# Costa's Minimal Surface

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# Outline

# The Weierstrass Representation

## Theorem (The Weierstrass Representation Theorem)

*Let $f$ and $g$ be functions on a simply connected domain $U \subset \mathbb{C}$, where $g$ is meromorphic and $f$ is holomorphic, such that wherever $g$ has a pole of order $m$, $f$ has a zero of order at least $2m$ (or equivalently, such that the product $fg^2$ is holomorphic). Fix $z_0 \in U$, and let $c_1, c_2, c_3$ be constants. Then the surface with coordinates $(x_1, x_2, x_3)$ is* ***minimal****, where the $x_k$ are defined as follows:*

$$x_k(z) = \mathfrak{Re}\left\{\int_{z_0}^{z} \varphi_k(w)\,\mathrm{d}w\right\} + c_k, \quad k = 1, 2, 3.$$

$$\varphi_1 = \frac{f\left(1 - g^2\right)}{2}, \quad \varphi_2 = \frac{\mathrm{i}f\left(1 + g^2\right)}{2}, \quad \varphi_3 = fg.$$

# A Basic Example

From the functions

$$f(z) = -e^{-z} \text{ and } g(z) = -e^{z}$$

we obtain (up to constants)

$$\begin{cases} x_1(u, v) = \cosh u \cos v, \\ x_2(u, v) = \cosh u \sin v, \\ x_3(u, v) = u, \end{cases}$$

which describes the catenoid.

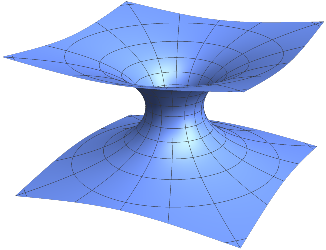


Figure: A catenoid

# The Weierstrass $\wp$ and $\zeta$

We choose the lattice $\mathbb{Z}[\mathrm{i}] = \{m + \mathrm{i}n : m, n \in \mathbb{Z}\}$ so that the Weierstrass $\wp$ and $\zeta$ functions are defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \left( \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Clearly, $\zeta'(z) = -\wp(z)$. Let us denote

$$e_1 = \wp(\tfrac{1}{2}), \quad e_2 = \wp\left(\tfrac{\mathrm{i}}{2}\right), \quad e_3 = \wp\left(\tfrac{1+\mathrm{i}}{2}\right).$$

**Please don't confuse $\zeta$ with the Riemann zeta function.**

# Identities of $\wp$ and $\zeta$

1. $\wp(z+m+\mathrm{i}n)=\wp(z)$ for all $m,n\in\mathbb{Z}$.
2. $\wp(z_1+z_2)=\frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2-\wp(z_1)-\wp(z_2)$.
3. $\wp'(z)^2=\left[4\wp(z)^2-g_2\right]\wp(z)$.
4. $\wp'(z)^2=4\wp(z)\left[\wp(z)^2-e_1^2\right]$.
5. $\wp\left(z+\frac{1}{2}\right)=e_1+\frac{2e_1^2}{\wp(z)-e_1}$.
6. $\wp\left(z+\frac{\mathrm{i}}{2}\right)=e_2+\frac{2e_2^2}{\wp(z)-e_2}=-e_1+\frac{2e_1^2}{\wp(z)+e_1}$.
7. $\wp\left(z-\frac{1}{2}\right)-\wp\left(z-\frac{\mathrm{i}}{2}\right)-2e_1=\frac{16e_1^3\wp(z)}{\wp'(z)^2}$.

# Identities of $\wp$ and $\zeta$

8. $\zeta(z+m+\mathrm{i}n)=\zeta(z)+2m\zeta\left(\frac{1}{2}\right)+2n\zeta\left(\frac{\mathrm{i}}{2}\right)$ for all $m,n\in\mathbb{Z}$.
9. $\mathrm{i}\zeta(\mathrm{i}z)=\zeta(z)$.
10. $\zeta\left(\frac{1}{2}\right)=\mathrm{i}\zeta\left(\frac{\mathrm{i}}{2}\right)=\frac{\pi}{2}$.
11. $\zeta\left(\frac{1+\mathrm{i}}{2}\right)=\frac{(1-\mathrm{i})\pi}{2}$.
12. $\zeta(z+u)-\zeta(z)-\zeta(u)=\frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}$.

1. Clear from the definition of $\wp$.
2. A well-known addition formula that can be found in most textbooks on elliptic functions. So is 12.
3. Corollary 2.3 in Chapter 9 of *Complex Analysis* by Stein and Shakarchi gives the identity

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

where

$$g_2 = 60 \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \frac{1}{\omega^4} \quad \text{and} \quad g_3 = 140 \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \frac{1}{\omega^6}.$$

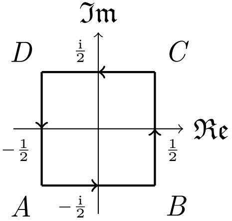
In our case, $g_3 = 0$ since

$$(m - \mathrm{i}n)^6 + (m + \mathrm{i}n)^6 + (n - \mathrm{i}m)^6 + (n + \mathrm{i}m)^6 = 0.$$

4. It is known that $1/2$, $\mathrm{i}/2$ and $(1+\mathrm{i})/2$ are the roots of the cubic polynomial $\left[4\wp(z)^2 - g_2\right]\wp(z)$, and $\wp\left(\frac{1+\mathrm{i}}{2}\right) = e_3 = 0$. Hence $4e_1^2 = g_2$, and the identity follows from 3.
5. Apply 2 and then 4.
6. Apply 2 and then 4. Note that $e_2 = -e_1$.
7. By 1 $\wp\left(z - \frac{1}{2}\right) - \wp\left(z - \frac{\mathrm{i}}{2}\right) = \wp\left(z + \frac{1}{2}\right) - \wp\left(z + \frac{\mathrm{i}}{2}\right)$. Then combine 5 and 6 to get the identity.
8. Since $\zeta'(z) = -\wp(z)$ and $\wp(z+1) = \wp(z)$, the two functions $\zeta(z+1)$ and $\zeta(z)$ differ by a constant, say $\zeta(z+1) = \zeta(z) + c$. Take $z = -\frac{1}{2}$ and use the fact that $\zeta$ is odd to get $c = 2\zeta\left(\frac{1}{2}\right)$. The same argument gives $\zeta(z+\mathrm{i}) = \zeta(z) + 2\zeta\left(\frac{\mathrm{i}}{2}\right)$.
9. Clear from the definition of $\zeta$ and the fact that $\mathrm{i}\mathbb{Z}[\mathrm{i}] = \mathbb{Z}[\mathrm{i}]$.

10. The residue theorem gives

$$\int_{ABCDA} \zeta(z)\,\mathrm{d}z = 2\pi\mathrm{i}.$$




On the other hand, by 8 we have

$$\int_{CD} \zeta(z)\,\mathrm{d}z = \int_{BA} \zeta(z)\,\mathrm{d}z - 2\zeta\left(\tfrac{\mathrm{i}}{2}\right), \quad \int_{BC} \zeta(z)\,\mathrm{d}z = \int_{AD} \zeta(z)\,\mathrm{d}z + 2\mathrm{i}\zeta\left(\tfrac{1}{2}\right).$$

Combining these equations gives $\zeta\left(\frac{1}{2}\right) + \mathrm{i}\zeta\left(\frac{\mathrm{i}}{2}\right) = \pi$. Then use 9.

11. Take $z = -\frac{1+\mathrm{i}}{2}$ and $m = n = 1$ in 8 and use the fact that $\zeta$ is odd to get $\zeta\left(\frac{1+\mathrm{i}}{2}\right) = \zeta\left(\frac{1}{2}\right) + \zeta\left(\frac{\mathrm{i}}{2}\right)$. Then 10 applies.

# The Weierstrass Data

Costa's minimal surface is defined as a Weierstrass patch using the functions

$$f(z) = \wp(z) \quad \text{and} \quad g(z) = \frac{A}{\wp'(z)}.$$

In order that Costa's minimal surface has no self-intersections, we need to take[a]

$$A = 2\sqrt{2\pi}\, e_1 \approx 34.46707.$$

[a] D. Hoffman and W. Meeks, A complete minimal surface in $\mathbb{R}^3$ with genus one and three ends, *J. Differential Geometry 21* (1985), 109–127.

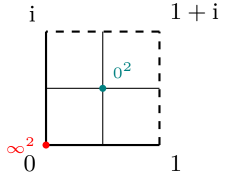



Figure: Zeros and poles of $f$

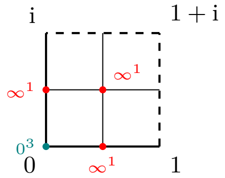



Figure: Zeros and poles of $g$

**We shall use $\zeta$ to express the coordinates without integrals.**

Using 7 we obtain

$$
\begin{aligned}
f(w)\left[1-g(w)^2\right] &= \wp(w) - \frac{A^2\wp(w)}{\wp'(w)^2} \\
&= \wp(w) - \frac{A^2}{16e_1^3}\left[\wp\left(w-\tfrac{1}{2}\right) - \wp\left(w-\tfrac{\mathrm{i}}{2}\right) - 2e_1\right] \\
&= \wp(w) - \frac{\pi}{2e_1}\left[\wp\left(w-\tfrac{1}{2}\right) - \wp\left(w-\tfrac{\mathrm{i}}{2}\right) - 2e_1\right] \\
&= \wp(w) + \pi - \frac{\pi}{2e_1}\left[\wp\left(w-\tfrac{1}{2}\right) - \wp\left(w-\tfrac{\mathrm{i}}{2}\right)\right].
\end{aligned}
$$

Take $z_0 = \frac{1+\mathrm{i}}{2}$. Integrating both sides and using 10 and 11, we get

$$
\begin{aligned}
&\int_{z_0}^{z} f(w)\left[1 - g(w)^2\right] \mathrm{d}w \\
&= \left\{ -\zeta(w) + \pi w + \frac{\pi}{2e_1}\left[\zeta\left(w - \tfrac{1}{2}\right) - \zeta\left(w - \tfrac{\mathrm{i}}{2}\right)\right] \right\} \Bigg|_{z_0}^{z} \\
&= -\zeta(z) + \pi z + \frac{\pi}{2e_1}\left[\zeta\left(z - \tfrac{1}{2}\right) - \zeta\left(z - \tfrac{\mathrm{i}}{2}\right)\right] \\
&\quad + \zeta\left(\tfrac{1+\mathrm{i}}{2}\right) - \frac{\pi(1+\mathrm{i})}{2} - \frac{\pi}{2e_1}\left[\zeta\left(\tfrac{\mathrm{i}}{2}\right) - \zeta\left(\tfrac{1}{2}\right)\right] \\
&= -\zeta(z) + \pi z + \frac{\pi}{2e_1}\left[\zeta\left(z - \tfrac{1}{2}\right) - \zeta\left(z - \tfrac{\mathrm{i}}{2}\right)\right] - \mathrm{i}\pi + \frac{\pi^2(1+\mathrm{i})}{4e_1}.
\end{aligned}
$$

Dividing by 2 and taking the real part, we get $x_1$.

Similarly,

$$f(w)\left[1+g(w)^2\right]=\wp(w)-\pi+\frac{\pi}{2e_1}\left[\wp\left(w-\tfrac{1}{2}\right)-\wp\left(w-\tfrac{\mathrm{i}}{2}\right)\right]$$

and then

$$\begin{aligned}&\int_{z_0}^{z} f(w)\left[1+g(w)^2\right]\mathrm{d}w\\&=\left\{-\zeta(w)-\pi w-\frac{\pi}{2e_1}\left[\zeta\left(w-\tfrac{1}{2}\right)-\zeta\left(w-\tfrac{\mathrm{i}}{2}\right)\right]\right\}\Bigg|_{z_0}^{z}\\&=-\zeta(z)-\pi z-\frac{\pi}{2e_1}\left[\zeta\left(z-\tfrac{1}{2}\right)-\zeta\left(z-\tfrac{\mathrm{i}}{2}\right)\right]+\pi-\frac{\pi^2(1+\mathrm{i})}{4e_1}.\end{aligned}$$

From this we can find $x_2$.

Using 4 we obtain

$$
\begin{aligned}
\int_{z_0}^{z} f(w)g(w)\,\mathrm{d}w &= A\int_{z_0}^{z} \frac{\wp(w)}{\wp'(w)}\,\mathrm{d}w = \frac{A}{4}\int_{z_0}^{z} \frac{\wp'(w)\,\mathrm{d}w}{\wp(w)^2 - e_1^2} \\
&= \frac{A}{8e_1}\int_{z_0}^{z} \left( \frac{\wp'(w)}{\wp(w) - e_1} - \frac{\wp'(w)}{\wp(w) + e_1} \right) \mathrm{d}w \\
&= \frac{\sqrt{2\pi}}{4} \log\left( \frac{\wp(w) - e_1}{\wp(w) + e_1} \right) \Bigg|_{z_0}^{z} \\
&= \frac{\sqrt{2\pi}}{4} \left\{ \log\left( \frac{\wp(z) - e_1}{\wp(z) + e_1} \right) - \log\left( \frac{e_3 - e_1}{e_3 + e_1} \right) \right\} \\
&= \frac{\sqrt{2\pi}}{4} \left\{ \log\left( \frac{\wp(z) - e_1}{\wp(z) + e_1} \right) - \pi \mathrm{i} \right\}.
\end{aligned}
$$

Taking the real part gives $x_3$.

# Parametric Equations

Costa's minimal surface is given by $(x_1, x_2, x_3)$ where

$$\begin{cases} x_1(u,v) = & \frac{1}{2}\mathfrak{Re}\Big\{-\zeta(u+\mathrm{i}v) + \pi u + \frac{\pi^2}{4e_1} \\ & \qquad + \frac{\pi}{2e_1}\left[\zeta\left(u+\mathrm{i}v-\frac{1}{2}\right) - \zeta\left(u+\mathrm{i}v-\frac{\mathrm{i}}{2}\right)\right]\Big\}, \\ x_2(u,v) = & \frac{1}{2}\mathfrak{Re}\Big\{-\mathrm{i}\zeta(u+\mathrm{i}v) + \pi v + \frac{\pi^2}{4e_1} \\ & \qquad - \frac{\pi \mathrm{i}}{2e_1}\left[\zeta\left(u+\mathrm{i}v-\frac{1}{2}\right) - \zeta\left(u+\mathrm{i}v-\frac{\mathrm{i}}{2}\right)\right]\Big\}, \\ x_3(u,v) = & \frac{\sqrt{2\pi}}{4}\log\left|\frac{\wp(u+\mathrm{i}v)-e_1}{\wp(u+\mathrm{i}v)+e_1}\right|. \end{cases}$$

# What the Surface Looks Like

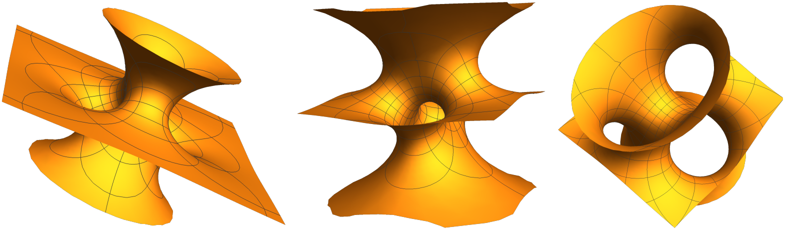


Figure: Zoom in on the Costa's surface (left to right)

# Parameter Domain

By 8 and 10 we have $\zeta(z+1) = \zeta(z) + \pi$, hence

$$\begin{aligned} x_1(u+1, v) =& \frac{1}{2}\mathfrak{Re}\left\{ -\zeta(u+\mathrm{i}v) - \pi + \pi u + \pi + \frac{\pi^2}{4e_1} \right. \\ & \left. + \frac{\pi}{2e_1}\left[\zeta\left(u+\mathrm{i}v-\tfrac{1}{2}\right) + \pi - \zeta\left(u+\mathrm{i}v-\tfrac{\mathrm{i}}{2}\right) - \pi\right] \right\} \\ =& x_1(u, v). \end{aligned}$$

Similarly, one can show that $x_1(u, v+1) = x_1(u, v)$ and

$$\begin{aligned} x_2(u+1, v) &= x_2(u, v), & x_2(u, v+1) &= x_2(u, v), \\ x_3(u+1, v) &= x_3(u, v), & x_3(u, v+1) &= x_3(u, v). \end{aligned}$$

Therefore, we may restrict $u$ and $v$ to the unit square $[0, 1) \times [0, 1)$.

# Two Catenoidal Ends and One Planar End

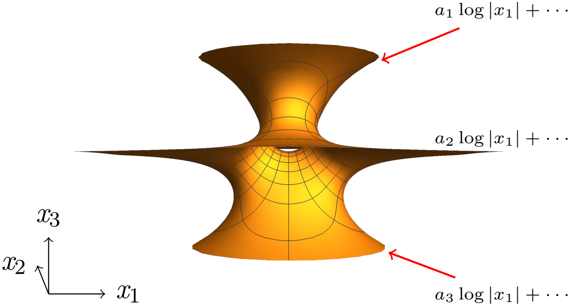



Figure: Front view of the Costa's surface

# Key Observations (i)

- $\wp(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$.

$$\overline{\wp(x)} = \frac{1}{x^2} + \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \left( \frac{1}{(x - \overline{\omega})^2} - \frac{1}{\overline{\omega}^2} \right) = \wp(x).$$

- $\zeta(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$.

$$\overline{\zeta(x)} = \frac{1}{x} + \sum_{\omega \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \left( \frac{1}{x - \overline{\omega}} + \frac{1}{\overline{\omega}} + \frac{x}{\overline{\omega}^2} \right) = \zeta(x).$$

# Key Observations (ii)

$$x_2(u,0) = \frac{1}{2}\mathfrak{Re}\left\{-\mathrm{i}\zeta(u) + \frac{\pi^2}{4e_1} - \frac{\pi\mathrm{i}}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\}$$
$$= \frac{1}{2}\left\{\frac{\pi^2}{4e_1} - \frac{\pi}{2e_1}\mathfrak{Im}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right\}.$$

By 12 we have

$$\zeta\left(u-\tfrac{\mathrm{i}}{2}\right) = \zeta(u) + \zeta\left(-\tfrac{\mathrm{i}}{2}\right) + \frac{1}{2}\frac{\wp'(u) - \wp'\left(\frac{\mathrm{i}}{2}\right)}{\wp(u) - \wp\left(\frac{\mathrm{i}}{2}\right)}.$$

Since $\wp\left(\frac{\mathrm{i}}{2}\right) = e_2 = -e_1 \in \mathbb{R}$ and $\wp'\left(\frac{\mathrm{i}}{2}\right) = 0$, we obtain

$$\mathfrak{Im}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\} = \mathfrak{Im}\left\{\zeta\left(-\tfrac{\mathrm{i}}{2}\right)\right\} = \mathfrak{Im}\left\{\tfrac{\pi\mathrm{i}}{2}\right\} = \tfrac{\pi}{2}.$$

Hence $x_2(u,0) = 0$.

# Key Observations (iii)

$$
\begin{aligned}
& x_2\left(u, \tfrac{1}{2}\right) \\
=& \frac{1}{2}\mathfrak{Re}\left\{-\mathrm{i}\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)+\frac{\pi}{2}+\frac{\pi^2}{4e_1}-\frac{\pi\mathrm{i}}{2e_1}\left[\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)-\zeta(u)\right]\right\} \\
=& \frac{1}{2}\left\{\mathfrak{Im}\left\{\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)\right\}+\frac{\pi}{2}+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\mathfrak{Im}\left\{\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)\right\}\right\}.
\end{aligned}
$$

As in the case with (ii), we have

$$
\begin{gathered}
\mathfrak{Im}\left\{\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)\right\}=\mathfrak{Im}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)-\pi\mathrm{i}\right\}=\frac{\pi}{2}-\pi=-\frac{\pi}{2}, \\
\mathfrak{Im}\left\{\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)\right\}=\mathfrak{Im}\left\{\zeta\left(\left(u-\tfrac{1}{2}\right)+\tfrac{\mathrm{i}}{2}\right)\right\}=-\frac{\pi}{2}.
\end{gathered}
$$

Therefore, $x_2\left(u, \tfrac{1}{2}\right)=0$.

# Key Observations (iv)

- $x_1(u,0) \to -\infty$ as $u \searrow 0$.
- $x_1(u,0) \to -\infty$ as $u \nearrow \frac{1}{2}$.
- $x_1(u,0) \to +\infty$ as $u \searrow \frac{1}{2}$.
- $x_1(u,0) \to +\infty$ as $u \nearrow 1$.
- $x_3(u,0) \to 0$ as $u \to 0$.
- $x_3(u,0) \to -\infty$ as $u \to \frac{1}{2}$.

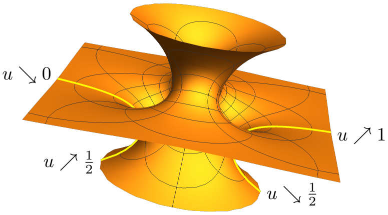



Figure: The curve $v = 0$

# Proof of (iv)

When $u \searrow 0$,

$$
\begin{aligned}
& x_1(u,0) \\
=& \frac{1}{2}\mathfrak{Re}\left\{-\zeta(u) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\
=& \frac{1}{2}\left\{-\zeta(u) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \mathfrak{Re}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
\sim& \frac{1}{2}\left\{-\frac{1}{u} + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(-\tfrac{1}{2}\right) - \mathfrak{Re}\left\{\zeta\left(-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
\sim& -\frac{1}{2u} \to -\infty.
\end{aligned}
$$

# Proof of (iv)

When $u \nearrow \frac{1}{2}$,

$$
\begin{aligned}
& x_1(u,0) \\
=& \frac{1}{2}\mathfrak{Re}\left\{-\zeta(u) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\
=& \frac{1}{2}\left\{-\zeta(u) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \mathfrak{Re}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
\sim& \frac{1}{2}\left\{-\zeta\left(\tfrac{1}{2}\right) + \frac{\pi}{2} + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right) - \mathfrak{Re}\left\{\zeta\left(\tfrac{1-\mathrm{i}}{2}\right)\right\}\right]\right\} \\
\sim& \frac{\pi}{4e_1}\zeta\left(u-\tfrac{1}{2}\right) \\
\sim& \frac{\pi}{4e_1}\frac{1}{u-\frac{1}{2}} \to -\infty.
\end{aligned}
$$

# Proof of (iv)

When $u \searrow \frac{1}{2}$,

$$
\begin{aligned}
& x_1(u,0) \\
=& \frac{1}{2}\mathfrak{Re}\left\{-\zeta(u)+\pi u+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right)-\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\
=& \frac{1}{2}\left\{-\zeta(u)+\pi u+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right)-\mathfrak{Re}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
\sim& \frac{1}{2}\left\{-\zeta\left(\tfrac{1}{2}\right)+\frac{\pi}{2}+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right)-\mathfrak{Re}\left\{\zeta\left(\tfrac{1-\mathrm{i}}{2}\right)\right\}\right]\right\} \\
\sim& \frac{\pi}{4e_1}\zeta\left(u-\tfrac{1}{2}\right) \\
\sim& \frac{\pi}{4e_1}\frac{1}{u-\frac{1}{2}} \to +\infty.
\end{aligned}
$$

# Proof of (iv)

When $u \nearrow 1$,

$$
\begin{aligned}
& x_1(u,0) \\
=& \frac{1}{2}\mathfrak{Re}\left\{-\zeta(u)+\pi u+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right)-\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\
=& \frac{1}{2}\left\{-\zeta(u)+\pi u+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(u-\tfrac{1}{2}\right)-\mathfrak{Re}\left\{\zeta\left(u-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
\sim& \frac{1}{2}\left\{-\frac{1}{u-1}-\pi+\pi+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(\tfrac{1}{2}\right)-\mathfrak{Re}\left\{\zeta\left(1-\tfrac{\mathrm{i}}{2}\right)\right\}\right]\right\} \\
\sim& -\frac{1}{2(u-1)} \to +\infty.
\end{aligned}
$$

# Proof of (iv)

$$x_3(u,0) = \frac{\sqrt{2\pi}}{4} \log \left| \frac{\wp(u) - e_1}{\wp(u) + e_1} \right| .$$

- Since $\wp(u) \sim \frac{1}{u^2}$ as $u \to 0$, we have $x_3(u,0) \to 0$ as $u \to 0$.
- Since $\wp\left(\frac{1}{2}\right) = e_1$, we have $x_3(u,0) \to -\infty$ as $u \to \frac{1}{2}$.

# The Coefficients $a_2$ and $a_3$

It is obvious from observation (iv) that $a_2 = 0$.
To find $a_3$, first note that

$$a_3 = \lim_{u \searrow \frac{1}{2}} \frac{x_3(u,0)}{\log x_1(u,0)} = \frac{\sqrt{2\pi}}{4} \lim_{u \searrow 0} \frac{\log \left| \frac{\wp(u+1/2)-e_1}{\wp(u+1/2)+e_1} \right|}{\log \frac{\pi}{4 e_1 u}}.$$

Using 5 we have

$$\frac{\wp\left(u+\frac{1}{2}\right)-e_1}{\wp\left(u+\frac{1}{2}\right)+e_1} = 1 - \frac{2e_1}{\wp\left(u+\frac{1}{2}\right)+e_1} = 1 - \frac{2e_1}{2e_1 + \frac{2e_1^2}{\wp(u)-e_1}}$$

$$= \frac{e_1}{\wp(u)} \sim e_1 u^2 \quad \text{as } u \to 0.$$

# The Coefficients $a_2$ and $a_3$

Now

$$
\begin{aligned}
a_3 &= \frac{\sqrt{2\pi}}{4} \lim_{u \searrow 0} \frac{\log\left(e_1 u^2\right)}{\log \frac{\pi}{4 e_1 u}} \\
&= \frac{\sqrt{2\pi}}{4} \lim_{u \searrow 0} \frac{2 \log u + \log e_1}{-\log u + \log \frac{\pi}{4 e_1}} \\
&= -\sqrt{\frac{\pi}{2}} \approx -1.25331.
\end{aligned}
$$

# Key Observations (v)

- $x_1\left(u, \frac{1}{2}\right) \to -\infty$ as $u \searrow 0$.
- $x_1\left(u, \frac{1}{2}\right) \to +\infty$ as $u \nearrow 1$.

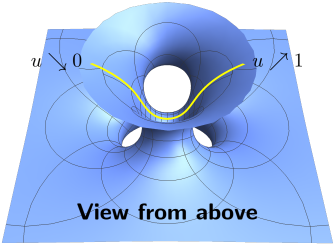



Figure: The curve $v = \frac{1}{2}$

# Proof of (v)

$$
\begin{aligned}
& x_1\left(u, \tfrac{1}{2}\right) \\
=&\frac{1}{2}\mathfrak{Re}\left\{-\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)+\pi u+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)-\zeta(u)\right]\right\}.
\end{aligned}
$$

When $u \searrow 0$,

$$
\begin{aligned}
x_1\left(u, \tfrac{1}{2}\right) &\sim \frac{1}{2}\mathfrak{Re}\left\{-\zeta\left(\tfrac{\mathrm{i}}{2}\right)+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)-\zeta(u)\right]\right\} \\
&\sim \frac{1}{2}\left\{\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left(-\frac{\pi}{2}-\frac{1}{u}\right)\right\} \\
&\sim -\frac{\pi}{4e_1 u} \to -\infty.
\end{aligned}
$$

# Proof of (v)

$$
\begin{aligned}
& x_1\left(u, \tfrac{1}{2}\right) \\
=& \frac{1}{2}\mathfrak{Re}\left\{-\zeta\left(u+\tfrac{\mathrm{i}}{2}\right)+\pi u+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(u+\tfrac{\mathrm{i}}{2}-\tfrac{1}{2}\right)-\zeta(u)\right]\right\}.
\end{aligned}
$$

When $u \nearrow 1$,

$$
\begin{aligned}
x_1\left(u, \tfrac{1}{2}\right) &\sim \frac{1}{2}\mathfrak{Re}\left\{-\zeta\left(1+\tfrac{\mathrm{i}}{2}\right)+\pi+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\zeta\left(\tfrac{1+\mathrm{i}}{2}\right)-\zeta(u)\right]\right\} \\
&\sim \frac{1}{2}\left\{-\pi+\pi+\frac{\pi^2}{4e_1}+\frac{\pi}{2e_1}\left[\frac{\pi}{2}-\zeta(u-1)-\pi\right]\right\} \\
&\sim \frac{\pi}{4e_1(1-u)} \to +\infty.
\end{aligned}
$$

# The Coefficient $a_1$

As before, we write $a_1$ as

$$a_1 = \lim_{u \nearrow 1} \frac{x_3\left(u, \frac{1}{2}\right)}{\log x_1\left(u, \frac{1}{2}\right)} = \frac{\sqrt{2\pi}}{4} \lim_{u \nearrow 1} \frac{\log \left| \frac{\wp(u+\mathrm{i}/2)-e_1}{\wp(u+\mathrm{i}/2)+e_1} \right|}{\log \frac{\pi}{4e_1(1-u)}}.$$

Using 6 we have

$$\begin{aligned} \frac{\wp\left(u+\frac{\mathrm{i}}{2}\right)-e_1}{\wp\left(u+\frac{\mathrm{i}}{2}\right)+e_1} &= 1 - \frac{2e_1}{\wp\left(u+\frac{\mathrm{i}}{2}\right)+e_1} = 1 - \frac{2e_1}{\frac{2e_1^2}{\wp(u)+e_1}} \\ &= -\frac{\wp(u)}{e_1} \sim -\frac{1}{e_1(1-u)^2} \quad \text{as } u \to 1. \end{aligned}$$

# The Coefficient $a_1$

Now

$$\begin{aligned} a_1 &= \frac{\sqrt{2\pi}}{4} \lim_{u \nearrow 1} \frac{\log \frac{1}{e_1(1-u)^2}}{\log \frac{\pi}{4e_1(1-u)}} \\ &= \frac{\sqrt{2\pi}}{4} \lim_{u \nearrow 1} \frac{-2\log(1-u) - \log e_1}{-\log(1-u) + \log \frac{\pi}{4e_1}} \\ &= \sqrt{\frac{\pi}{2}} \approx 1.25331. \end{aligned}$$

# Straight Lines on the Surface

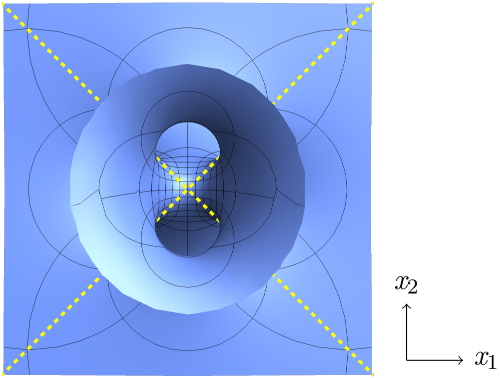



Figure: Vertical view of the Costa's surface

# The Weierstrass $\wp$ on the Unit Square

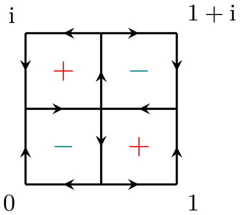



Figure: Sign of $\mathfrak{Im}(\wp)$. Arrows in direction of increasing $\mathfrak{Re}(\wp)$.

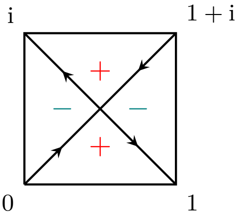



Figure: Sign of $\mathfrak{Re}(\wp)$. Arrows in direction of increasing $\mathfrak{Im}(\wp)$.

Since $\wp$ is imaginary on the diagonals of the unit square,

$$x_3(u,u) = \frac{\sqrt{2\pi}}{4} \log \frac{|\wp(u+\mathrm{i}u) - e_1|}{|\wp(u+\mathrm{i}u) + e_1|} = 0.$$

Moreover, with 9 and $\zeta(\bar{z}) = \overline{\zeta(z)}$ we see that

$$\begin{aligned} x_2(u,u) =& \tfrac{1}{2}\mathfrak{Re}\Big\{ -\mathrm{i}\zeta(u+\mathrm{i}u) + \pi u + \tfrac{\pi^2}{4e_1} \\ & - \tfrac{\pi \mathrm{i}}{2e_1}\left[\zeta\left(u+\mathrm{i}u-\tfrac{1}{2}\right) - \zeta\left(u+\mathrm{i}u-\tfrac{\mathrm{i}}{2}\right)\right]\Big\} \\ =& \tfrac{1}{2}\mathfrak{Re}\Big\{ -\zeta(u-\mathrm{i}u) + \pi u + \tfrac{\pi^2}{4e_1} \\ & - \tfrac{\pi}{2e_1}\left[\zeta\left(u-\mathrm{i}u+\tfrac{\mathrm{i}}{2}\right) - \zeta\left(u-\mathrm{i}u-\tfrac{1}{2}\right)\right]\Big\} \\ =& x_1(u,u). \end{aligned}$$

As before, we can show that as $u \searrow 0$

$$\begin{aligned} x_1(u,u) &= x_2(u,u) \\ &\sim \tfrac{1}{2}\mathfrak{Re}\left\{-\tfrac{1}{u+\mathrm{i}u} + \tfrac{\pi^2}{4e_1} + \tfrac{\pi}{2e_1}\left[\zeta\left(-\tfrac{1}{2}\right) - \zeta\left(-\tfrac{\mathrm{i}}{2}\right)\right]\right\} \\ &\sim -\tfrac{1}{4u} \to -\infty, \end{aligned}$$

and as $u \nearrow 1$

$$x_1(u,u) = x_2(u,u) \sim \tfrac{1}{4(1-u)} \to +\infty.$$

Therefore the straight line $(x, x, 0)$ with $x \in \mathbb{R}$ lies on the surface. By reflection in the $x_2$-$x_3$ plane, we find the other straight line on the surface.

# Straight Lines Imply Symmetry

### Theorem (Schwarz Reflection Principle for Minimal Surfaces)

*A minimal surface which contains a straight line on its boundary can be analytically extended by reflection across the line.*

### Corollary

*If a minimal surface contains a straight line, then it is invariant under rotation by $\pi$ about that line.*

The symmetry group of Costa's surface is the dihedral group generated by

- Reflection in the $x_1$-$x_3$ plane; and
- Rotation about the $x_3$-axis by $\frac{\pi}{2}$ followed by reflection in the $x_1$-$x_2$ plane.

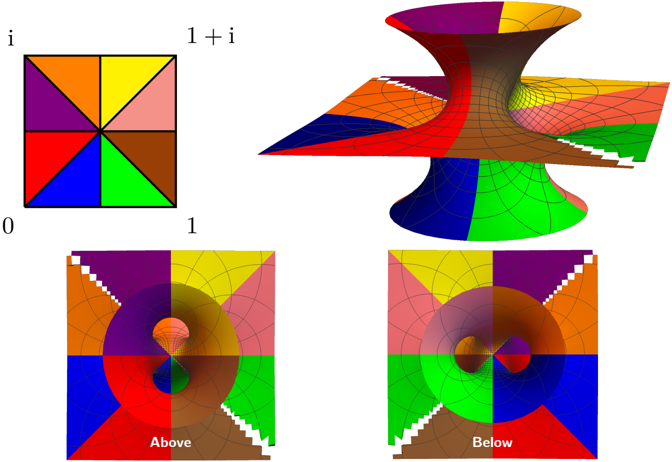



Figure: Eight fundamental triangles corresponding to congruent pieces of the surface.

# The End