf{r}}(s) \wedge \ddot{\mathbf{r}}(s)| = \kappa \left(\frac{ds}{dt}\right)^3, \\ (\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) &= \left(\frac{ds}{dt}\right)^6 (\dot{\mathbf{r}}(s), \ddot{\mathbf{r}}(s), \ddot{\mathbf{r}}(s)). \end{aligned}$$

于是

$$\kappa(t) = \frac{|\mathbf{r}' \wedge \mathbf{r}''|}{|\mathbf{r}'|^3}, \quad \tau(t) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{|\mathbf{r}' \wedge \mathbf{r}''|^2}.$$

□

题 13 设曲线

$$\mathbf{r}(t) = \begin{cases} \left(e^{-\frac{1}{t^2}}, t, 0 \right), & t < 0, \\ (0, 0, 0), & t = 0, \\ \left(0, t, e^{-\frac{1}{t^2}} \right), & t > 0. \end{cases}$$

(1) 证明: $\mathbf{r}(t)$ 是一条正则曲线, 且在 $t=0$ 处曲率 $\kappa=0$;

(2) 求 $\mathbf{r}(t)$ ($t \neq 0$ 时) 的 Frenet 标架, 并讨论 $t \rightarrow 0$ 时 Frenet 标架的极限.

解 (1) 定义函数 $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ 观察可知, 当 $x \neq 0$ 时, $f(x)$ 有任意阶导数:

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) \cdot e^{-\frac{1}{x^2}}, \quad n = 1, 2, \dots,$$

其中 $P_n(z)$ 是 $3n$ 次整系数多项式. 由

$$\frac{f(x) - f(0)}{x - 0} = \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} \rightarrow 0, \quad x \rightarrow 0$$

知 $f'(0) = 0$. 再假设 $f(x)$ 在 $x=0$ 处直至 n 阶导数都为 0. 则

$$\frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \frac{\frac{1}{x} P_n\left(\frac{1}{x}\right)}{e^{\frac{1}{x^2}}} \rightarrow 0, \quad x \rightarrow 0,$$

即 $f^{(n+1)}(0) = 0$. 故由数学归纳法可知 $f(x)$ 在 $x=0$ 处各阶导数均为 0. 于是 $f \in C^\infty(\mathbb{R})$, 由此可知曲线 $\mathbf{r}(t)$ 的每个分量都是 C^∞ 函数. 又

$$\mathbf{r}'(t) = \begin{cases} \left(\frac{2}{t^3} e^{-\frac{1}{t^2}}, 1, 0 \right), & t < 0, \\ (0, 1, 0), & t = 0, \\ \left(0, 1, \frac{2}{t^3} e^{-\frac{1}{t^2}} \right), & t > 0, \end{cases}$$

因此 $|\mathbf{r}'(t)| \neq 0, \forall t$. 故 $\mathbf{r}(t)$ 是一条正则曲线. 从而 $\mathbf{r}(t)$ 的曲率函数也是 C^∞ 函数. 当 $t < 0$ 时,

$$\mathbf{r}''(t) = \left(\left(\frac{4}{t^6} - \frac{6}{t^4} \right) e^{-\frac{1}{t^2}}, 0, 0 \right),$$

因此当 $t < 0$ 时, 曲率

$$\kappa(t) = \frac{|\mathbf{r}' \wedge \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|6t^2 - 4| e^{-\frac{1}{t^2}}}{t^6 \left(1 + \frac{4}{t^6} e^{-\frac{2}{t^2}} \right)^{\frac{3}{2}}}.$$

由 $\kappa(t)$ 的光滑性即得

$$\kappa(0) = \lim_{t \rightarrow 0^-} \kappa(t) = \lim_{t \rightarrow 0^-} \frac{|6t^2 - 4| e^{-\frac{1}{t^2}}}{t^6 \left(1 + \frac{4}{t^6} e^{-\frac{2}{t^2}} \right)^{\frac{3}{2}}} = \lim_{t \rightarrow 0^-} \frac{|6t^2 - 4| \cdot \frac{\frac{1}{t^6}}{e^{\frac{1}{t^2}}}}{\left(1 + \frac{\frac{4}{t^6}}{e^{\frac{2}{t^2}}} \right)^{\frac{3}{2}}} = 0.$$

(2) 当 $t \neq 0$ 时,

$$\mathbf{t}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \begin{cases} \left(\frac{\frac{2}{t^3} e^{-\frac{1}{t^2}}}{\sqrt{1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}}}, \frac{1}{\sqrt{1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}}}, 0 \right), & t < 0, \\ \left(0, \frac{1}{\sqrt{1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}}}, \frac{\frac{2}{t^3} e^{-\frac{1}{t^2}}}{\sqrt{1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}}} \right), & t > 0. \end{cases}$$

$$\mathbf{b}(t) = \frac{\mathbf{r}'(t) \wedge \mathbf{r}''(t)}{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|} = \begin{cases} \left(0, 0, \frac{\frac{6}{t^4} - \frac{4}{t^6}}{\left|\frac{6}{t^4} - \frac{4}{t^6}\right|}\right), & t < 0, \\ \left(\frac{\frac{4}{t^6} - \frac{6}{t^4}}{\left|\frac{4}{t^6} - \frac{6}{t^4}\right|}, 0, 0\right), & t > 0 \end{cases} = \begin{cases} \operatorname{sgn}(6t^2 - 4)(0, 0, 1), & t < 0, \\ \operatorname{sgn}(4 - 6t^2)(1, 0, 0), & t > 0. \end{cases}$$

$$\mathbf{n}(t) = \mathbf{b}(t) \wedge \mathbf{t}(t) = \begin{cases} \frac{\operatorname{sgn}(6t^2 - 4)}{\sqrt{1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}}} \left(-1, \frac{2}{t^3} e^{-\frac{1}{t^2}}, 0\right), & t < 0, \\ \frac{\operatorname{sgn}(6t^2 - 4)}{\sqrt{1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}}} \left(0, \frac{2}{t^3} e^{-\frac{1}{t^2}}, -1\right), & t > 0. \end{cases}$$

当 $t \rightarrow 0$ 时, $\operatorname{sgn}(6t^2 - 4) = -1$. 因此

$$\begin{aligned} \lim_{t \rightarrow 0^-} \mathbf{t}(t) &= (0, 1, 0) = \lim_{t \rightarrow 0^+} \mathbf{t}(t), \\ \lim_{t \rightarrow 0^-} \mathbf{b}(t) &= (0, 0, -1), \quad \lim_{t \rightarrow 0^+} \mathbf{b}(t) = (1, 0, 0), \\ \lim_{t \rightarrow 0^-} \mathbf{n}(t) &= (1, 0, 0), \quad \lim_{t \rightarrow 0^+} \mathbf{n}(t) = (0, 0, 1). \end{aligned}$$

故 $\mathbf{r}(t)$ 的 Frenet 标架在 $t \rightarrow 0^-$ 和 $t \rightarrow 0^+$ 时极限不相等. □

题 14 设弧长参数曲线 $\mathbf{r}(s)$ 的曲率 $\kappa > 0$, 挠率 $\tau > 0$, $\mathbf{b}(s)$ 是 C 的副法向量, 定义曲线 \tilde{C} :

$$\tilde{\mathbf{r}}(s) = \int_0^s \mathbf{b}(u) \, du.$$

- (1) 证明: s 是曲线 \tilde{C} 的弧长参数且 $\tilde{\kappa} = \tau$, $\tilde{\tau} = \kappa$;
- (2) 求 \tilde{C} 的 Frenet 标架.

证明 因为 $|\tilde{\mathbf{r}}'(s)| = |\mathbf{b}(s)| = 1$, 所以 s 是曲线 \tilde{C} 的弧长参数. 因为

$$\tilde{\mathbf{r}}'(s) = \mathbf{b}(s), \quad \tilde{\mathbf{r}}''(s) = \mathbf{b}'(s) = -\tau(s)\mathbf{n}(s),$$

所以

$$\tilde{\kappa}(s) = |\tilde{\mathbf{r}}''(s)| = |-\tau(s)| = \tau(s).$$

又曲线 \tilde{C} 的单位切向量为 $\mathbf{b}(s)$, 主法向量为 $-\mathbf{n}(s)$, 所以它的副法向量为

$$\mathbf{b}(s) \wedge (-\mathbf{n}(s)) = \mathbf{t}(s).$$

即 \tilde{C} 的 Frenet 标架为 $\{\tilde{\mathbf{r}}'(s); \mathbf{b}(s), -\mathbf{n}(s), \mathbf{t}(s)\}$. 故 \tilde{C} 的挠率

$$\tilde{\tau} = \langle \tilde{\mathbf{n}}'(s), \tilde{\mathbf{b}}(s) \rangle = \langle -\dot{\mathbf{n}}(s), \mathbf{t}(s) \rangle = \langle \kappa(s)\mathbf{t}(s) - \tau\mathbf{b}(s), \mathbf{t}(s) \rangle = \kappa(s).$$

□

题 15 给定曲线 $\mathbf{r}(s)$, 它的曲率和挠率分别是 κ, τ ; $\mathbf{r}(s)$ 的单位切向量 $\mathbf{t}(s)$ 可视作单位球面 S^2 上的一条曲线, 称为曲线 $\mathbf{r}(s)$ 的切线像. 证明: 曲线 $\tilde{\mathbf{r}}(s) = \mathbf{t}(s)$ 的曲率、挠率分别为

$$\tilde{\kappa} = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}, \quad \tilde{\tau} = \frac{\frac{d}{ds} \left(\frac{\tau}{\kappa}\right)}{\kappa \left[1 + \left(\frac{\tau}{\kappa}\right)^2\right]}.$$

证明 由 Frenet 公式,

$$\begin{aligned}\tilde{\mathbf{r}}'(s) &= \dot{\mathbf{t}}(s) = \kappa(s)\mathbf{n}(s), \\ \tilde{\mathbf{r}}''(s) &= \dot{\kappa}(s)\mathbf{n}(s) + \kappa(s)\dot{\mathbf{n}}(s) = \dot{\kappa}(s)\mathbf{n}(s) + \kappa(s)[- \kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s)] = -\kappa^2\mathbf{t} + \dot{\kappa}\mathbf{n} + \kappa\tau\mathbf{b}, \\ \tilde{\mathbf{r}}'''(s) &= -2\kappa\dot{\kappa}\mathbf{t} - \kappa^2\dot{\mathbf{t}} + \ddot{\kappa}\mathbf{n} + \dot{\kappa}\dot{\mathbf{n}} + \dot{\kappa}\tau\mathbf{b} + \kappa\dot{\tau}\mathbf{b} + \kappa\tau\dot{\mathbf{b}} = -3\kappa\dot{\kappa}\mathbf{t} + (\ddot{\kappa} - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\dot{\kappa}\tau + \kappa\dot{\tau})\mathbf{b}.\end{aligned}$$

于是

$$\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s) = \kappa^2\tau\mathbf{t} + \kappa^3\mathbf{b}, \quad |\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)| = \sqrt{\kappa^6 + \kappa^4\tau^2}.$$

从而曲线 $\tilde{\mathbf{r}}(s)$ 的曲率

$$\tilde{\kappa}(s) = \frac{|\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)|}{|\tilde{\mathbf{r}}'(s)|^3} = \frac{\sqrt{\kappa^6 + \kappa^4\tau^2}}{\kappa^3} = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}.$$

又

$$(\tilde{\mathbf{r}}'(s), \tilde{\mathbf{r}}''(s), \tilde{\mathbf{r}}'''(s)) = \kappa^4\dot{\tau} - \kappa^3\dot{\kappa}\tau,$$

所以曲线 $\tilde{\mathbf{r}}(s)$ 的挠率

$$\tilde{\tau}(s) = \frac{(\tilde{\mathbf{r}}'(s), \tilde{\mathbf{r}}''(s), \tilde{\mathbf{r}}'''(s))}{|\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)|^2} = \frac{\kappa^4\dot{\tau} - \kappa^3\dot{\kappa}\tau}{\kappa^6 + \kappa^4\tau^2} = \frac{\kappa\dot{\tau} - \dot{\kappa}\tau}{\kappa^3 + \kappa\tau^2} = \frac{\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)}{\kappa\left[1 + \left(\frac{\tau}{\kappa}\right)^2\right]}.$$

□

题 16 求满足 $\tau = c\kappa$ (c 为常数, $\kappa > 0$) 的曲线.

解 由 Frenet 公式,

$$\begin{pmatrix} \dot{\mathbf{t}}(s) \\ \dot{\mathbf{n}}(s) \\ \dot{\mathbf{b}}(s) \end{pmatrix} = \begin{pmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}.$$

作换元 $t(s) = \int_0^s \kappa(u) du$, 则 $dt = \kappa(s) ds$, 从而

$$\begin{pmatrix} \mathbf{t}'(t) \\ \mathbf{n}'(t) \\ \mathbf{b}'(t) \end{pmatrix} = \begin{pmatrix} & 1 & \\ -1 & & c \\ & -c & \end{pmatrix} \begin{pmatrix} \mathbf{t}(t) \\ \mathbf{n}(t) \\ \mathbf{b}(t) \end{pmatrix}.$$

由此可知

$$\mathbf{n}''(t) = -\mathbf{t}'(t) + c\mathbf{b}'(t) = -(c^2 + 1)\mathbf{n}(t).$$

令 $\omega = \sqrt{1 + c^2}$, 则

$$\mathbf{n}(t) = \cos \omega t \mathbf{v}_1 + \sin \omega t \mathbf{v}_2,$$

其中 $\mathbf{v}_1, \mathbf{v}_2$ 为常向量. 进而由 $\mathbf{t}'(t) = \mathbf{n}(t)$ 可解得

$$\mathbf{t}(t) = \frac{1}{\omega} (\sin \omega t \mathbf{v}_1 - \cos \omega t \mathbf{v}_2 + c\mathbf{v}_3),$$

其中 \mathbf{v}_3 为常向量. 于是再由 $\mathbf{b}'(t) = -c\mathbf{n}(t)$ 可解得

$$\mathbf{b}(t) = -\frac{c}{\omega} (\sin \omega t \mathbf{v}_1 - \cos \omega t \mathbf{v}_2) + \frac{1}{\omega} \mathbf{v}_3.$$

于是

$$\begin{pmatrix} \mathbf{t}(0) \\ \mathbf{n}(0) \\ \mathbf{b}(0) \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{\omega} & \frac{c}{\omega} \\ 1 & \\ \frac{c}{\omega} & \frac{1}{\omega} \end{pmatrix}}_P \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix}.$$

注意到 $\det(P) = \frac{1+c^2}{\omega^2} = 1$, 所以为使 $\{\mathbf{t}(0), \mathbf{n}(0), \mathbf{b}(0)\}$ 是单位正交右手系, 只需选取 $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ 为单位正交右手系. 由 $\mathbf{r}'(t) = \mathbf{t}(t)$ 可得曲线方程

$$\begin{aligned} \mathbf{r}(t) &= \frac{1}{\omega} \left(\int_0^s \sin \omega t(u) du \mathbf{v}_1 - \int_0^s \cos \omega t(u) du \mathbf{v}_2 + cs \mathbf{v}_3 \right) + \mathbf{v}_4 \\ &= \frac{1}{\omega} \left[\int_0^s \sin \left(\sqrt{1+c^2} \int_0^u \kappa(t) dt \right) du \mathbf{v}_1 - \int_0^s \cos \left(\sqrt{1+c^2} \int_0^u \kappa(t) dt \right) du \mathbf{v}_2 + cs \mathbf{v}_3 \right] + \mathbf{v}_4, \end{aligned}$$

其中 \mathbf{v}_4 为常向量. □

题 17 证明: 曲线 $\mathbf{r}(t) = (t + \sqrt{3} \sin t, 2 \cos t, \sqrt{3}t - \sin t)$ 与曲线 $\tilde{\mathbf{r}}(t) = (2 \cos \frac{t}{2}, 2 \sin \frac{t}{2}, -t)$ 是合同的.

证明 曲线 $\tilde{\mathbf{r}}(t) = (2 \cos \frac{t}{2}, 2 \sin \frac{t}{2}, -t)$ 是圆柱螺旋线, 作换元 $u = \frac{t}{2}$ 可得 $\tilde{\mathbf{r}}(u) = (2 \cos u, 2 \sin u, -2u)$. 由例 (3.2) 知其曲率 $\kappa_2 \equiv \frac{1}{4}$, 挠率 $\tau_2 \equiv -\frac{1}{4}$. 而

$$\begin{aligned} \mathbf{r}'(t) &= (1 + \sqrt{3} \cos t, -2 \sin t, \sqrt{3} - \cos t), \\ \mathbf{r}''(t) &= (-\sqrt{3} \sin t, -2 \cos t, \sin t), \\ \mathbf{r}'''(t) &= (-\sqrt{3} \cos t, 2 \sin t, \cos t), \\ \mathbf{r}'(t) \wedge \mathbf{r}''(t) &= (2\sqrt{3} \cos t - 2, -4 \sin t, -2 \cos t - 2\sqrt{3}), \\ \langle \mathbf{r}', \mathbf{r}'', \mathbf{r}''' \rangle &= \langle \mathbf{r}''', \mathbf{r}' \wedge \mathbf{r}'' \rangle = -8, \end{aligned}$$

故曲线 $\mathbf{r}(t)$ 的曲率

$$\kappa_1(t) = \frac{|\mathbf{r}' \wedge \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{4\sqrt{2}}{16\sqrt{2}} = \frac{1}{4},$$

挠率

$$\tau_1(t) = \frac{\langle \mathbf{r}', \mathbf{r}'', \mathbf{r}''' \rangle}{|\mathbf{r}' \wedge \mathbf{r}''|^2} = \frac{-8}{32} = -\frac{1}{4}.$$

因此 $\kappa_1 \equiv \kappa_2, \tau_1 \equiv \tau_2$. 由曲线论基本定理得两曲线合同. □

第三章 曲面的局部理论

§3.1 正则曲面、第一基本形式、Gauss 曲率、法曲率、第二基本形式

题 18 求 xy 平面的曲线 $\mathbf{r}(t) = (x(t), y(t))$, 沿 E^3 的常方向 \mathbf{a} 平行移动所得的曲面的参数表示式.

解 所得曲面参数表示为 $\tilde{\mathbf{r}}(u, v) = (x(u), y(u), 0) + v\mathbf{a}$. □

题 19 证明: 曲面 $F\left(\frac{y}{x}, \frac{z}{x}\right) = 0$ 的任意切平面过原点.

证明 曲面 $F\left(\frac{y}{x}, \frac{z}{x}\right) = 0$ 在其上一点 $P_0(x_0, y_0, z_0)$ 处切平面方程为

$$\langle (\nabla F)_{P_0}, (x - x_0, y - y_0, z - z_0) \rangle = 0.$$

代入

$$\nabla F = \left(-F_1 \frac{y}{x^2} - F_2 \frac{z}{x^2}, \frac{F_1}{x}, \frac{F_2}{x} \right)$$

化简即得 P_0 处切平面方程

$$(F_1 y_0 + F_2 z_0)z - F_1 x_0 y - F_2 x_0 z = 0,$$

可见其过原点. □

题 20 设曲面 S 与平面 Π 相交于点 P , 且 S 位于 Π 的同一侧, 证明: Π 是曲面 S 在点 P 的切平面.

证明 设曲面 S 的参数表示为 $\mathbf{r} = \mathbf{r}(u, v)$, $P = \mathbf{r}(u_0, v_0)$. 取平面 Π 的单位法向量 \mathbf{n}_Π , 其方向指向曲面 S 一侧. 考虑曲面 S 上点到平面 Π 的高度函数

$$h(u, v) = \langle \mathbf{r}(u, v) - \mathbf{r}(u_0, v_0), \mathbf{n}_\Pi \rangle,$$

它在 (u_0, v_0) 处有极小值, 因此

$$h_u(u_0, v_0) = \langle \mathbf{r}_u(u_0, v_0), \mathbf{n}_\Pi \rangle = 0,$$

$$h_v(u_0, v_0) = \langle \mathbf{r}_v(u_0, v_0), \mathbf{n}_\Pi \rangle = 0.$$

这说明 \mathbf{n}_Π 与 $\mathbf{r}_u(u_0, v_0) \wedge \mathbf{r}_v(u_0, v_0)$ 平行, 故 Π 是曲面 S 在点 P 的切平面. □

题 21 求曲面 $z = f(x, y)$ 的第一基本形式.

解 该曲面以 x, y 为参数的表示为 $\mathbf{r}(x, y) = (x, y, f(x, y))$. 则由

$$\mathbf{r}_x = (1, 0, f_x), \quad \mathbf{r}_y = (0, 1, f_y)$$

可得

$$E = \langle \mathbf{r}_x, \mathbf{r}_x \rangle = 1 + f_x^2, \quad F = \langle \mathbf{r}_x, \mathbf{r}_y \rangle = f_x f_y, \quad G = \langle \mathbf{r}_y, \mathbf{r}_y \rangle = 1 + f_y^2.$$

故该曲面的第一基本形式为

$$I = (1 + f_x^2) dx \otimes dx + f_x f_y (dx \otimes dy + dy \otimes dx) + (1 + f_y^2) dy \otimes dy.$$

□

题 22 使 $F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0$ 的参数 (u, v) 称为曲面的正交参数系. 给定一个曲面 S 以及它的一个参数表示 $\mathbf{r} = \mathbf{r}(u, v)$, 证明: 对曲面 S 上任一点 $P_0 = P(u_0, v_0)$, 存在 P_0 的邻域 D 以及 D 的新参数 (s, t) , 使得 (s, t) 是曲面 S 的正交参数系.

证明 先证明一个比正交参数系的存在性更一般的引理.

引理 设 $\mathbf{a}(u, v)$ 与 $\mathbf{b}(u, v)$ 是正则参数曲面 $S: \mathbf{r} = \mathbf{r}(u, v)$ 上两个处处线性无关的光滑向量场, 则对曲面 S 上任意一点 P , 存在 P 的邻域 U 及其上的新参数化 $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}(\tilde{u}, \tilde{v})$, 使得 $\tilde{\mathbf{r}}_{\tilde{u}}, \tilde{\mathbf{r}}_{\tilde{v}}$ 分别与 \mathbf{a}, \mathbf{b} 平行.

【引理的证明: 设

$$\begin{cases} \mathbf{a}(u, v) = a_1(u, v)\mathbf{r}_u + a_2(u, v)\mathbf{r}_v, \\ \mathbf{b}(u, v) = b_1(u, v)\mathbf{r}_u + b_2(u, v)\mathbf{r}_v, \end{cases}$$

则由 $\mathbf{a}(u, v)$ 与 $\mathbf{b}(u, v)$ 处处线性无关可得

$$\Delta := \begin{vmatrix} a_1(u, v) & a_2(u, v) \\ b_1(u, v) & b_2(u, v) \end{vmatrix} \neq 0.$$

根据一次微分式积分因子的存在性定理, 对于曲面上任意一点 P , 存在 P 的邻域 U 和定义在 U 上的处处非零的光滑函数 ξ, η , 使得

$$\begin{cases} d\tilde{u} = \xi(b_2 du - b_1 dv), \\ d\tilde{v} = \eta(-a_2 du + a_1 dv). \end{cases}$$

于是

$$\begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix} = \begin{pmatrix} \xi b_2(u, v) & -\eta a_2(u, v) \\ -\xi b_1(u, v) & \eta a_1(u, v) \end{pmatrix},$$

且由 $\xi\eta\Delta \neq 0$ 可知该矩阵可逆, 说明 \tilde{u}, \tilde{v} 是曲面 S 在邻域 U 上的新参数. 此时

$$\begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}} \\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}^{-1} = \frac{1}{\xi\eta\Delta} \begin{pmatrix} \eta a_1(u, v) & \eta a_2(u, v) \\ \xi b_1(u, v) & \xi b_2(u, v) \end{pmatrix},$$

故

$$\begin{cases} \tilde{\mathbf{r}}_u = \mathbf{r}_u \frac{\partial u}{\partial \tilde{u}} + \mathbf{r}_v \frac{\partial v}{\partial \tilde{u}} = \frac{1}{\xi\Delta} (a_1\mathbf{r}_u + a_2\mathbf{r}_v) = \frac{1}{\xi\Delta} \mathbf{a}, \\ \tilde{\mathbf{r}}_v = \mathbf{r}_u \frac{\partial u}{\partial \tilde{v}} + \mathbf{r}_v \frac{\partial v}{\partial \tilde{v}} = \frac{1}{\eta\Delta} (b_1\mathbf{r}_u + b_2\mathbf{r}_v) = \frac{1}{\eta\Delta} \mathbf{b}. \end{cases}$$

引理得证.】

对 $\{\mathbf{r}_u, \mathbf{r}_v\}$ 作 Gram-Schmidt 标准正交化:

$$\begin{aligned} \mathbf{e}_1 &= \frac{\mathbf{r}_u}{|\mathbf{r}_u|} = \frac{\mathbf{r}_u}{\sqrt{E}}, \\ \mathbf{b} &= \mathbf{r}_v - \langle \mathbf{r}_v, \mathbf{e}_1 \rangle \mathbf{e}_1 = \mathbf{r}_v - \frac{F}{E} \mathbf{r}_u, \\ \mathbf{e}_2 &= \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{\sqrt{G - \frac{F^2}{E}}} \left(\mathbf{r}_v - \frac{F}{E} \mathbf{r}_u \right) = \frac{1}{\sqrt{EG - F^2}} \left(-\frac{F}{\sqrt{E}} \mathbf{r}_u + \sqrt{E} \mathbf{r}_v \right). \end{aligned}$$

将 $\mathbf{e}_1 = \mathbf{e}_1(u, v)$ 与 $\mathbf{e}_2 = \mathbf{e}_2(u, v)$ 视作曲面 S 上的两个单位正交向量场, 则由引理知存在点 P 的邻域 D 以及 D 的新参数化 $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}(s, t)$, 使得 $\tilde{\mathbf{r}}_s$ 与 \mathbf{e}_1 平行, $\tilde{\mathbf{r}}_t$ 与 \mathbf{e}_2 平行, 故 (s, t) 是 S 的正交参数系. \square

题 23 证明: 在曲面的任意一点, 任何两个相互正交的切向的法曲率之和为常数.

证明 对给定点选取充分小的邻域合适的参数化使得曲面方程在其上可写为 $\mathbf{r}(x, y) = (x, y, f(x, y))$, 且该点坐标为 $(0, 0, 0)$, $f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = 0$. 记 $\mathbf{e}_1 = \mathbf{r}_x(0, 0) = (1, 0, 0)$, $\mathbf{e}_2 = \mathbf{r}_y(0, 0) = (0, 1, 0)$. 任取曲面在该点处的两个单位切向量 $\mathbf{v}_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$ 与 $\mathbf{v}_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$, 则由 Euler 公式,

$$\kappa_n(\mathbf{v}_1) + \kappa_n(\mathbf{v}_2) = \kappa_n(\mathbf{e}_1) \cos^2\theta + \kappa_n(\mathbf{e}_2) \sin^2\theta + \kappa_n(\mathbf{e}_1) \sin^2\theta + \kappa_n(\mathbf{e}_2) \cos^2\theta = \kappa_n(\mathbf{e}_1) + \kappa_n(\mathbf{e}_2).$$

\square

题 24 设曲面 S 由方程 $x^2 + y^2 - f(z) = 0$ 给定, f 满足 $f(0) = 0, f'(0) \neq 0$, 证明: S 在点 $(0, 0, 0)$ 的法曲率为常数.

证明 由于 $f'(0) \neq 0$, 由反函数定理, 存在 $f(0)$ 的一个邻域 U 及函数 $g: f(U) \rightarrow U$, 使得 $g = (f|_U)^{-1}$. 于是曲面在点 $(0, 0, 0)$ 附近可表示为 $\mathbf{r}(x, y) = (x, y, g(x^2 + y^2))$. 我们有

$$\mathbf{r}_x = (1, 0, 2xg'(x^2 + y^2)), \quad \mathbf{r}_y = (0, 1, 2yg'(x^2 + y^2)).$$

记 $h(x, y) = g(x^2 + y^2)$, 则

$$h_x(0, 0) = (2xg'(x^2 + y^2))_{(x,y)=(0,0)} = 0,$$

$$h_y(0, 0) = (2yg'(x^2 + y^2))_{(x,y)=(0,0)} = 0,$$

$$h_{xy}(0, 0) = (4xyg''(x^2 + y^2))_{(x,y)=(0,0)} = 0.$$

下设

$$\mathbf{e}_1 = \mathbf{r}_x(0, 0) = (1, 0, 0), \quad \mathbf{e}_2 = \mathbf{r}_y(0, 0) = (0, 1, 0),$$

则由 Euler 公式, 只需证 $\kappa_{\mathbf{n}}(\mathbf{e}_1) = \kappa_{\mathbf{n}}(\mathbf{e}_2)$, 其中 $\mathbf{n} = \mathbf{e}_1 \wedge \mathbf{e}_2 = (0, 0, 1)$. 因为平面 $\{\mathbf{e}_1, \mathbf{n}\}$ 截曲面所得曲线可表示为 $\tilde{\mathbf{r}}(t) = (t, g(t^2))$, 该参数化满足曲线在点 $(0, 0)$ 处切向量恰为 \mathbf{e}_1 , 因此 S 在点 $(0, 0, 0)$ 处沿 \mathbf{e}_1 方向的法曲率即该曲线在点 $(0, 0)$ 处带符号曲率:

$$\kappa_{\mathbf{n}}(\mathbf{e}_1) = \left(\frac{1 \cdot [2g'(x^2) + 4x^2g''(x^2)]}{[1^2 + (2xg'(x^2))^2]^{\frac{3}{2}}} \right)_{x=0} = 2g'(0).$$

同理可得 S 在点 $(0, 0, 0)$ 处沿 \mathbf{e}_2 方向的法曲率 $\kappa_{\mathbf{n}}(\mathbf{e}_2) = 2g'(0)$. 因此 S 在点 $(0, 0, 0)$ 处沿各方向的法曲率均为 $\frac{2}{f'(0)}$. \square

题 25 求曲面 $z = f(x, y)$ 的 Gauss 曲率.

解 曲面以 x, y 为参数的表示为 $\mathbf{r} = \mathbf{r}(x, y) = (x, y, f(x, y))$. 由

$$\begin{cases} \mathbf{r}_x = (1, 0, f_x), \\ \mathbf{r}_y = (0, 1, f_y) \end{cases}$$

可得

$$\mathbf{r}_x \wedge \mathbf{r}_y = (1, 0, f_x) \wedge (0, 1, f_y) = (-f_x, -f_y, 1).$$

从而与 $\mathbf{r}_x, \mathbf{r}_y$ 构成右手系的单位法向量

$$\mathbf{n}(x, y) = \frac{\mathbf{r}_x \wedge \mathbf{r}_y}{|\mathbf{r}_x \wedge \mathbf{r}_y|} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

由于 $\mathbf{r}_x \wedge \mathbf{r}_y$ 与 $\mathbf{n}_x \wedge \mathbf{n}_y$ 平行, 为计算 Gauss 曲率, 只需考虑这两个向量的第 3 个分量之比

$$\mathbf{n}_x^1 \mathbf{n}_y^2 - \mathbf{n}_x^2 \mathbf{n}_y^1.$$

而

$$\begin{aligned} \mathbf{n}_x^1 &= \frac{\partial}{\partial x} \left(\frac{-f_x}{\sqrt{f_x^2 + f_y^2 + 1}} \right) = \frac{-f_{xx} \sqrt{f_x^2 + f_y^2 + 1} + f_x \frac{f_x f_{xx} + f_y f_{yx}}{\sqrt{f_x^2 + f_y^2 + 1}}}{f_x^2 + f_y^2 + 1} = \frac{-f_{xx} (f_y^2 + 1) + f_x f_y f_{yx}}{(f_x^2 + f_y^2 + 1)^{\frac{3}{2}}}, \\ \mathbf{n}_x^2 &= \frac{\partial}{\partial x} \left(\frac{-f_y}{\sqrt{f_x^2 + f_y^2 + 1}} \right) = \frac{-f_{yx} \sqrt{f_x^2 + f_y^2 + 1} + f_y \frac{f_x f_{xx} + f_y f_{yx}}{\sqrt{f_x^2 + f_y^2 + 1}}}{f_x^2 + f_y^2 + 1} = \frac{-f_{yx} (f_x^2 + 1) + f_x f_y f_{xx}}{(f_x^2 + f_y^2 + 1)^{\frac{3}{2}}}, \\ \mathbf{n}_y^1 &= \frac{\partial}{\partial y} \left(\frac{-f_x}{\sqrt{f_x^2 + f_y^2 + 1}} \right) = \frac{-f_{xy} \sqrt{f_x^2 + f_y^2 + 1} + f_x \frac{f_x f_{xy} + f_y f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}}}{f_x^2 + f_y^2 + 1} = \frac{-f_{xy} (f_y^2 + 1) + f_x f_y f_{yy}}{(f_x^2 + f_y^2 + 1)^{\frac{3}{2}}}, \\ \mathbf{n}_y^2 &= \frac{\partial}{\partial y} \left(\frac{-f_y}{\sqrt{f_x^2 + f_y^2 + 1}} \right) = \frac{-f_{yy} \sqrt{f_x^2 + f_y^2 + 1} + f_y \frac{f_x f_{xy} + f_y f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}}}{f_x^2 + f_y^2 + 1} = \frac{-f_{yy} (f_x^2 + 1) + f_x f_y f_{xy}}{(f_x^2 + f_y^2 + 1)^{\frac{3}{2}}}, \end{aligned}$$

代入即得

$$\begin{aligned} \kappa(x, y, f(x, y)) &= \mathbf{n}_x^1 \mathbf{n}_y^2 - \mathbf{n}_x^2 \mathbf{n}_y^1 \\ &= \frac{[f_x f_y f_{yx} - f_{xx} (f_y^2 + 1)] [f_x f_y f_{xy} - f_{yy} (f_x^2 + 1)] - [f_x f_y f_{xx} - f_{yx} (f_x^2 + 1)] [f_x f_y f_{yy} - f_{xy} (f_y^2 + 1)]}{(f_x^2 + f_y^2 + 1)^3} \\ &= \frac{(f_x^2 + f_y^2 + 1) f_{xx} f_{yy} - (f_x^2 + f_y^2 + 1) f_{xy}^2}{(f_x^2 + f_y^2 + 1)^3} \\ &= \frac{f_{xx} f_{yy} - f_{xy}^2}{(f_x^2 + f_y^2 + 1)^2}. \end{aligned}$$

□

题 26 求曲面 $z = f(x, y)$ 的第二基本形式.

解 由曲面方程 $\mathbf{r}(x, y) = (x, y, f(x, y))$ 可得

$$\begin{aligned} \mathbf{r}_x &= (1, 0, f_x), \quad \mathbf{r}_y = (0, 1, f_y), \\ \mathbf{r}_{xx} &= (0, 0, f_{xx}), \quad \mathbf{r}_{xy} = (0, 0, f_{xy}), \quad \mathbf{r}_{yy} = (0, 0, f_{yy}). \end{aligned}$$

于是

$$\begin{aligned} \mathbf{n}(x, y) &= \frac{\mathbf{r}_x \wedge \mathbf{r}_y}{|\mathbf{r}_x \wedge \mathbf{r}_y|} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}, \\ L &= \langle \mathbf{r}_{xx}, \mathbf{n} \rangle = \frac{f_{xx}}{\sqrt{f_x^2 + f_y^2 + 1}}, \\ M &= \langle \mathbf{r}_{xy}, \mathbf{n} \rangle = \frac{f_{xy}}{\sqrt{f_x^2 + f_y^2 + 1}}, \\ N &= \langle \mathbf{r}_{yy}, \mathbf{n} \rangle = \frac{f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}}. \end{aligned}$$

故曲面的第二基本形式

$$\text{II} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} [f_{xx} dx \otimes dx + f_{xy} (dx \otimes dy + dy \otimes dx) + f_{yy} dy \otimes dy].$$

□

题 27 设曲面 S_1 和 S_2 的交线 C 的曲率为 κ , 曲线 C 在曲面 S_i 上的法曲率为 k_i ($i = 1, 2$); 若沿 C , S_1 和 S_2 法向的夹角为 θ , 证明:

$$\kappa^2 \sin^2 \theta = k_1^2 + k_2^2 - 2k_1 k_2 \cos \theta.$$

证明 设曲线 C 的弧长参数表示为 $\mathbf{r} = \mathbf{r}(s)$, $\mathbf{n}_i(s)$ 为 $\mathbf{r}(s)$ 在曲面 S_i 上的法向量 (定向选取同题目条件), 则

$$\begin{aligned} k_1^2 + k_2^2 - 2k_1 k_2 \cos \theta &= \langle \dot{\mathbf{t}}(s), \mathbf{n}_1(s) \rangle^2 + \langle \dot{\mathbf{t}}(s), \mathbf{n}_2(s) \rangle^2 - 2 \langle \dot{\mathbf{t}}(s), \mathbf{n}_1(s) \rangle^2 \langle \dot{\mathbf{t}}(s), \mathbf{n}_2(s) \rangle^2 \langle \mathbf{n}_1(s), \mathbf{n}_2(s) \rangle \\ &= \left| \langle \dot{\mathbf{t}}(s), \mathbf{n}_1(s) \rangle \mathbf{n}_2(s) - \langle \dot{\mathbf{t}}(s), \mathbf{n}_2(s) \rangle \mathbf{n}_1(s) \right|^2 \\ &\stackrel{\text{题 1}}{=} \left| \dot{\mathbf{t}}(s) \wedge (\mathbf{n}_2(s) \wedge \mathbf{n}_1(s)) \right|^2 \\ &= \left| \dot{\mathbf{t}}(s) \wedge (\pm \sin \theta \mathbf{t}(s)) \right|^2 \\ &= \kappa^2 |\mathbf{n}(s) \wedge \mathbf{t}(s)|^2 \sin^2 \theta \\ &= \kappa^2 \sin^2 \theta. \end{aligned}$$

□

§3.2 平均曲率、极小曲面、曲面的局部外蕴几何、脐点

题 28 求曲面 $z = f(x, y)$ 的平均曲率.

解 曲面方程为 $\mathbf{r}(x, y) = (x, y, f(x, y))$. 由

$$\mathbf{r}_x = (1, 0, f_x), \quad \mathbf{r}_y = (0, 1, f_y)$$

可得

$$\begin{aligned} \mathbf{n}(x, y) &= \frac{\mathbf{r}_x \wedge \mathbf{r}_y}{|\mathbf{r}_x \wedge \mathbf{r}_y|} = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} (-f_x, -f_y, 1), \\ \mathbf{r}_{xx} &= (0, 0, f_{xx}), \quad \mathbf{r}_{xy} = (0, 0, f_{xy}), \quad \mathbf{r}_{yy} = (0, 0, f_{yy}). \end{aligned}$$

于是

$$\begin{aligned} E &= \langle \mathbf{r}_x, \mathbf{r}_x \rangle = 1 + f_x^2, \quad F = \langle \mathbf{r}_x, \mathbf{r}_y \rangle = f_x f_y, \quad G = \langle \mathbf{r}_y, \mathbf{r}_y \rangle = 1 + f_y^2, \\ L &= \langle \mathbf{r}_{xx}, \mathbf{n} \rangle = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}}, \quad M = \langle \mathbf{r}_{xy}, \mathbf{n} \rangle = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}}, \quad N = \langle \mathbf{r}_{yy}, \mathbf{n} \rangle = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}. \end{aligned}$$

故曲面的平均曲率

$$\begin{aligned} H(x, y) &= \frac{1}{2} \operatorname{tr} \left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \right) = \frac{LG - 2MF + NE}{2(EG - F^2)} \\ &= \frac{f_{xx}(1 + f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1 + f_x^2)}{2(1 + f_x^2 + f_y^2)^{\frac{3}{2}}}. \end{aligned}$$

□

题 29 求曲面 $\mathbf{r}(u, v) = (a(u+v), b(u-v), 4uv)$ 的 Gauss 曲率、平均曲率、主曲率及对应的主方向.

解 由

$$\mathbf{r}_u = (a, b, 4v), \quad \mathbf{r}_v = (a, -b, 4u)$$

可得

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_u \wedge \mathbf{r}_v}{|\mathbf{r}_u \wedge \mathbf{r}_v|} = \frac{1}{\sqrt{4b^2(u+v)^2 + 4a^2(u-v)^2 + a^2b^2}}(2b(u+v), 2a(v-u), -ab),$$

$$\mathbf{r}_{uu} = (0, 0, 0), \quad \mathbf{r}_{uv} = (0, 0, 4), \quad \mathbf{r}_{vv} = (0, 0, 0).$$

于是

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = a^2 + b^2 + 16v^2, \quad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = a^2 - b^2 + 16uv, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = a^2 + b^2 + 16u^2,$$

$$L = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle = 0, \quad M = \langle \mathbf{r}_{uv}, \mathbf{n} \rangle = \frac{-4ab}{\sqrt{4b^2(u+v)^2 + 4a^2(u-v)^2 + a^2b^2}}, \quad N = \langle \mathbf{r}_{vv}, \mathbf{n} \rangle = 0.$$

故曲面的 Gauss 曲率

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\frac{-16a^2b^2}{4b^2(u+v)^2 + 4a^2(u-v)^2 + a^2b^2}}{(a^2 + b^2 + 16v^2)(a^2 + b^2 + 16u^2) - (a^2 - b^2 + 16uv)^2}$$

$$= \frac{-4a^2b^2}{[a^2b^2 + 4a^2(u-v)^2 + 4b^2(u+v)^2]^2},$$

平均曲率

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{\frac{8ab(a^2 - b^2 + 16uv)}{\sqrt{4b^2(u+v)^2 + 4a^2(u-v)^2 + a^2b^2}}}{2[(a^2 + b^2 + 16v^2)(a^2 + b^2 + 16u^2) - (a^2 - b^2 + 16uv)^2]}$$

$$= \frac{ab(a^2 - b^2 + 16uv)}{[a^2b^2 + 4a^2(u-v)^2 + 4b^2(u+v)^2]^{\frac{3}{2}}}.$$

解关于 k 的方程 $k^2 - 2Hk + K = 0$ 得主曲率 $k_1 = H + \sqrt{H^2 - K}$, $k_2 = H - \sqrt{H^2 - K}$, 由

$$H^2 - K = \frac{a^2b^2[(a^2 + b^2 + 16u^2)(a^2 + b^2 + 16v^2)]}{[a^2b^2 + 4a^2(u-v)^2 + 4b^2(u+v)^2]^3}$$

不妨设 $ab \geq 0$ (若 $ab < 0$ 则 k_1, k_2 互换, 不影响结果). 代入即得

$$k_1 = \frac{ab \left[a^2 - b^2 + 16uv + \sqrt{(a^2 + b^2 + 16u^2)(a^2 + b^2 + 16v^2)} \right]}{[a^2b^2 + 4a^2(u-v)^2 + 4b^2(u+v)^2]^{\frac{3}{2}}},$$

$$k_2 = \frac{ab \left[a^2 - b^2 + 16uv - \sqrt{(a^2 + b^2 + 16u^2)(a^2 + b^2 + 16v^2)} \right]}{[a^2b^2 + 4a^2(u-v)^2 + 4b^2(u+v)^2]^{\frac{3}{2}}}.$$

Weingarten 变换在自然标架 $\{\mathbf{r}_u, \mathbf{r}_v\}$ 下的矩阵表示为

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{M}{EG - F^2} \begin{pmatrix} -F & G \\ E & -F \end{pmatrix},$$

于是该矩阵与特征值 k_1, k_2 相伴的特征空间的一个基即两方程

$$\begin{pmatrix} \pm \sqrt{(a^2 + b^2 + 16u^2)(a^2 + b^2 + 16v^2)} & a^2 + b^2 + 16u^2 \\ a^2 + b^2 + 16v^2 & \pm \sqrt{(a^2 + b^2 + 16u^2)(a^2 + b^2 + 16v^2)} \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$$

的各一个非零解:

$$\begin{aligned}\xi_1 &= \sqrt{a^2 + b^2 + 16u^2} \mathbf{r}_u + \sqrt{a^2 + b^2 + 16v^2} \mathbf{r}_v, \\ \xi_2 &= \sqrt{a^2 + b^2 + 16u^2} \mathbf{r}_u - \sqrt{a^2 + b^2 + 16v^2} \mathbf{r}_v,\end{aligned}$$

这即是分别与主曲率 k_1, k_2 对应的主方向. □

题 30 曲面 S 上的一条曲线 C 称为曲率线, 是指 C 在每点的切向量都是曲面 S 在该点的一个主方向. 证明: 曲线 $C: \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ 是曲率线当且仅当沿着 C , $\frac{d\mathbf{n}}{dt}$ 与 $\frac{d\mathbf{r}}{dt}$ 平行.

证明 曲线 $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ 是曲率线 $\iff \mathbf{r}'(t) = u'(t)\mathbf{r}_u + v'(t)\mathbf{r}_v$ 是 Weingarten 变换的特征向量 \iff 存在 $\lambda(t)$, 使得 $\lambda(t)\mathbf{r}'(t) = \mathcal{W}(\mathbf{r}'(t)) = u'(t)\mathcal{W}(\mathbf{r}_u) + v'(t)\mathcal{W}(\mathbf{r}_v) = -u'(t)\mathbf{n}_u - v'(t)\mathbf{n}_v = -\mathbf{n}'(t) \iff \mathbf{n}'(t)$ 与 $\mathbf{r}'(t)$ 平行. □

题 31 曲面 S 上的一个切向称为渐近方向, 是指沿此方向的法曲率为 0; S 上一条曲线 C 称为渐近线, 是指它的切向均为渐近方向. 证明: 曲面 $S: \mathbf{r} = \mathbf{r}(u, v)$ 的参数曲线是渐近线当且仅当 $L = N = 0$.

证明 曲面 $\mathbf{r} = \mathbf{r}(u, v)$ 的参数曲线 $\mathbf{r}_1(u) = \mathbf{r}(u, v_0)$ 和 $\mathbf{r}_2(v) = \mathbf{r}(u_0, v)$ 都是渐近线 $\iff \kappa_n(\mathbf{r}_u) = \kappa_n(\mathbf{r}_v) = 0 \iff \begin{cases} \text{II}(\mathbf{r}_u, \mathbf{r}_u) = L = 0, \\ \text{II}(\mathbf{r}_v, \mathbf{r}_v) = N = 0. \end{cases}$ □

题 32 求曲面 $\mathbf{r}(u, v) = (u^3, v^3, u + v)$ 上抛物点的轨迹.

解 由

$$\mathbf{r}_u = (3u^2, 0, 1), \quad \mathbf{r}_v = (0, 3v^2, 1)$$

可得

$$\begin{aligned}\mathbf{r}_{uu} &= (6u, 0, 0), \quad \mathbf{r}_{uv} = (0, 0, 0), \quad \mathbf{r}_{vv} = (0, 6v, 0), \\ \mathbf{n}(u, v) &= \frac{\mathbf{r}_u \wedge \mathbf{r}_v}{|\mathbf{r}_u \wedge \mathbf{r}_v|} = \frac{1}{\sqrt{u^4 + v^4 + 9u^4v^4}} (-v^2, -u^2, 3u^2v^2).\end{aligned}$$

于是

$$L = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle = \frac{-6uv^2}{\sqrt{u^4 + v^4 + 9u^4v^4}}, \quad M = \langle \mathbf{r}_{uv}, \mathbf{n} \rangle = 0, \quad N = \langle \mathbf{r}_{vv}, \mathbf{n} \rangle = \frac{-6u^2v}{\sqrt{u^4 + v^4 + 9u^4v^4}}.$$

因此曲面上抛物点的轨迹方程为

$$LN - M^2 = 0 \iff uv = 0,$$

其轨迹即两条三次曲线

$$\begin{cases} y = z^3, \\ x = 0 \end{cases} \quad \text{与} \quad \begin{cases} x = z^3, \\ y = 0. \end{cases}$$

□

题 33 设 P 是曲面 S 上的一点. 证明: 当 P 不是脐点时, S 的主曲率 k_1, k_2 是 P 附近的光滑函数; 当 P 是脐点时, 主曲率是 P 附近的连续函数.

证明 由

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}$$

可得

$$k_{1,2} = H \pm \sqrt{H^2 - K},$$

而

$$H^2 - K = \frac{(k_1 - k_2)^2}{4} \geq 0,$$

等号成立当且仅当 $k_1 = k_2$ 即 P 是脐点. 由此可见, 主曲率 k_1, k_2 总是 P 附近的连续函数, 并在 P 不是脐点时是 P 附近的光滑函数. \square

题 34 设曲面 $S: \mathbf{r} = \mathbf{r}(u, v)$ 上没有抛物点, \mathbf{n} 是 S 的法向量; 曲面 $\tilde{S}: \tilde{\mathbf{r}} = \tilde{\mathbf{r}}(u, v) = \mathbf{r}(u, v) + \lambda \mathbf{n}(u, v)$ (常数 λ 充分小) 称为 S 的平行曲面.

(1) 证明: 曲面 S 和 \tilde{S} 在对应点的切平面平行;

(2) 可以选取 \tilde{S} 的单位法向量 $\tilde{\mathbf{n}}$, 使得 \tilde{S} 的 Gauss 曲率和平均曲率分别为

$$\tilde{K} = \frac{K}{1 - 2\lambda H + \lambda^2 K}, \quad \tilde{H} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}.$$

解 (1) 由

$$\tilde{\mathbf{r}}_u = \mathbf{r}_u + \lambda \mathbf{n}_u, \quad \tilde{\mathbf{r}}_v = \mathbf{r}_v + \lambda \mathbf{n}_v$$

可知

$$\tilde{\mathbf{r}}_u \wedge \tilde{\mathbf{r}}_v = \mathbf{r}_u \wedge \mathbf{r}_v + \lambda \mathbf{r}_u \wedge \mathbf{n}_v + \lambda \mathbf{n}_u \wedge \mathbf{r}_v + \lambda^2 \mathbf{n}_u \wedge \mathbf{n}_v.$$

注意到 $\mathbf{r}_u \wedge \mathbf{n}_v$ 、 $\mathbf{n}_u \wedge \mathbf{r}_v$ 及 $\mathbf{n}_u \wedge \mathbf{n}_v$ 均与 $\mathbf{r}_u \wedge \mathbf{r}_v$ 共线, 因此 $\tilde{\mathbf{r}}_u \wedge \tilde{\mathbf{r}}_v$ 与 $\mathbf{r}_u \wedge \mathbf{r}_v$ 共线, 即 S 和 \tilde{S} 在对应点的切平面平行.

(2) 由 (1) 知 $\tilde{\mathbf{n}} = \pm \mathbf{n}$. 假设 $\tilde{\mathbf{n}} = \mathbf{n}$. 设曲面 S 和 \tilde{S} 的 Weingarten 变换在各自的自然标架 $\{\mathbf{r}_u, \mathbf{r}_v\}$ 和 $\{\tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v\}$ 下的矩阵表示分别为 A 和 \tilde{A} , 则由

$$\begin{aligned} A \begin{pmatrix} \mathbf{r}_u & \mathbf{r}_v \end{pmatrix} &= \begin{pmatrix} -\mathbf{n}_u & -\mathbf{n}_v \end{pmatrix}, \\ \begin{pmatrix} \tilde{\mathbf{r}}_u & \tilde{\mathbf{r}}_v \end{pmatrix} &= \begin{pmatrix} \mathbf{r}_u & \mathbf{r}_v \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{n}_u & \mathbf{n}_v \end{pmatrix} = (I - \lambda A) \begin{pmatrix} \mathbf{r}_u & \mathbf{r}_v \end{pmatrix}, \\ \tilde{A} \begin{pmatrix} \tilde{\mathbf{r}}_u & \tilde{\mathbf{r}}_v \end{pmatrix} &= \begin{pmatrix} -\mathbf{n}_u & -\mathbf{n}_v \end{pmatrix} \end{aligned}$$

可得

$$\tilde{A} = (I - \lambda A)^{-1} A.$$

于是 \tilde{S} 的 Gauss 曲率

$$\tilde{K} = \det(\tilde{A}) = \frac{\det(A)}{\det(I - \lambda A)} = \frac{K}{\lambda^2 \det\left(\frac{1}{\lambda} I - A\right)} = \frac{K}{\lambda^2 \left(\frac{1}{\lambda^2} - \frac{2H}{\lambda} + K\right)} = \frac{K}{1 - 2\lambda H + \lambda^2 K},$$

平均曲率

$$\tilde{H} = \frac{\tilde{k}_1 + \tilde{k}_2}{2} = \frac{\frac{k_1}{1 - \lambda k_1} + \frac{k_2}{1 - \lambda k_2}}{2} = \frac{k_1 + k_2 - 2\lambda k_1 k_2}{2\lambda^2 k_1 k_2 - 2\lambda(k_1 + k_2) + 2} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}.$$

故 $\tilde{\mathbf{n}} = \mathbf{n}$ 即满足要求. \square

题 35 设 $\mathbf{r} = \mathbf{r}(u, v)$ 是无脐点曲面 S 的一个参数表示, 证明: 曲面 S 的参数曲线 $u = \text{常数}$ 和 $v = \text{常数}$ 是曲率线的充要条件是 $F = M = 0$.

证明 \Rightarrow : 若曲面 S 的参数曲线是曲率线, 则有

$$-\mathbf{n}_u = k_1 \mathbf{r}_u, \quad -\mathbf{n}_v = k_2 \mathbf{r}_v,$$

其中 k_1, k_2 是曲面 S 的主曲率, 且由曲面 S 无脐点知 $k_1 \neq k_2$. 于是由 Euler 定理知 \mathbf{r}_u 与 \mathbf{r}_v 正交, 即 $F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0$. 进而 $M = -\langle \mathbf{r}_u, \mathbf{n}_v \rangle = \langle \mathbf{r}_u, k_2 \mathbf{r}_v \rangle = 0$.

\Leftarrow : 若 $F = M = 0$, 则 \mathbf{r}_u 与 \mathbf{r}_v 、 \mathbf{r}_u 与 \mathbf{n}_v 均正交, 而 $\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}_v$ 共面, 因此存在 $k_2 \in \mathbb{R}$, 使得 $-\mathbf{n}_v = k_2 \mathbf{r}_v$, 即 \mathbf{r}_v 是一个主方向. 同理 \mathbf{r}_u 也是一个主方向. 故曲面 S 的参数曲线是曲率线. \square

题 36 若曲面 $z = f(x) + g(y)$ 是极小曲面且非平面, 证明: 除相差一个常数外, 它可以写成

$$z = \frac{1}{a} \ln \frac{\cos ay}{\cos ax}.$$

这个曲面称为 Scherk 曲面.

证明 曲面方程为 $\mathbf{r}(x, y) = (x, y, f(x) + g(y))$. 由

$$\mathbf{r}_x = (1, 0, f'(x)), \quad \mathbf{r}_y = (0, 1, g'(y))$$

可得

$$\begin{aligned} \mathbf{r}_{xx} &= (0, 0, f''(x)), \quad \mathbf{r}_{xy} = (0, 0, 0), \quad \mathbf{r}_{yy} = (0, 0, g''(y)), \\ \mathbf{n}(x, y) &= \frac{\mathbf{r}_x \wedge \mathbf{r}_y}{|\mathbf{r}_x \wedge \mathbf{r}_y|} = \frac{1}{\sqrt{(f'(x))^2 + (g'(y))^2 + 1}} (-f'(x), -g'(y), 1). \end{aligned}$$

于是

$$\begin{aligned} E &= \langle \mathbf{r}_x, \mathbf{r}_x \rangle = 1 + (f'(x))^2, \quad F = \langle \mathbf{r}_x, \mathbf{r}_y \rangle = f'(x)g'(y), \quad G = \langle \mathbf{r}_y, \mathbf{r}_y \rangle = 1 + (g'(y))^2, \\ L &= \langle \mathbf{r}_{xx}, \mathbf{n} \rangle = \frac{f''(x)}{\sqrt{(f'(x))^2 + (g'(y))^2 + 1}}, \quad M = \langle \mathbf{r}_{xy}, \mathbf{n} \rangle = 0, \quad N = \langle \mathbf{r}_{yy}, \mathbf{n} \rangle = \frac{g''(y)}{\sqrt{(f'(x))^2 + (g'(y))^2 + 1}}. \end{aligned}$$

故曲面的平均曲率

$$H = \frac{1}{2} \frac{LG - 2MF + NE}{EG - F^2} = \frac{f''(x) + f''(x)(g'(y))^2 + g''(y) + g''(y)(f'(x))^2}{2 \left[1 + (f'(x))^2 + (g'(y))^2 \right]^{\frac{3}{2}}} = 0,$$

即

$$f''(x) + f''(x)(g'(y))^2 + g''(y) + g''(y)(f'(x))^2 = 0.$$

分离变量即得

$$\frac{f''(x)}{1 + (f'(x))^2} = -\frac{g''(y)}{1 + (g'(y))^2} \equiv C,$$

也即

$$(\arctan f'(x))' = -(\arctan g'(y))' = a.$$

由于曲面非平面, $a \neq 0$, 可解得

$$f(x) = -\frac{1}{a} \ln [\cos(ax + c_1)] + c_2,$$

$$g(y) = \frac{1}{a} \ln [\cos(ay + c_3)] + c_4,$$

其中 c_1, c_2, c_3, c_4 为任意常数. 故

$$z = f(x) + g(y) = \frac{1}{a} \ln \frac{\cos(ay + c_3)}{\cos(ax + c_1)} + c_2 + c_4.$$

□

第四章 标架与曲面论基本定理

§4.1 自然标架运动方程、曲面结构方程、曲面论基本定理

题 37 定义 Riemann 记号

$$R_{\delta\alpha\beta\gamma} := -g_{\delta\xi} \left(\frac{\partial \Gamma_{\alpha\beta}^{\xi}}{\partial u^{\gamma}} - \frac{\partial \Gamma_{\alpha\gamma}^{\xi}}{\partial u^{\beta}} + \Gamma_{\alpha\beta}^{\eta} \Gamma_{\eta\gamma}^{\xi} - \Gamma_{\alpha\gamma}^{\eta} \Gamma_{\eta\beta}^{\xi} \right).$$

(1) 证明:

$$R_{\delta\alpha\beta\gamma} = \frac{1}{2} \left(-\frac{\partial^2 g_{\delta\beta}}{\partial u^{\gamma} \partial u^{\alpha}} + \frac{\partial^2 g_{\alpha\beta}}{\partial u^{\gamma} \partial u^{\delta}} + \frac{\partial^2 g_{\delta\gamma}}{\partial u^{\beta} \partial u^{\alpha}} - \frac{\partial^2 g_{\alpha\gamma}}{\partial u^{\beta} \partial u^{\delta}} \right) + \Gamma_{\alpha\beta}^{\eta} \Gamma_{\eta\delta\gamma} - \Gamma_{\alpha\gamma}^{\eta} \Gamma_{\eta\delta\beta},$$

其中 $\Gamma_{\eta\delta\gamma} := g_{\eta\xi} \Gamma_{\delta\gamma}^{\xi}$ 为第二类 Christoffel 符号.

(2) 计算所有 Christoffel 符号 $\{\Gamma_{\beta\gamma}^{\alpha} \mid \alpha, \beta, \gamma = 1, 2\}$ 用 E, F, G 及其偏导数表示的表达式.

(3) 利用 (1) 和 (2) 证明:

$$4(EG - F^2) R_{1212} = E(E_v G_v - 2F_u G_v + (G_v)^2) + F(E_u G_v - E_v G_u - 2E_v F_v + F_u F_v - 2F_u G_u) \\ + G(E_u G_u - 2E_u F_v + (E_v)^2) - 2(EG - F^2)(E_{vv} - 2F_{uv} + G_{uu}).$$

由此及 Gauss 绝妙定理即得 $R_{1212} = (EG - F^2) K$.

(4) 设 $\mathbf{r} : D \rightarrow \mathbb{R}^3, (u, v) \mapsto \mathbf{r}(u, v)$ 为一正则曲面片且其参数化满足 $F = M = 0$. 利用 (2) 证明 Codazzi 方程组

$$\begin{cases} \frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} = \Gamma_{12}^{\xi} b_{\xi 1} - \Gamma_{11}^{\xi} b_{\xi 2}, \\ \frac{\partial b_{21}}{\partial u^2} - \frac{\partial b_{22}}{\partial u^1} = \Gamma_{22}^{\xi} b_{\xi 1} - \Gamma_{21}^{\xi} b_{\xi 2} \end{cases}$$

用记号 E, F, G, L, M, N 可表为

$$L_v = HE_v, \quad N_u = HG_u,$$

其中 H 为平均曲率.

证明 (1) 我们有

$$R_{\delta\alpha\beta\gamma} = -\frac{\partial \Gamma_{\delta\alpha\beta}}{\partial u^{\gamma}} + \frac{\partial \Gamma_{\delta\alpha\gamma}}{\partial u^{\beta}} + \frac{\partial g_{\delta\xi}}{\partial u^{\gamma}} \Gamma_{\alpha\beta}^{\xi} - \frac{\partial g_{\delta\xi}}{\partial u^{\beta}} \Gamma_{\alpha\gamma}^{\xi} - \Gamma_{\alpha\beta}^{\eta} \Gamma_{\delta\eta\gamma} + \Gamma_{\alpha\gamma}^{\eta} \Gamma_{\delta\eta\beta} \\ = -\frac{\partial \Gamma_{\delta\alpha\beta}}{\partial u^{\gamma}} + \frac{\partial \Gamma_{\delta\alpha\gamma}}{\partial u^{\beta}} - \Gamma_{\alpha\beta}^{\eta} \left(\Gamma_{\delta\eta\gamma} - \frac{\partial g_{\delta\eta}}{\partial u^{\gamma}} \right) + \Gamma_{\alpha\gamma}^{\eta} \left(\Gamma_{\delta\eta\beta} - \frac{\partial g_{\delta\eta}}{\partial u^{\beta}} \right) \\ = -\frac{1}{2} \frac{\partial}{\partial u^{\gamma}} (g_{\alpha\delta, \beta} + g_{\delta\beta, \alpha} - g_{\alpha\beta, \delta}) + \frac{1}{2} \frac{\partial}{\partial u^{\beta}} (g_{\alpha\delta, \gamma} + g_{\delta\gamma, \alpha} - g_{\alpha\gamma, \delta}) \\ - \Gamma_{\alpha\beta}^{\eta} \left[\frac{1}{2} (g_{\eta\delta, \gamma} + g_{\delta\gamma, \eta} - g_{\eta\gamma, \delta}) - g_{\delta\eta, \gamma} \right] + \Gamma_{\alpha\gamma}^{\eta} \left[\frac{1}{2} (g_{\eta\delta, \beta} + g_{\delta\beta, \eta} - g_{\eta\beta, \delta}) - g_{\delta\eta, \beta} \right]$$

$$\begin{aligned}
&= -\frac{1}{2} \frac{\partial}{\partial u^\gamma} (g_{\alpha\delta,\beta} + g_{\delta\beta,\alpha} - g_{\alpha\beta,\delta}) + \frac{1}{2} \frac{\partial}{\partial u^\beta} (g_{\alpha\delta,\gamma} + g_{\delta\gamma,\alpha} - g_{\alpha\gamma,\delta}) \\
&\quad + \Gamma_{\alpha\beta}^\eta \left[\frac{1}{2} (g_{\eta\delta,\gamma} - g_{\delta\gamma,\eta} + g_{\eta\gamma,\delta}) \right] - \Gamma_{\alpha\gamma}^\eta \left[\frac{1}{2} (g_{\eta\delta,\beta} - g_{\delta\beta,\eta} + g_{\eta\beta,\delta}) \right] \\
&= -\frac{1}{2} \frac{\partial}{\partial u^\gamma} (g_{\alpha\delta,\beta} + g_{\delta\beta,\alpha} - g_{\alpha\beta,\delta}) + \frac{1}{2} \frac{\partial}{\partial u^\beta} (g_{\alpha\delta,\gamma} + g_{\delta\gamma,\alpha} - g_{\alpha\gamma,\delta}) + \Gamma_{\alpha\beta}^\eta \Gamma_{\eta\delta\gamma} - \Gamma_{\alpha\gamma}^\eta \Gamma_{\eta\delta\beta} \\
&= \frac{1}{2} \left(-\frac{\partial^2 g_{\delta\beta}}{\partial u^\gamma \partial u^\alpha} + \frac{\partial^2 g_{\alpha\beta}}{\partial u^\gamma \partial u^\delta} + \frac{\partial^2 g_{\delta\gamma}}{\partial u^\beta \partial u^\alpha} - \frac{\partial^2 g_{\alpha\gamma}}{\partial u^\beta \partial u^\delta} \right) + \Gamma_{\alpha\beta}^\eta \Gamma_{\eta\delta\gamma} - \Gamma_{\alpha\gamma}^\eta \Gamma_{\eta\delta\beta}.
\end{aligned}$$

(2) 我们有

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{1}{2} g^{1\xi} (2g_{1\xi,1} - g_{11,\xi}) = \frac{1}{2} \frac{GE_u - 2FF_u + FE_v}{EG - F^2}, \\
\Gamma_{11}^2 &= \frac{1}{2} g^{2\xi} (2g_{1\xi,1} - g_{11,\xi}) = \frac{1}{2} \frac{-EE_v + 2EF_u - FE_u}{EG - F^2}, \\
\Gamma_{12}^1 &= \Gamma_{21}^1 = \frac{1}{2} g^{1\xi} (g_{1\xi,2} + g_{2\xi,1} - g_{12,\xi}) = \frac{1}{2} \frac{GE_v - FG_u}{EG - F^2}, \\
\Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{2} g^{2\xi} (g_{1\xi,2} + g_{2\xi,1} - g_{12,\xi}) = \frac{1}{2} \frac{EG_u - FE_v}{EG - F^2}, \\
\Gamma_{22}^1 &= \frac{1}{2} g^{1\xi} (2g_{2\xi,2} - g_{22,\xi}) = \frac{1}{2} \frac{-FG_v + 2GF_v - GG_u}{EG - F^2}, \\
\Gamma_{22}^2 &= \frac{1}{2} g^{2\xi} (2g_{2\xi,2} - g_{22,\xi}) = \frac{1}{2} \frac{EG_v - 2FF_v + FG_u}{EG - F^2}.
\end{aligned}$$

(3) 利用 (1) 可见 (每个) Gauss 方程

$$R_{\delta\alpha\beta\gamma} = -b_{\alpha\beta} b_{\gamma\delta} + b_{\alpha\gamma} b_{\beta\delta}$$

等号两端均关于 β, γ 反称, 关于 δ, α 反称, 而关于 $(\delta, \alpha), (\beta, \gamma)$ 对称. 因此只需考虑 $(\delta, \alpha, \beta, \gamma) = (1, 2, 1, 2)$ 的情形, 即方程

$$R_{1212} = b_{11} b_{22} - b_{12}^2.$$

先计算 $\Gamma_{112}, \Gamma_{212}, \Gamma_{111}, \Gamma_{211}$ 用 E, F, G 及其偏导数表示的表达式:

$$\begin{aligned}
\Gamma_{112} &= g_{1\xi} \Gamma_{12}^\xi = \frac{E(GE_v - FG_u) + F(EG_u - FE_v)}{2(EG - F^2)} = \frac{E_v}{2}, \\
\Gamma_{212} &= g_{2\xi} \Gamma_{12}^\xi = \frac{F(GE_v - FG_u) + G(EG_u - FE_v)}{2(EG - F^2)} = \frac{G_u}{2}, \\
\Gamma_{111} &= g_{1\xi} \Gamma_{11}^\xi = \frac{E(GE_u - 2FF_u + FE_v) - F(EE_v - 2EF_u + FE_u)}{2(EG - F^2)} = \frac{E_u}{2}, \\
\Gamma_{211} &= g_{2\xi} \Gamma_{11}^\xi = \frac{F(GE_u - 2FF_u + FE_v) - G(EE_v - 2EF_u + FE_u)}{2(EG - F^2)} = \frac{2F_u - E_v}{2}.
\end{aligned}$$

由 (1) 可得

$$R_{1212} = \frac{1}{2} (-E_{vv} + 2F_{uv} - G_{uu}) + \Gamma_{21}^1 \Gamma_{112} + \Gamma_{21}^2 \Gamma_{212} - \Gamma_{22}^1 \Gamma_{111} - \Gamma_{22}^2 \Gamma_{211}.$$

因此

$$\begin{aligned}
4(EG - F^2) R_{1212} &= 2(EG - F^2) (-E_{vv} + 2F_{uv} - G_{uu}) + (GE_v - FG_u) E_v + (EG_u - FE_v) G_u \\
&\quad + (FG_v - 2GF_v + GG_u) E_u - (EG_v - 2FF_v + FG_u) (2F_u - E_v) \\
&= E(E_v G_v - 2F_u G_v + (G_u)^2) + F(E_u G_v - E_v G_u - 2E_v F_v + 4F_u F_v - 2F_u G_u) \\
&\quad + G(E_u G_u - 2E_u F_v + (E_v)^2) - 2(EG - F^2) (E_{vv} - 2F_{uv} + G_{uu}).
\end{aligned}$$

由 Gauss 绝妙定理, 上式右端即 $4(EG - F^2)^2 K$, 因此 $R_{1212} = (EG - F^2) K$.

(4) 由 (2) 及 $F = M = 0$ 可得

$$\begin{aligned}\Gamma_{12}^{\xi} b_{\xi 1} &= \Gamma_{12}^1 b_{11} + \Gamma_{12}^2 b_{21} = \frac{LE_v}{2E}, \\ \Gamma_{11}^{\xi} b_{\xi 2} &= \Gamma_{11}^1 b_{12} + \Gamma_{11}^2 b_{22} = \frac{-NE_v}{2G}, \\ \Gamma_{22}^{\xi} b_{\xi 1} &= \Gamma_{22}^1 b_{11} + \Gamma_{22}^2 b_{21} = \frac{-LG_u}{2E}, \\ \Gamma_{21}^{\xi} b_{\xi 2} &= \Gamma_{21}^1 b_{12} + \Gamma_{21}^2 b_{22} = \frac{NG_u}{2G}.\end{aligned}$$

故此时 Codazzi 方程组等价于

$$\begin{cases} L_v = \frac{(LG + NE)E_v}{2EG}, \\ N_u = \frac{(LG + NE)G_u}{2EG}. \end{cases}$$

注意到平均曲率

$$H = \frac{1}{2} \frac{LG - 2MF + NE}{EG - F^2} = \frac{LG + NE}{2EG},$$

因此 Codazzi 方程组可表为

$$L_v = HE_v, \quad N_u = HG_u.$$

□

§4.2 正交活动标架运动方程、曲面上的微分形式和外微分、正交活动标架结构方程

题 38 在旋转曲面 $\mathbf{r}(u, v) = (u \cos v, u \sin v, f(u))$ 上建立正交标架场 $\{\mathbf{e}_1, \mathbf{e}_2\}$ 并求相应的诸微分形式.

解 由 $\mathbf{r}(u, v)$ 是旋转曲面可不妨设 $u > 0$. 因为 $\mathbf{r}_u = (\cos v, \sin v, f'(u))$ 与 $\mathbf{r}_v = (-u \sin v, u \cos v, 0)$ 正交, 所以可取

$$\mathbf{e}_1 = \frac{\mathbf{r}_u}{|\mathbf{r}_u|} = \frac{1}{\sqrt{E}} \mathbf{r}_u, \quad \mathbf{e}_2 = \frac{\mathbf{r}_v}{|\mathbf{r}_v|} = \frac{1}{\sqrt{G}} \mathbf{r}_v, \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2,$$

其中

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = 1 + (f'(u))^2, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = u^2.$$

由

$$d\mathbf{r} = \mathbf{r}_u du + \mathbf{r}_v dv$$

可得

$$\begin{aligned}\omega^1 &= \langle d\mathbf{r}, \mathbf{e}_1 \rangle = \left\langle \mathbf{r}_u du + \mathbf{r}_v dv, \frac{1}{\sqrt{E}} \mathbf{r}_u \right\rangle = \sqrt{E} du = \sqrt{1 + (f'(u))^2} du, \\ \omega^2 &= \langle d\mathbf{r}, \mathbf{e}_2 \rangle = \left\langle \mathbf{r}_u du + \mathbf{r}_v dv, \frac{1}{\sqrt{G}} \mathbf{r}_v \right\rangle = \sqrt{G} dv = u dv, \\ \omega^1 \wedge \omega^2 &= \sqrt{EG} du \wedge dv.\end{aligned}$$

又

$$\begin{aligned}d\omega^1 &= d(\sqrt{E} du) = d(\sqrt{E}) \wedge du = \left(\left(\frac{\sqrt{E}}{u} \right)_u du + \left(\frac{\sqrt{E}}{v} \right)_v dv \right) \wedge du = \left(\frac{\sqrt{E}}{v} \right)_v dv \wedge du \\ &= -\frac{\left(\frac{\sqrt{E}}{v} \right)_v}{\sqrt{EG}} \omega^1 \wedge \omega^2,\end{aligned}$$

$$\begin{aligned} d\omega^2 &= d(\sqrt{G} dv) = d(\sqrt{G}) \wedge dv = \left(\left(\frac{\sqrt{G}}{u} \right)_u du + \left(\frac{\sqrt{G}}{v} \right)_v dv \right) \wedge dv = \left(\frac{\sqrt{G}}{u} \right)_u du \wedge dv \\ &= \frac{\left(\frac{\sqrt{G}}{u} \right)_u}{\sqrt{EG}} \omega^1 \wedge \omega^2, \end{aligned}$$

因此

$$\omega_1^2 = -\frac{\left(\frac{\sqrt{E}}{v} \right)_v}{\sqrt{EG}} \omega^1 + \frac{\left(\frac{\sqrt{G}}{u} \right)_u}{\sqrt{EG}} \omega^2 = -\frac{\left(\frac{\sqrt{E}}{v} \right)_v}{\sqrt{G}} du + \frac{\left(\frac{\sqrt{G}}{u} \right)_u}{\sqrt{E}} dv = \frac{1}{\sqrt{1+(f'(u))^2}} dv.$$

再由

$$\begin{aligned} d\mathbf{e}_1 &= d\left(\frac{1}{\sqrt{E}} \mathbf{r}_u \right) = \left[\left(\frac{1}{\sqrt{E}} \right)_u du + \left(\frac{1}{\sqrt{E}} \right)_v dv \right] \mathbf{r}_u + \frac{1}{\sqrt{E}} (\mathbf{r}_{uu} du + \mathbf{r}_{uv} dv) \\ &= \left[\left(\frac{1}{\sqrt{E}} \right)_u \mathbf{r}_u + \frac{1}{\sqrt{E}} \mathbf{r}_{uu} \right] du + \left[\left(\frac{1}{\sqrt{E}} \right)_v \mathbf{r}_u + \frac{1}{\sqrt{E}} \mathbf{r}_{uv} \right] dv, \\ d\mathbf{e}_2 &= d\left(\frac{1}{\sqrt{G}} \mathbf{r}_v \right) = \left[\left(\frac{1}{\sqrt{G}} \right)_u du + \left(\frac{1}{\sqrt{G}} \right)_v dv \right] \mathbf{r}_v + \frac{1}{\sqrt{G}} (\mathbf{r}_{vu} du + \mathbf{r}_{vv} dv) \\ &= \left[\left(\frac{1}{\sqrt{G}} \right)_u \mathbf{r}_v + \frac{1}{\sqrt{G}} \mathbf{r}_{vu} \right] du + \left[\left(\frac{1}{\sqrt{G}} \right)_v \mathbf{r}_v + \frac{1}{\sqrt{G}} \mathbf{r}_{vv} \right] dv \end{aligned}$$

可得

$$\begin{aligned} \omega_1^3 &= \langle d\mathbf{e}_1, \mathbf{e}_3 \rangle = \frac{\langle \mathbf{r}_{uu}, \mathbf{e}_3 \rangle}{\sqrt{E}} du + \frac{\langle \mathbf{r}_{uv}, \mathbf{e}_3 \rangle}{\sqrt{E}} dv = \frac{L}{\sqrt{E}} du + \frac{M}{\sqrt{E}} dv, \\ \omega_2^3 &= \langle d\mathbf{e}_2, \mathbf{e}_3 \rangle = \frac{\langle \mathbf{r}_{vu}, \mathbf{e}_3 \rangle}{\sqrt{G}} du + \frac{\langle \mathbf{r}_{vv}, \mathbf{e}_3 \rangle}{\sqrt{G}} dv = \frac{M}{\sqrt{G}} du + \frac{N}{\sqrt{G}} dv. \end{aligned}$$

由

$$\begin{aligned} \mathbf{r}_{uu} &= (0, 0, f''(u)), \quad \mathbf{r}_{uv} = (-\sin v, \cos v, 0), \quad \mathbf{r}_{vv} = (-u \cos v, -u \sin v, 0), \\ \mathbf{r}_u \wedge \mathbf{r}_v &= (-u \cos v f'(u), -u \sin v f'(u), u), \quad \mathbf{e}_3 = \frac{1}{\sqrt{1+(f'(u))^2}} (-\cos v f'(u), -\sin v f'(u), 1) \end{aligned}$$

可得

$$L = \langle \mathbf{r}_{uu}, \mathbf{e}_3 \rangle = \frac{f''(u)}{\sqrt{1+(f'(u))^2}}, \quad M = \langle \mathbf{r}_{uv}, \mathbf{e}_3 \rangle = 0, \quad N = \langle \mathbf{r}_{vv}, \mathbf{e}_3 \rangle = \frac{uf'(u)}{\sqrt{1+(f'(u))^2}}.$$

代入即得

$$\omega_1^3 = \frac{f''(u)}{1+(f'(u))^2} du, \quad \omega_2^3 = \frac{f'(u)}{\sqrt{1+(f'(u))^2}} dv.$$

□

题 39 设 (u, v) 是曲面 S 的正交参数, $\mathbf{e}_1 = \frac{\mathbf{r}_u}{\sqrt{E}}, \mathbf{e}_2 = \frac{\mathbf{r}_v}{\sqrt{G}}$, 证明: 方程

$$\begin{cases} d\omega_1^3 = \omega_1^2 \wedge \omega_2^3, \\ d\omega_2^3 = \omega_2^1 \wedge \omega_1^3 \end{cases}$$

与 Codazzi 方程

$$\begin{cases} \left(\frac{L}{\sqrt{E}} \right)_v - \left(\frac{M}{\sqrt{E}} \right)_u - N \frac{\left(\frac{\sqrt{E}}{G} \right)_v}{G} - M \frac{\left(\frac{\sqrt{G}}{\sqrt{EG}} \right)_u}{\sqrt{EG}} = 0, \\ \left(\frac{N}{\sqrt{G}} \right)_u - \left(\frac{M}{\sqrt{G}} \right)_v - L \frac{\left(\frac{\sqrt{G}}{E} \right)_u}{E} - M \frac{\left(\frac{\sqrt{E}}{\sqrt{EG}} \right)_v}{\sqrt{EG}} = 0 \end{cases}$$

等价.

证明 在题 38 中已求得正交参数系下

$$\omega_1^2 = -\frac{(\sqrt{E})_v}{\sqrt{G}} du + \frac{(\sqrt{G})_u}{\sqrt{E}} dv, \quad \omega_1^3 = \frac{L}{\sqrt{E}} du + \frac{M}{\sqrt{E}} dv, \quad \omega_2^3 = \frac{M}{\sqrt{G}} du + \frac{N}{\sqrt{G}} dv.$$

因此

$$\begin{aligned} d\omega_1^3 &= \left[\left(\frac{M}{\sqrt{E}} \right)_u - \left(\frac{L}{\sqrt{E}} \right)_v \right] du \wedge dv, \\ d\omega_2^3 &= \left[\left(\frac{N}{\sqrt{G}} \right)_u - \left(\frac{M}{\sqrt{G}} \right)_v \right] du \wedge dv, \\ \omega_1^2 \wedge \omega_2^3 &= - \left[\frac{(\sqrt{E})_v}{\sqrt{G}} \frac{N}{\sqrt{G}} + \frac{(\sqrt{G})_u}{\sqrt{E}} \frac{M}{\sqrt{G}} \right] du \wedge dv, \\ \omega_2^1 \wedge \omega_1^3 &= \left[\frac{(\sqrt{E})_v}{\sqrt{G}} \frac{M}{\sqrt{E}} + \frac{(\sqrt{G})_u}{\sqrt{E}} \frac{L}{\sqrt{E}} \right] du \wedge dv. \end{aligned}$$

由此可见两个方程组等价. \square

题 40 设 M 为正则曲面片, $p \in M$ 不是脐点, 则存在 p 点的邻域 $U \subset M$, 使得 U 上存在正交活动标架 $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, 满足 $\mathbf{e}_1, \mathbf{e}_2$ 为主方向.

证明 设 $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 是 M 的正交活动标架, 其中 $\mathbf{e}_3 = \mathbf{n}$. 因为 p 不是脐点, 总可以令 $\mathbf{e}_1(p), \mathbf{e}_2(p)$ 在切平面内旋转使得它们均不是主方向. 由连续性, $k_1 \neq k_2$ 在 p 点的一个邻域内成立. 设 Weingarten 变换在 $\{\mathbf{e}_1, \mathbf{e}_2\}$ 下的矩阵表示为 $\begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix}$. 由假设及此矩阵的对称性, 有 $h_1^2(p) = h_2^1(p) \neq 0$, 由连续性, 在 p 点的一个邻域内有 $h_1^2 = h_2^1 \neq 0$. 结合 $h_1^1 h_2^2 - h_1^2 h_2^1 = k_1 k_2, h_1^1 + h_2^2 = k_1 + k_2$ 可注意到在此邻域内,

$$\mathbf{v}_1 = h_1^2 \mathbf{e}_1 + (k_1 - h_1^1) \mathbf{e}_2, \quad \mathbf{v}_2 = (k_2 - h_2^2) \mathbf{e}_1 + h_1^2 \mathbf{e}_2$$

为该矩阵的两个特征向量 (由 $h_1^2 \neq 0$ 知 $\mathbf{v}_1, \mathbf{v}_2 \neq \mathbf{0}$), 故可取 $\tilde{\mathbf{e}}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \tilde{\mathbf{e}}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \tilde{\mathbf{e}}_3 = \mathbf{e}_3$, 则 $\{\mathbf{r}; \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ 为满足题意的正交活动标架. \square

题 41 设 $\{\mathbf{e}_1, \mathbf{e}_2\}$ 是曲面的正交标架, $\mathbf{e}_1, \mathbf{e}_2$ 是曲面的主方向, k_1, k_2 是相应的主曲率. 证明: 这时曲面的 Codazzi 方程等价于

$$dk_1 \wedge \omega^1 = (k_2 - k_1) \omega_1^2 \wedge \omega^2, \quad dk_2 \wedge \omega^2 = (k_1 - k_2) \omega_2^1 \wedge \omega^1.$$

证明 由已知, $\begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$. 故此时 Codazzi 方程组为

$$\begin{cases} d(k_1 \omega^1) = \omega_1^2 \wedge (k_2 \omega^2), \\ d(k_2 \omega^2) = \omega_2^1 \wedge (k_1 \omega^1). \end{cases}$$

也即

$$\begin{cases} dk_1 \wedge \omega^1 = k_2 \omega_1^2 \wedge \omega^2 - k_1 d\omega^1, \\ dk_2 \wedge \omega^2 = k_1 \omega_2^1 \wedge \omega^1 - k_2 d\omega^2. \end{cases}$$

再代入 $d\omega^1 = \omega^2 \wedge \omega_2^1, d\omega^2 = \omega^1 \wedge \omega_1^2$ 即得题中形式. \square

题 42 验证: $\omega^1 \wedge \omega^2 = \sqrt{EG - F^2} du^1 \wedge du^2$.

证明 对 $\{\mathbf{r}_1, \mathbf{r}_2\}$ 作 Gram-Schmidt 标准正交化:

$$\begin{aligned} \mathbf{e}_1 &= \frac{\mathbf{r}_1}{|\mathbf{r}_1|} = \frac{\mathbf{r}_1}{\sqrt{E}}, \\ \mathbf{b} &= \mathbf{r}_2 - \langle \mathbf{r}_2, \mathbf{e}_1 \rangle \mathbf{e}_1 = \mathbf{r}_2 - \frac{F}{E} \mathbf{r}_1, \\ \mathbf{e}_2 &= \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{\sqrt{G - \frac{F^2}{E}}} \left(\mathbf{r}_2 - \frac{F}{E} \mathbf{r}_1 \right) = \frac{1}{\sqrt{EG - F^2}} \left(-\frac{F}{\sqrt{E}} \mathbf{r}_1 + \sqrt{E} \mathbf{r}_2 \right). \end{aligned}$$

故

$$\begin{aligned} \omega^1 &= \langle d\mathbf{r}, \mathbf{e}_1 \rangle = \langle \mathbf{r}_1 du^1 + \mathbf{r}_2 du^2, \mathbf{e}_1 \rangle = \sqrt{E} du^1 + \frac{F}{\sqrt{E}} du^2, \\ \omega^2 &= \langle d\mathbf{r}, \mathbf{e}_2 \rangle = \langle \mathbf{r}_1 du^1 + \mathbf{r}_2 du^2, \mathbf{e}_2 \rangle = \sqrt{\frac{EG - F^2}{E}} du^2, \\ \omega^1 \wedge \omega^2 &= \sqrt{EG - F^2} du^1 \wedge du^2. \end{aligned}$$

□

题 43 若正则曲面片 M 的 Gauss 曲率 $K \equiv 0$ 且无脐点, 则 M 是可展曲面.

证明 由题 40, 可取 M 上的正交活动标架 $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 使 $\mathbf{e}_1, \mathbf{e}_2$ 是 M 的主方向. 设对应的主曲率为 k_1, k_2 . 由 M 无脐点知 $k_1 \neq k_2$, 不妨设 $k_1 = 0, k_2 \neq 0$, 则 $\begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \begin{pmatrix} 0 \\ k_2 \omega^2 \end{pmatrix}$. 此时由 Codazzi 方程得

$$d\omega_1^3 = \omega_1^2 \wedge \omega_2^3 = \omega_1^2 \wedge (k_2 \omega^2) = k_2 \omega_1^2 \wedge \omega^2 = 0,$$

因为 $k_2 \neq 0$, 所以 $\omega_1^2 \wedge \omega^2 = 0, \omega_1^2 = f \omega^2$. 下面证明, 曲面 M 上满足 $\omega^2 \equiv 0$ 的曲线族是直线族. 将正交活动标架运动方程限制在任一这样的曲线上, 由 $\omega_1^2 = f \omega^2 = 0$ 得

$$d\mathbf{e}_1 = \omega_1^2 \mathbf{e}_2 + \omega_1^3 \mathbf{e}_3 = 0.$$

而

$$d\mathbf{r} = \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2 = \omega^1 \mathbf{e}_1,$$

说明此曲线的切向量为 E^3 中的常向量, 因此它是直线. 故 M 为直纹面, 结合 Gauss 曲率 $K \equiv 0$ 知 M 是可展曲面. □

题 44 设曲面 $S: \mathbf{r}(u^1, u^2)$ 有参数变换 $u^\alpha = u^\alpha(\tilde{u}^1, \tilde{u}^2), \alpha = 1, 2$. 记 $a_i^\alpha = \frac{\partial u^\alpha}{\partial \tilde{u}^i}, \tilde{a}_\alpha^i = \frac{\partial \tilde{u}^i}{\partial u^\alpha} (1 \leq \alpha, i \leq 2)$, S 在参数 $(\tilde{u}^1, \tilde{u}^2)$ 下的第一、第二基本形式为 $\{\tilde{g}_{ij}\}, \{\tilde{b}_{ij}\}$, 证明:

- (1) $\tilde{g}_{ij} = g_{\alpha\beta} a_i^\alpha a_j^\beta, \tilde{b}_{ij} = \text{sgn}(\det(a_\beta^\alpha)) b_{\alpha\beta} a_i^\alpha a_j^\beta, g^{\alpha\beta} = \tilde{g}^{ij} a_i^\alpha a_j^\beta;$
- (2) $\tilde{\Gamma}_{ij}^k = \Gamma_{\alpha\beta}^\gamma a_i^\alpha a_j^\beta \tilde{a}_\gamma^k + \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \tilde{a}_\alpha^k.$

证明 (1) 记 $\tilde{\mathbf{r}}(\tilde{u}^1, \tilde{u}^2) = \mathbf{r}(u^1(\tilde{u}^1, \tilde{u}^2), u^2(\tilde{u}^1, \tilde{u}^2))$, 则 $\tilde{\mathbf{r}}_i = \mathbf{r}_\alpha \frac{\partial u^\alpha}{\partial \tilde{u}^i} = a_i^\alpha \mathbf{r}_\alpha$, 从而

$$\tilde{g}_{ij} = \langle \tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j \rangle = \langle a_i^\alpha \mathbf{r}_\alpha, a_j^\beta \mathbf{r}_\beta \rangle = a_i^\alpha a_j^\beta \langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle = g_{\alpha\beta} a_i^\alpha a_j^\beta.$$

又

$$\tilde{\mathbf{r}}_1 \wedge \tilde{\mathbf{r}}_2 = (a_1^\alpha \mathbf{r}_\alpha) \wedge (a_2^\beta \mathbf{r}_\beta) = \det(a_\beta^\alpha) \mathbf{r}_1 \wedge \mathbf{r}_2,$$

因此

$$\tilde{\mathbf{n}} = \frac{\tilde{\mathbf{r}}_1 \wedge \tilde{\mathbf{r}}_2}{|\tilde{\mathbf{r}}_1 \wedge \tilde{\mathbf{r}}_2|} = \operatorname{sgn}(\det(a_\beta^\alpha)) \mathbf{n}.$$

记 $a_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial \tilde{u}^i \partial \tilde{u}^j}$, 则 $\tilde{\mathbf{r}}_{ij} = \frac{\partial}{\partial \tilde{u}^j} (a_i^\alpha \mathbf{r}_\alpha) = a_{ij}^\alpha \mathbf{r}_\alpha + a_i^\alpha a_j^\beta \mathbf{r}_{\alpha\beta}$, 于是

$$\tilde{b}_{ij} = \langle \tilde{\mathbf{r}}_{ij}, \tilde{\mathbf{n}} \rangle = \langle a_{ij}^\alpha \mathbf{r}_\alpha + a_i^\alpha a_j^\beta \mathbf{r}_{\alpha\beta}, \operatorname{sgn}(\det(a_\beta^\alpha)) \mathbf{n} \rangle = \operatorname{sgn}(\det(a_\beta^\alpha)) b_{\alpha\beta} a_i^\alpha a_j^\beta.$$

设 $G = (g_{\alpha\beta})_{\alpha\beta}$, $\tilde{G} = (\tilde{g}_{ij})_{ij}$, $A = (a_{i\alpha})_{i\alpha}$, 则由 $\tilde{g}_{ij} = a_i^\alpha g_{\alpha\beta} a_j^\beta$ 可知 $\tilde{G} = AGA^T$, 因此 $\tilde{G}^{-1} = A^{-T}G^{-1}A^{-1}$, 从而 $G^{-1} = A^T \tilde{G}^{-1} A$, 即 $g^{\alpha\beta} = \tilde{g}^{ij} a_i^\alpha a_j^\beta$.

(2) 一方面, 由曲面自然标架运动方程,

$$\frac{\partial \tilde{\mathbf{r}}_i}{\partial \tilde{u}^j} = \tilde{\Gamma}_{ij}^k \tilde{\mathbf{r}}_k + \tilde{b}_{ij} \tilde{\mathbf{n}}.$$

另一方面,

$$\begin{aligned} \frac{\partial \tilde{\mathbf{r}}_i}{\partial \tilde{u}^j} &= \frac{\partial}{\partial \tilde{u}^j} \left(\frac{\partial \mathbf{r}}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \tilde{u}^i} \right) \\ &= \frac{\partial}{\partial \tilde{u}^j} \left(\frac{\partial \mathbf{r}}{\partial u^\alpha} \right) \frac{\partial u^\alpha}{\partial \tilde{u}^i} + \frac{\partial \mathbf{r}}{\partial u^\alpha} \frac{\partial^2 u^\alpha}{\partial \tilde{u}^j \partial \tilde{u}^i} \\ &= \frac{\partial u^\beta}{\partial \tilde{u}^j} \frac{\partial}{\partial u^\beta} \left(\frac{\partial \mathbf{r}}{\partial u^\alpha} \right) \frac{\partial u^\alpha}{\partial \tilde{u}^i} + \frac{\partial \mathbf{r}}{\partial u^\alpha} \frac{\partial^2 u^\alpha}{\partial \tilde{u}^j \partial \tilde{u}^i} \\ &= a_j^\beta a_i^\alpha (\Gamma_{\alpha\beta}^\gamma \mathbf{r}_\gamma + b_{\alpha\beta} \mathbf{n}) + \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \mathbf{r}_\alpha \\ &= \left(\Gamma_{\gamma\beta}^\alpha a_j^\beta a_i^\gamma + \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \right) \mathbf{r}_\alpha + a_j^\beta a_i^\gamma b_{\gamma\beta} \mathbf{n} \\ &= \left(\Gamma_{\gamma\beta}^\alpha a_j^\beta a_i^\gamma + \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \right) \tilde{a}_\alpha^k \tilde{\mathbf{r}}_k + a_j^\beta a_i^\gamma b_{\gamma\beta} \mathbf{n}, \end{aligned}$$

故

$$\tilde{\Gamma}_{ij}^k = \Gamma_{\gamma\beta}^\alpha a_j^\beta a_i^\gamma \tilde{a}_\alpha^k + \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \tilde{a}_\alpha^k = \Gamma_{\alpha\beta}^\gamma a_i^\alpha a_j^\beta \tilde{a}_\gamma^k + \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \tilde{a}_\alpha^k.$$

□

第五章 曲面的内蕴几何学

§5.1 测地线、测地曲率、协变导数和平行移动

题 45 在球面 $\mathbf{r} = (a \cos u \cos v, a \cos u \sin v, a \sin u)$ 上,

(1) 证明: 曲线的测地曲率可以表示为

$$k_g = \frac{d\theta}{ds} - \sin u \frac{dv}{ds},$$

其中 s 是曲线 $\mathbf{r}(u(s), v(s))$ 的弧长参数, θ 是曲线与经线 (u 线) 的夹角;

(2) 求曲面纬圆的测地曲率.

解 (1) 由

$$\mathbf{r}_u = (-a \sin u \cos v, -a \sin u \sin v, a \cos u), \quad \mathbf{r}_v = (-a \cos u \sin v, a \cos u \cos v, 0)$$

可知

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = a^2, \quad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = a^2 \cos^2 u.$$

因此 (u, v) 是曲面的正交参数, 根据 Liouville 公式, 曲线的测地曲率

$$\begin{aligned} k_g &= \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}} \frac{\partial \ln E}{\partial v} \cos \theta + \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial u} \sin \theta \\ &= \frac{d\theta}{ds} - \frac{1}{2a \cos u} \frac{\partial \ln(a^2)}{\partial v} \cos \theta + \frac{1}{2a} \frac{\partial \ln(a^2 \cos^2 u)}{\partial u} \sin \theta \\ &= \frac{d\theta}{ds} + \frac{1}{a} \frac{\partial \ln(a \cos u)}{\partial u} \sin \theta \\ &= \frac{d\theta}{ds} - \frac{1}{a} \sin \theta \tan u. \end{aligned}$$

由于 s 是曲线 $\mathbf{r}(s) = \mathbf{r}(u(s), v(s))$ 的弧长参数,

$$1 = \langle \dot{\mathbf{r}}(s), \dot{\mathbf{r}}(s) \rangle = \left\langle \frac{du}{ds} \mathbf{r}_u + \frac{dv}{ds} \mathbf{r}_v, \frac{du}{ds} \mathbf{r}_u + \frac{dv}{ds} \mathbf{r}_v \right\rangle = E \left(\frac{du}{ds} \right)^2 + G \left(\frac{dv}{ds} \right)^2,$$

进而

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \left(\frac{\langle \mathbf{r}_u, \frac{du}{ds} \mathbf{r}_u + \frac{dv}{ds} \mathbf{r}_v \rangle}{\sqrt{E}} \right)^2 = 1 - \left(\frac{E \frac{du}{ds}}{\sqrt{E}} \right)^2 = 1 - E \left(\frac{du}{ds} \right)^2 = G \left(\frac{dv}{ds} \right)^2,$$

即

$$\sin \theta = \sqrt{G} \frac{dv}{ds} = a \cos u \frac{dv}{ds}.$$

将其代入测地曲率表达式中即得

$$k_g = \frac{d\theta}{ds} - \frac{1}{a} \cdot a \cos u \frac{dv}{ds} \cdot \tan u = \frac{d\theta}{ds} - \sin u \frac{dv}{ds}.$$

(2) 纬圆 $\mathbf{r}(u_0, v)$ 与经线夹角 $\theta = \frac{\pi}{2}$, 代入 (1) 证明过程中得到的公式就有

$$k_g = \frac{d\theta}{ds} - \frac{1}{a} \tan u_0 = -\frac{1}{a} \tan u_0.$$

□

题 46 设 S 是 E^3 的曲面, \mathbf{n} 是 S 的单位法向量场, $\mathbf{r}(t)$ 是曲面 S 上的正则曲线. 若 $\mathbf{v} = \mathbf{v}(t), \mathbf{w} = \mathbf{w}(t)$ 是沿曲线 $\mathbf{r}(t)$ 的曲面的单位切向量场, θ 是 \mathbf{v} 和 \mathbf{w} 的夹角, 证明:

$$\left\langle \frac{D\mathbf{w}}{dt}, \mathbf{n} \wedge \mathbf{w} \right\rangle - \left\langle \frac{D\mathbf{v}}{dt}, \mathbf{n} \wedge \mathbf{v} \right\rangle = \frac{d\theta}{dt}.$$

证明 由混合积的轮换对称性,

$$\begin{aligned} & \left\langle \frac{D\mathbf{w}}{dt}, \mathbf{n} \wedge \mathbf{w} \right\rangle - \left\langle \frac{D\mathbf{v}}{dt}, \mathbf{n} \wedge \mathbf{v} \right\rangle \\ &= \left(\frac{D\mathbf{w}}{dt}, \mathbf{n}, \mathbf{w} \right) - \left(\frac{D\mathbf{v}}{dt}, \mathbf{n}, \mathbf{v} \right) \\ &= \left(\mathbf{n}, \mathbf{w}, \frac{D\mathbf{w}}{dt} \right) - \left(\mathbf{n}, \mathbf{v}, \frac{D\mathbf{v}}{dt} \right) \\ &= \left\langle \mathbf{n}, \mathbf{w} \wedge \frac{D\mathbf{w}}{dt} \right\rangle - \left\langle \mathbf{n}, \mathbf{v} \wedge \frac{D\mathbf{v}}{dt} \right\rangle \\ &= \left\langle \mathbf{n}, \mathbf{w} \wedge \left(\frac{d\mathbf{w}}{dt} - \left\langle \frac{d\mathbf{w}}{dt}, \mathbf{n} \right\rangle \mathbf{n} \right) \right\rangle - \left\langle \mathbf{n}, \mathbf{v} \wedge \left(\frac{d\mathbf{v}}{dt} - \left\langle \frac{d\mathbf{v}}{dt}, \mathbf{n} \right\rangle \mathbf{n} \right) \right\rangle \end{aligned}$$

$$= \left\langle \mathbf{n}, \mathbf{w} \wedge \frac{d\mathbf{w}}{dt} - \mathbf{v} \wedge \frac{d\mathbf{v}}{dt} \right\rangle.$$

由于 \mathbf{v}, \mathbf{w} 是沿曲线 $\mathbf{r}(t)$ 的曲面的单位切向量场, 可取 $\mathbf{e}_1 = \mathbf{v}, \mathbf{e}_2 = \mathbf{n} \wedge \mathbf{e}_1$, 则 $\mathbf{w} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2$. 由

$$\frac{d\mathbf{w}}{dt} = -\sin \theta \frac{d\theta}{dt} \mathbf{e}_1 + \cos \theta \frac{d\mathbf{e}_1}{dt} + \cos \theta \frac{d\theta}{dt} \mathbf{e}_2 + \sin \theta \frac{d\mathbf{e}_2}{dt}$$

得

$$\begin{aligned} \mathbf{w} \wedge \frac{d\mathbf{w}}{dt} &= \cos^2 \theta \mathbf{e}_1 \wedge \frac{d\mathbf{e}_1}{dt} + \cos^2 \theta \frac{d\theta}{dt} \mathbf{n} + \sin \theta \cos \theta \mathbf{e}_1 \wedge \frac{d\mathbf{e}_2}{dt} + \sin^2 \theta \frac{d\theta}{dt} \mathbf{n} \\ &\quad + \sin \theta \cos \theta \mathbf{e}_2 \wedge \frac{d\mathbf{e}_1}{dt} + \sin^2 \theta \mathbf{e}_2 \wedge \frac{d\mathbf{e}_2}{dt}. \end{aligned}$$

又

$$\mathbf{v} \wedge \frac{d\mathbf{v}}{dt} = \mathbf{e}_1 \wedge \frac{d\mathbf{e}_1}{dt},$$

所以

$$\mathbf{w} \wedge \frac{d\mathbf{w}}{dt} - \mathbf{v} \wedge \frac{d\mathbf{v}}{dt} = \sin^2 \theta \left(\mathbf{e}_2 \wedge \frac{d\mathbf{e}_2}{dt} - \mathbf{e}_1 \wedge \frac{d\mathbf{e}_1}{dt} \right) + \sin \theta \cos \theta \left(\mathbf{e}_1 \wedge \frac{d\mathbf{e}_2}{dt} - \mathbf{e}_2 \wedge \frac{d\mathbf{e}_1}{dt} \right) + \frac{d\theta}{dt} \mathbf{n}.$$

注意到

$$\frac{d\mathbf{e}_2}{dt} = \frac{d}{dt}(\mathbf{n} \wedge \mathbf{e}_1) = \underbrace{\frac{d\mathbf{n}}{dt} \wedge \mathbf{e}_1}_{\text{与 } \mathbf{n} \text{ 平行}} + \underbrace{\mathbf{n} \wedge \frac{d\mathbf{e}_1}{dt}}_{\text{与 } \mathbf{e}_1 \text{ 平行}},$$

因此若设

$$\frac{d\mathbf{e}_1}{dt} = a\mathbf{e}_2 + b\mathbf{n},$$

则

$$\frac{d\mathbf{e}_2}{dt} = \frac{d\mathbf{n}}{dt} \wedge \mathbf{e}_1 + a\mathbf{n} \wedge \mathbf{e}_2 = \frac{d\mathbf{n}}{dt} \wedge \mathbf{e}_1 - a\mathbf{e}_1,$$

故可设

$$\frac{d\mathbf{e}_2}{dt} = -a\mathbf{e}_1 + c\mathbf{n}.$$

于是

$$\mathbf{w} \wedge \frac{d\mathbf{w}}{dt} - \mathbf{v} \wedge \frac{d\mathbf{v}}{dt} = \sin^2 \theta (a\mathbf{n} + c\mathbf{e}_1 - a\mathbf{n} + b\mathbf{e}_2) + \sin \theta \cos \theta (-c\mathbf{e}_2 + b\mathbf{e}_1) + \frac{d\theta}{dt} \mathbf{n},$$

从而

$$\left\langle \frac{D\mathbf{w}}{dt}, \mathbf{n} \wedge \mathbf{w} \right\rangle - \left\langle \frac{D\mathbf{v}}{dt}, \mathbf{n} \wedge \mathbf{v} \right\rangle = \left\langle \mathbf{n}, \mathbf{w} \wedge \frac{d\mathbf{w}}{dt} - \mathbf{v} \wedge \frac{d\mathbf{v}}{dt} \right\rangle = \left\langle \mathbf{n}, \frac{d\theta}{dt} \mathbf{n} \right\rangle = \frac{d\theta}{dt}.$$

□

题 47 设曲线 C 是旋转曲面 $\mathbf{r}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$ 上的一条测地线, θ 是曲线 C 与经线的夹角, 证明: 沿 C 有 $f(u) \sin \theta = \text{常数}$.

证明 由曲面是旋转曲面可不妨设 $f(u) > 0$. 设 s 是曲线 C 的弧长参数. 由

$$\mathbf{r}_u = (f'(u) \cos v, f'(u) \sin v, g'(u)), \quad \mathbf{r}_v = (-f(u) \sin v, f(u) \cos v, 0)$$

可知

$$E = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = (f'(u))^2 + (g'(u))^2, \quad F = \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0, \quad G = \langle \mathbf{r}_v, \mathbf{r}_v \rangle = (f(u))^2.$$

因此 (u, v) 是曲面的正交参数, 根据 Liouville 公式, 测地线 C 满足

$$0 = k_g = \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}} \frac{\partial \ln E}{\partial v} \cos \theta + \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial u} \sin \theta,$$

即

$$\begin{aligned} \frac{d\theta}{ds} &= \frac{1}{2\sqrt{G}} \frac{\partial \ln E}{\partial v} \cos \theta - \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial u} \sin \theta = -\frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial u} \sin \theta \\ &= -\frac{1}{\sqrt{(f'(u))^2 + (g'(u))^2}} \frac{\partial \ln f(u)}{\partial u} \sin \theta = -\frac{f'(u)}{f(u)\sqrt{(f'(u))^2 + (g'(u))^2}} \sin \theta. \end{aligned}$$

又 $\cos \theta = \sqrt{E} \frac{du}{ds}$, 于是沿曲线 C 有

$$\begin{aligned} \frac{d}{ds} (f(u) \sin \theta) &= f'(u) \frac{du}{ds} \sin \theta + f(u) \cos \theta \frac{d\theta}{ds} \\ &= f'(u) \cdot \frac{\cos \theta}{\sqrt{(f'(u))^2 + (g'(u))^2}} \cdot \sin \theta + f(u) \cos \theta \left(-\frac{f'(u)}{f(u)\sqrt{(f'(u))^2 + (g'(u))^2}} \sin \theta \right) \\ &= 0, \end{aligned}$$

即沿曲线 C 有 $f(u) \sin \theta = \text{常数}$. □

§5.2 法坐标系、Gauss 引理、测地线局部最短性、常 Gauss 曲率曲面

题 48 设曲面的第一基本形式为 $I = du \otimes du + G(u, v) dv \otimes dv$, 且 $G(0, v) = 1$, $G_u(0, v) = 0$, 证明:

$$G(u, v) = 1 - u^2 K(0, v) + o(u^2).$$

证明 设曲面参数化为 $\mathbf{r} = \mathbf{r}(u, v)$. 取正交活动标架 $\mathbf{e}_1 = \mathbf{r}_u, \mathbf{e}_2 = \frac{\mathbf{r}_v}{\sqrt{G}}$, 则 $\omega^1 = du, \omega^2 = \sqrt{G} dv$. 由

$$d\omega^1 = d^2u = 0, \quad d\omega^2 = d(\sqrt{G} dv) = (\sqrt{G})_u du \wedge dv = \frac{(\sqrt{G})_u}{\sqrt{G}} \omega^1 \wedge \omega^2$$

可知

$$\omega_1^2 = \frac{(\sqrt{G})_u}{\sqrt{G}} \omega^2 = (\sqrt{G})_u dv.$$

因此

$$d\omega_1^2 = (\sqrt{G})_{uu} du \wedge dv = \frac{(\sqrt{G})_{uu}}{\sqrt{G}} \omega^1 \wedge \omega^2.$$

故 Gauss 曲率

$$K = -\frac{d\omega_1^2}{\omega^1 \wedge \omega^2} = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$

由 $(\sqrt{G})_u = \frac{G_u}{2\sqrt{G}}$ 得 $G_u = 2\sqrt{G} (\sqrt{G})_u$, 进而

$$G_{uu} = \frac{\partial}{\partial u} (2\sqrt{G} (\sqrt{G})_u) = 2 \left[(\sqrt{G})_{uu} \right]^2 + 2\sqrt{G} (\sqrt{G})_{uu} = \frac{G_u^2}{2G} - 2KG.$$

于是

$$G_{uu}(0, v) = \frac{(G_u(0, v))^2}{2G(0, v)} - 2K(0, v)G(0, v) = -2K(0, v).$$

对每一个固定的 v , 将 $G(u, v)$ 在 $u = 0$ 处作 Taylor 展开,

$$G(u, v) = G(0, v) + uG_u(0, v) + \frac{u^2}{2}G_{uu}(0, v) + o(u^2) = 1 - u^2K(0, v) + o(u^2).$$

□

题 49 证明: 在常 Gauss 曲率曲面上, 测地圆具有常测地曲率.

证明 设以曲面上一点 p 为中心的测地极坐标系参数为 (ρ, θ) . 设测地圆为 $\mathbf{c}(s) = \mathbf{r}(\rho_0, \theta(s))$, 其中 s 为弧长参数. 注意到测地圆与 ρ -曲线夹角始终为 $\frac{\pi}{2}$.

① 若 $K \equiv 0$, 则 $I = d\rho \otimes d\rho + \rho^2 d\theta \otimes d\theta$. 由 Liouville 公式,

$$k_g = \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial \rho} = \frac{1}{2} \frac{\partial \ln(\rho^2)}{\partial \rho} \Big|_{\rho=\rho_0} = \frac{1}{\rho_0}.$$

② 若 $K \equiv \frac{1}{a^2} > 0$ ($a > 0$), 则 $I = d\rho \otimes d\rho + a^2 \sin^2 \frac{\rho}{a} d\theta \otimes d\theta$. 由 Liouville 公式,

$$k_g = \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial \rho} = \frac{1}{2} \frac{\partial \ln(a^2 \sin^2 \frac{\rho}{a})}{\partial \rho} \Big|_{\rho=\rho_0} = \frac{1}{a} \cot \frac{\rho_0}{a}.$$

③ 若 $K \equiv -\frac{1}{a^2} < 0$ ($a > 0$), 则 $I = d\rho \otimes d\rho + a^2 \sinh^2 \frac{\rho}{a} d\theta \otimes d\theta$. 由 Liouville 公式,

$$k_g = \frac{1}{2\sqrt{E}} \frac{\partial \ln G}{\partial \rho} = \frac{1}{2} \frac{\partial \ln(a^2 \sinh^2 \frac{\rho}{a})}{\partial \rho} \Big|_{\rho=\rho_0} = \frac{1}{a} \coth \frac{\rho_0}{a}.$$

□

题 50 (测地平行坐标系的 Gauss 引理) 设 C 为曲面 M 上的一条曲线, 过 C 上每一点取与之正交的测地线, 得一族测地线, 连接这族测地线在 C 同一侧到与 C 交点弧长相等的点所得曲线与这族测地线正交.

证明 取曲线 C 的弧长参数 v , 不妨设 $p \in C$ 使 $C(0) = p$. 再取每一条测地线的弧长参数 u , 使得曲线 C 恰满足 $u = 0$. 这就得到 p 点附近曲面的一个参数化 $\mathbf{r} = \mathbf{r}(u, v)$. 由于这族测地线与 $C(v)$ 正交,

$$F(0, v) = \langle \mathbf{r}_u, \mathbf{r}_v \rangle |_{u=0} = 0.$$

又

$$\frac{\partial F}{\partial u} = \frac{\partial}{\partial u} \langle \mathbf{r}_u, \mathbf{r}_v \rangle = \langle \mathbf{r}_{uu}, \mathbf{r}_v \rangle + \langle \mathbf{r}_u, \mathbf{r}_{vu} \rangle,$$

注意到 \mathbf{r}_{uu} 为 u -曲线的曲率向量, 由 u -曲线是测地线可知 \mathbf{r}_{uu} 在 p 点处曲面切平面投影为 0, 从而 $\langle \mathbf{r}_{uu}, \mathbf{r}_v \rangle = 0$; 而

$$\langle \mathbf{r}_u, \mathbf{r}_{uu} \rangle = \frac{1}{2} \frac{\partial}{\partial v} \langle \mathbf{r}_u, \mathbf{r}_u \rangle = 0,$$

因此 $\frac{\partial F}{\partial u} = 0$, $F \equiv F(0, v) = 0$. □

第六章 微分流形基础

§6.1 抽象曲面

题 51 对于集合 $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, 令 $U_1 = S^2 \setminus \{(0, 0, 1)\}$, ϕ_1 为从 $(0, 0, 1)$ 点出发的球极投影; $U_2 = S^2 \setminus \{(0, 0, -1)\}$, ϕ_2 为从 $(0, 0, -1)$ 点出发的球极投影.

(1) 证明: $\{(U_1, \phi_1), (U_2, \phi_2)\}$ 是 S^2 的一个 C^∞ -坐标图册.

(2) 令 $U_0 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$. 定义映射

$$\phi_0 : U_0 \rightarrow \mathbb{R}^2, \quad (x, y, z) \mapsto (x, y).$$

证明: 坐标卡 (U_0, ϕ_0) 含在包括 $(U_1, \phi_1), (U_2, \phi_2)$ 的最大 C^∞ -坐标图册里.

证明 (1) 由

$$\begin{aligned} \phi_1 : U_1 \rightarrow \mathbb{R}^2, \quad (x, y, z) &\mapsto \left(\frac{x}{1-z}, \frac{y}{1-z} \right), \\ \phi_2 : U_2 \rightarrow \mathbb{R}^2, \quad (x, y, z) &\mapsto \left(\frac{x}{1+z}, \frac{y}{1+z} \right) \end{aligned}$$

以及它们的逆映射

$$\begin{aligned} \phi_1^{-1} : \mathbb{R}^2 \rightarrow U_1, \quad (u, v) &\mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right) \\ \phi_2^{-1} : \mathbb{R}^2 \rightarrow U_2, \quad (u, v) &\mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1} \right) \end{aligned}$$

均连续可知 ϕ_1, ϕ_2 均为微分同胚, 而 $U_1 \cup U_2 = S^2$, 因此 $\{(U_1, \phi_1), (U_2, \phi_2)\}$ 是 S^2 的一个坐标图册. 又

$$\begin{aligned} \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) &\rightarrow \phi_2(U_1 \cap U_2), \quad (u, v) \mapsto \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) \\ \phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) &\rightarrow \phi_1(U_1 \cap U_2), \quad (u, v) \mapsto \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) \end{aligned}$$

均光滑, 坐标卡 (U_1, ϕ_1) 与 (U_2, ϕ_2) 是 C^∞ -相容的. 故 $\{(U_1, \phi_1), (U_2, \phi_2)\}$ 是 S^2 的一个 C^∞ -坐标图册.

(2) 由

$$\begin{aligned} \phi_0 : U_0 \rightarrow \mathbb{R}^2, \quad (x, y, z) &\mapsto (x, y), \\ \phi_0^{-1} : \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} &\rightarrow U_0, \quad (x, y) \mapsto \left(x, y, \sqrt{1-x^2-y^2} \right) \end{aligned}$$

均连续可知 ϕ_0 为微分同胚. 又

$$\begin{aligned} \phi_1 \circ \phi_0^{-1} : \phi_0(U_0 \cap U_1) &\rightarrow \phi_1(U_0 \cap U_1), \quad (u, v) \mapsto \left(\frac{u}{1-\sqrt{1-u^2-v^2}}, \frac{v}{1-\sqrt{1-u^2-v^2}} \right), \\ \phi_2 \circ \phi_0^{-1} : \phi_0(U_0 \cap U_2) &\rightarrow \phi_2(U_0 \cap U_2), \quad (u, v) \mapsto \left(\frac{u}{1+\sqrt{1-u^2-v^2}}, \frac{v}{1+\sqrt{1-u^2-v^2}} \right), \\ \phi_0 \circ \phi_1^{-1} : \phi_1(U_0 \cap U_1) &\rightarrow \phi_0(U_0 \cap U_1), \quad (u, v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1} \right), \\ \phi_0 \circ \phi_2^{-1} : \phi_2(U_0 \cap U_2) &\rightarrow \phi_0(U_0 \cap U_2), \quad (u, v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1} \right) \end{aligned}$$

均光滑可知 (U_0, ϕ_0) 与 $(U_1, \phi_1), (U_2, \phi_2)$ 均 C^∞ -相容, 即 (U_0, ϕ_0) 含在包括 $(U_1, \phi_1), (U_2, \phi_2)$ 的最大 C^∞ -坐标图册里. \square

题 52 设 M 为一抽象光滑曲面, A 为一集合, $f : M \rightarrow A$ 为一个双射.

(1) 证明: 存在唯一方式使 A 成为抽象光滑流形且 f 是微分同胚.

(2) 利用 (1), 说明集合 $\{\mathbb{R}$ 中所有有限闭区间 $\}$ 有抽象光滑曲面结构.

证明 (1) 取 $\{(U_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$ 为 M 的极大 C^∞ -坐标图册. 我们断言 $\{(f(U_\lambda), \phi_\lambda \circ f^{-1}) \mid \lambda \in \Lambda\}$ 为 A 的极大 C^∞ -坐标图册, 检验如下. 由 f 是双射可知 $\bigcup_{\lambda \in \Lambda} f(U_\lambda) = A$. 再由 $\{(U_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$ 为 M 的 C^∞ -坐标图册可知

$$(\phi_{\lambda_1} \circ f^{-1}) \circ (\phi_{\lambda_2} \circ f^{-1})^{-1} = \phi_{\lambda_1} \circ \phi_{\lambda_2}^{-1} \text{ 光滑, } \forall \lambda_1, \lambda_2 \in \Lambda,$$

即 $\{(U_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$ 是 C^∞ -相容的. 而对任意与该坐标图册相容的坐标卡 $(f(U), \phi \circ f^{-1})$, 我们有 $\phi \circ \phi_\lambda^{-1}$ 与 $\phi_\lambda \circ \phi^{-1}$ 均光滑, $\forall \lambda \in \Lambda$, 即 (U, ϕ) 与坐标图册 $\{(U_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$ 是 C^∞ -相容的. 再由坐标图册 $\{(U_\lambda, \phi_\lambda) \mid \lambda \in \Lambda\}$ 的极大性可知存在 $\lambda_0 \in \Lambda$ 使得 $(U, \phi) = (U_0, \phi_0)$, 进而 $(f(U), \phi \circ f^{-1}) = (f(U_0), \phi_0 \circ f^{-1})$. 故 $\{(f(U_\lambda), \phi_\lambda \circ f^{-1}) \mid \lambda \in \Lambda\}$ 为 A 的极大 C^∞ -坐标图册. 又因为当 $f: M \rightarrow A$ 是微分同胚时, A 上的拓扑结构是唯一确定的, 所以 A 上的光滑结构是唯一的.

(2) 定义映射

$$g: \{\mathbb{R} \text{ 中所有有限闭区间}\} \rightarrow \mathbb{R}^2, \quad [x, y] \mapsto (x, y),$$

则 $\text{Im}(g) = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$ 是 \mathbb{R}^2 中开集, 显然是光滑流形. 因为 $g^{-1}: \text{Im}(g) \rightarrow \{\mathbb{R} \text{ 中所有有限闭区间}\}$ 为双射, 由 (1) 知 $\{\mathbb{R} \text{ 中所有有限闭区间}\}$ 有抽象光滑曲面结构. \square

题 53 设 S^2 为 \mathbb{R}^3 中单位球面, \mathbb{RP}^2 为实射影平面. 定义映射

$$f: S^2 \rightarrow \mathbb{RP}^2, \quad p \mapsto \{-p, p\}.$$

证明: f 的秩为 2.

证明 记 $U_i = \{(x^1 : x^2 : x^3) \in \mathbb{RP}^2 \mid x^i \neq 0\}$, $i = 1, 2, 3$. 定义坐标映射

$$\begin{aligned} \phi_1: U_1 &\rightarrow \mathbb{R}^2, & (x^1 : x^2 : x^3) &\mapsto \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right), \\ \phi_2: U_2 &\rightarrow \mathbb{R}^2, & (x^1 : x^2 : x^3) &\mapsto \left(\frac{x^3}{x^2}, \frac{x^1}{x^2}\right), \\ \phi_3: U_3 &\rightarrow \mathbb{R}^2, & (x^1 : x^2 : x^3) &\mapsto \left(\frac{x^1}{x^3}, \frac{x^2}{x^3}\right). \end{aligned}$$

则 $\{(U_i, \phi_i) \mid i = 1, 2, 3\}$ 是 \mathbb{RP}^2 的一个坐标图册.

记 $V_{i+} = \{(x^1, x^2, x^3) \mid x^i > 0\}$, $V_{i-} = \{(x^1, x^2, x^3) \mid x^i < 0\}$, $i = 1, 2, 3$. 定义坐标映射

$$\begin{aligned} \psi_{1+}: V_{1+} &\rightarrow \mathbb{R}^2, & (x^1, x^2, x^3) &\mapsto (x^2, x^3), \\ \psi_{1-}: V_{1-} &\rightarrow \mathbb{R}^2, & (x^1, x^2, x^3) &\mapsto (x^2, x^3), \end{aligned}$$

其余映射 $\psi_{2+}, \psi_{2-}, \psi_{3+}, \psi_{3-}$ 类似定义. 则 $\{(V_{i+}, \psi_{i+}), (V_{i-}, \psi_{i-}) \mid i = 1, 2, 3\}$ 是 S^2 的一个坐标图册.

由 $\phi_1 \circ f \circ \psi_{1+}^{-1}: (x, y) \mapsto \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}\right)$ 可知其 Jacobi 行列式

$$\det \begin{pmatrix} \frac{1-y^2}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{xy}{(1-x^2-y^2)^{\frac{3}{2}}} \\ \frac{xy}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{1-x^2}{(1-x^2-y^2)^{\frac{3}{2}}} \end{pmatrix} = \frac{1}{(1-x^2-y^2)^2} > 0$$

同理可得其他复合映射的 Jacobi 行列式不为 0, 因此 f 的秩为 2. \square

题 54 设 M 为一抽象光滑曲面, C_1, C_2 为 M 上两个不相交闭集. 证明: 存在光滑函数 $f: M \rightarrow \mathbb{R}$ 使

$$f(p) = \begin{cases} 1, & p \in C_1, \\ 0, & p \in C_2, \end{cases} \quad \forall p \in M.$$

证明 由 M 的底空间的正规性知存在 U_1, U_2 使得 $C_1 \subset U_1, C_2 \subset U_2$ 且 $U_1 \cap U_2 = \emptyset$. 下证存在函数 f 满足 $f|_{C_1} \equiv 1$ 且 $\overline{\text{supp } f} \subset U_1$. 设 $V_1 = U_1, V_2 = M \setminus C_1$, 并设 ψ_1, ψ_2 分别为从属于 V_1, V_2 的单位分解. 由 $\psi_1|_{C_1} \equiv 1$ 及 $\psi_1 + \psi_2 \equiv 1$ 可知 $\psi_2|_{C_1} \equiv 0, \psi_2|_{M \setminus U_1} \equiv 1$. 故 f 满足要求. \square

题 55 考虑映射

$$f: \mathbb{S}^2 \rightarrow \mathbb{R}^6, \quad (x, y, z) \mapsto (x^2, y^2, z^2, \sqrt{2}yz, \sqrt{2}zx, \sqrt{2}xy).$$

试判断 f 是否为浸入或嵌入.

解 f 的 Jacobi 矩阵为 $\begin{pmatrix} 2x & 0 & 0 & 0 & \sqrt{2}z & \sqrt{2}y \\ 0 & 2y & 0 & \sqrt{2}z & 0 & \sqrt{2}x \\ 0 & 0 & 2z & \sqrt{2}y & \sqrt{2}x & 0 \end{pmatrix}$. 由于 $(x, y, z) \in \mathbb{S}^2$, 不妨设 $x \neq 0$, 则由子式

$\begin{vmatrix} 2x & \sqrt{2}z & \sqrt{2}y \\ 0 & 0 & \sqrt{2}x \\ 0 & \sqrt{2}x & 0 \end{vmatrix} = -4x^3 \neq 0$ 可知该 Jacobi 矩阵行满秩, 同理可得其他情形下也是如此. 故 $\text{rank } f = 3, f$ 为浸入. 由于 \mathbb{S}^2 是紧集, f 为嵌入当且仅当 f 为单射, 但 $f(x, y, z) = f(-x, -y, -z)$, f 非单射, 故 f 非嵌入. \square

§6.2 切空间、流形上的微分形式、Riemann 度量

题 56 验证外微分 d 的如下性质: 对任意 $f, g \in \Omega^0(M) = C^\infty(M), \omega \in \Omega^1(M)$, 有

$$d(fg) = gdf + f dg, \quad d(f\omega) = df \wedge \omega + f d\omega.$$

证明 ① 由定义, 对任意 $p \in M$ 与 $v_p \in T_p M$, 有

$$d(fg)(v_p) = v_p(fg) = g(p)v_p(f) + f(p)v_p(g) = g(p)df(v_p) + f(p)dg(v_p),$$

即 $d(fg) = gdf + f dg$.

② 由于外微分 d 是线性的, 不妨设 $\omega = g dx^1$, 则

$$d(f\omega) = d(fg dx^1) = \partial_{x^2}(fg) dx^2 \wedge dx^1 = g \partial_{x^2} f dx^2 \wedge dx^1 + f \partial_{x^2} g dx^2 \wedge dx^1 = df \wedge \omega + f d\omega.$$

\square

题 57 设 M 为一抽象光滑曲面. 若存在坐标图册 $\{(U_\alpha, \phi_\alpha)\}$ 覆盖 M , 且对任一 $p \in U_\alpha \cap U_\beta$, 有

$$D_{\alpha\beta}(p) := \det(d(\phi_\alpha \circ \phi_\beta^{-1}(\phi_\beta(p)))) > 0,$$

证明: M 上存在处处非零连续 2-形式 ω .

证明 设 $\{f_\alpha\}$ 为从属于 $\{(U_\alpha, \phi_\alpha)\}$ 的单位分解, 构造如下 2-形式:

$$\omega := \sum_{\alpha} f_{\alpha} \omega^{\alpha}.$$

则对任一 $p \in M$, 上述求和为有限和, 且

$$\omega(p) := \sum_{\alpha} f_{\alpha}(p) \omega^{\alpha}(p) = \sum_{\alpha \in S} f_{\alpha}(p) \omega^{\alpha}(p),$$

其中 $S = \{\alpha \mid p \in \text{supp } f_{\alpha}\}$. 进一步有

$$\omega(p) := \left(\sum_{\alpha \in S} f_{\alpha}(p) D_{\alpha \alpha_0}(p) \right) \omega^{\alpha_0}(p),$$

其中 $\alpha_0 \in S$ 是一固定指标. 观察到 $f_{\alpha} \geq 0$, $\sum_{\alpha \in S} f_{\alpha}(p) = 1$, $D_{\alpha \alpha_0}(p) > 0$, 于是 $\omega(p) \neq 0$, 即 ω 是 M 上的处处非零的连续 2-形式. \square

题 58 设 M 为一抽象光滑曲面, g 为 M 的一个 Riemann 度量. 对任意 $p \in M$, $v_p, w_p \in T_p M$, 记其张成平行四边形的面积

$$\text{Area}(v_p, w_p) = \sqrt{g(v_p, v_p)g(w_p, w_p) - g(v_p, w_p)^2}.$$

M 上的光滑 2-形式 $\omega \in \Omega^2(M)$ 若满足

$$\omega(v_p, w_p) = \text{Area}(v_p, w_p) \text{ 或 } -\text{Area}(v_p, w_p), \quad \forall p \in M, \forall v_p, w_p \in T_p M,$$

则称其为 M 上一个面积形式. 证明: M 可定向当且仅当 M 上存在一个面积形式.

证明 \Leftarrow : 若 M 上存在一个面积形式 ω , 则 ω 是一个处处非零的光滑 2-形式, 从而 M 可定向.

\Rightarrow : 若 M 可定向, 设 ω 为 M 上的处处非零的光滑 2-形式. 对任意 $p \in M$, 取 $\{e_1, e_2\}$ 为 $T_p M$ 的一组标准正交基且 $\omega(e_1, e_2) > 0$. 令 $\omega_p = e_1^* \wedge e_2^*$, 则对任意 $v_p, w_p \in T_p M$,

$$\begin{aligned} \omega_p(v_p, w_p)^2 &= [e_1^*(v_p)e_2^*(w_p) - e_2^*(v_p)e_1^*(w_p)]^2 \\ &= e_1^*(v_p)^2 e_2^*(w_p)^2 + e_2^*(v_p)^2 e_1^*(w_p)^2 - 2e_1^*(v_p)e_2^*(v_p)e_1^*(w_p)e_2^*(w_p) \\ &= [e_1^*(v_p)^2 + e_2^*(v_p)^2] [e_1^*(w_p)^2 + e_2^*(w_p)^2] - [e_1^*(v_p)e_1^*(w_p) + e_2^*(v_p)e_2^*(w_p)]^2 \\ &= g(v_p, v_p)g(w_p, w_p) - g(v_p, w_p)^2 \\ &= \text{Area}(v_p, w_p)^2. \end{aligned}$$

只需再验证 p 处 ω_p 的上述定义不依赖于满足条件的 $\{e_1, e_2\}$ 的选取. 若 $\{\bar{e}_1, \bar{e}_2\}$ 是 $T_p M$ 的另一组标准正交基, 且 $\omega(\bar{e}_1, \bar{e}_2) > 0$, 则存在 $\theta \in [0, 2\pi)$ 与 $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, 使得 $(\bar{e}_1, \bar{e}_2) = (e_1, e_2)A$, 从而 $(\bar{e}_1^*, \bar{e}_2^*) = (e_1^*, e_2^*)A$. 于是 $\bar{\omega}_p := \bar{e}_1^* \wedge \bar{e}_2^* = \det(A)e_1^* \wedge e_2^* = e_1^* \wedge e_2^* = \omega_p$. 由 e_1, e_2 选取的光滑性可得 ω_p 的光滑性, 因此它是 M 上的面积形式. \square

第七章 整体微分几何

§7.1 度量完备曲面、Hilbert 定理

题 59 设 M 是 E^3 中的一个紧致曲面, φ, H, K 为 M 的支撑函数、平均曲率函数、Gauss 曲率函数, \mathbf{n} 为单位法向量场.

- (1) 证明: $\int_M dV = \int_M H\varphi dV$. (提示: 考察 $(\mathbf{r}, \mathbf{n}, d\mathbf{r})$)
- (2) 证明: $\int_M \mathbf{n} dV = \mathbf{0}$. (提示: 对任意常向量 \mathbf{a} , 考察 $(\mathbf{r}, d\mathbf{r}, \mathbf{a})$)
- (3) 证明: $\int_M H\mathbf{n} dV = \mathbf{0}$. (提示: 对任意常向量 \mathbf{a} , 考察 $(\mathbf{r}, \mathbf{n}, \mathbf{a})$)
- (4) 证明: $\int_M K\mathbf{n} dV = \mathbf{0}$. (提示: 对任意常向量 \mathbf{a} , 考察 $(\mathbf{n}, d\mathbf{n}, \mathbf{a})$)

A 附录

§1.1 曲面的联络形式

方程

$$\begin{cases} d\omega^1 = \omega^2 \wedge \omega_2^1, \\ d\omega^2 = \omega^1 \wedge \omega_1^2 \end{cases}$$

唯一确定了曲面的联络形式

$$\omega_1^2 = \frac{d\omega^1}{\omega^1 \wedge \omega^2} \omega^1 + \frac{d\omega^2}{\omega^1 \wedge \omega^2} \omega^2.$$

由此可算 Gauss 曲率

$$K = -\frac{d\omega_1^2}{\omega^1 \wedge \omega^2}.$$

例 A.1.1 用活动么正标架法计算下列第一基本形式的 Gauss 曲率:

$$(1) I = \frac{4}{(1-u^2-v^2)^2} (du \otimes du + dv \otimes dv). \quad \mathbf{【-1】}$$

$$(2) I = \frac{1}{v^2} (du \otimes du + dv \otimes dv). \quad \mathbf{【-1】}$$

$$(3) I = \frac{1}{4(u-v^2)} (du \otimes du - 4v du \otimes dv + 4u dv \otimes dv). \quad \mathbf{【0】}$$

§1.2 两个基本形式

$$I = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2,$$

$$II = \omega^1 \otimes \omega_1^3 + \omega^2 \otimes \omega_2^3.$$