复分析 (H) 作业

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习题 1.1.6 设 $|a|<1, |z|<1$. 证明:

(3) $\dfrac{||z|-|a||}{1-|a||z|} \leqslant \left|\dfrac{z-a}{1-\bar{a}z}\right| \leqslant \dfrac{|z|+|a|}{1+|a||z|}$.

证明 $\left|\dfrac{z-a}{1-\bar{a}z}\right|^2 = \dfrac{(z-a)(\bar{z}-\bar{a})}{(1-\bar{a}z)(1-a\bar{z})} = \dfrac{|z|^2+|a|^2-\bar{a}z-a\bar{z}}{1+|a|^2|z|^2-\bar{a}z-a\bar{z}}$. 令 $f(t) = \dfrac{\alpha+t}{\beta+t} = 1+\dfrac{\alpha-\beta}{\beta+t}$, 其中 $\alpha<\beta$, 则当 $t>-\beta$ 时, $f(t)$ 单调递增. 由于 $|a|<1, |z|<1$, 我们有 $\big(|z|^2-1\big)\big(|a|^2-1\big)>0$, 即 $|z|^2+|a|^2<1+|a|^2|z|^2$, 因此取 $\alpha=|z|^2+a^2, \beta=1+|a|^2|z|^2$, 则有

$$|a|^2|z|^2+1-2\operatorname{Re}(a\bar{z}) = [\operatorname{Re}(a\bar{z})-1]^2+[\operatorname{Im}(a\bar{z})]^2>0 \implies t=-\bar{z}-a\bar{z}>-\beta,$$

从而

$$\left|\frac{z-a}{1-\bar{a}z}\right|^2 = f(-\bar{a}z-a\bar{z}) = f(2\operatorname{Re}(-\bar{a}z)) < f(2|a||z|) = \frac{|z|^2+|a|^2+2|a||z|}{1+|a|^2|z|^2+2|a||z|} = \left(\frac{|z|+|a|}{1+|a||z|}\right)^2.$$

又

$$(|a||z|-1)^2>0 \implies 2|a||z|<1+|a|^2|z|^2 \implies t=-2|a||z|>-\beta,$$

因此

$$\left(\frac{|z|-|a|}{1-|a||z|}\right)^2 = \frac{|z|^2+|a|^2-2|a||z|}{1+|a|^2|z|^2-2|a||z|} = f(-2|a||z|) \leqslant f(-2\operatorname{Re}(\bar{a}z)) = \left|\frac{z-a}{1-\bar{a}z}\right|^2.$$

故

$$\frac{||z|-|a||}{1-|a||z|} \leqslant \left|\frac{z-a}{1-\bar{a}z}\right| \leqslant \frac{|z|+|a|}{1+|a||z|}.$$

□

习题 1.2.6 证明: 三点 z_1, z_2, z_3 共线的充要条件为

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0.$$

证明 记 $z_j = x_j + \mathrm{i}y_j$, 则 z_1, z_2, z_3 共线 $\iff \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} x_1 & x_1-\mathrm{i}y_1 & 1 \\ x_2 & x_2-\mathrm{i}y_2 & 1 \\ x_3 & x_3-\mathrm{i}y_3 & 1 \end{vmatrix} = 0 \iff$ $\begin{vmatrix} 2x_1 & x_1-\mathrm{i}y_1 & 1 \\ 2x_2 & x_2-\mathrm{i}y_2 & 1 \\ 2x_3 & x_3-\mathrm{i}y_3 & 1 \end{vmatrix} = 0 \iff \begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0.$ □

习题 1.2.12 设 z_1, z_2, z_3 是单位圆周上的三个点, 证明: 这三个点是一个正三角形三个顶点的充要条件为

$$z_1+z_2+z_3=0.$$

证明 不妨设沿逆时针方向次序为 z_1, z_2, z_3.

(⇒) 由于 z_1, z_2, z_3 恰三等分单位圆周, $z_2 = \omega z_1, z_2 = \omega^2 z_1$, 其中 $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$. 因此 $z_1 + z_2 + z_3 = z_1(1 + \omega + \omega^2) = 0$.

(⇐) 由于三点绕原点同方向旋转相同角度不影响正三角形的判定, 通过除以 z_1, 可不妨设 $z_1 = 1$, 则 $z_2 + z_3 = -1$. 因此 $|\mathrm{Im}\, z_2| = |\mathrm{Im}\, z_3|$, 再由 $|z_2| = |z_3| = 1$ 得 $\mathrm{Re}\, z_2 = \mathrm{Re}\, z_3 = -\dfrac{1}{2}$, 于是 $z_2 = \omega, z_3 = \omega^2$, z_1, z_2, z_3 构成正三角形的三个顶点. □

习题 1.3.1 证明: 在复数的球面表示下, z 和 $\dfrac{1}{\bar{z}}$ 的球面像关于复平面对称.

证明 在球极投影下, 对 $z \in \mathbb{C}$, 有

$$z \mapsto \left(\frac{z+\bar{z}}{|z|^2+1}, \frac{z-\bar{z}}{\mathrm{i}(|z|^2+1)}, \frac{|z|^2-1}{|z|^2+1}\right),$$
$$\frac{1}{\bar{z}} \mapsto \left(\frac{\frac{1}{\bar{z}}+\frac{1}{z}}{\left|\frac{1}{\bar{z}}\right|^2+1}, \frac{\frac{1}{\bar{z}}-\frac{1}{z}}{\mathrm{i}\left(\left|\frac{1}{\bar{z}}\right|^2+1\right)}, \frac{\left|\frac{1}{\bar{z}}\right|^2-1}{\left|\frac{1}{\bar{z}}\right|^2+1}\right) = \left(\frac{z+\bar{z}}{|z|^2+1}, \frac{z-\bar{z}}{\mathrm{i}(|z|^2+1)}, \frac{1-|z|^2}{|z|^2+1}\right).$$

故 z 和 $\dfrac{1}{\bar{z}}$ 的球面像关于复平面对称. □

习题 1.3.2 证明: 在复数的球面表示下, z 和 w 的球面像是直径对点当且仅当 $z\overline{w} = -1$.

证明 (⇐) 由习题 1.3.1, z 与 $\dfrac{1}{\bar{z}} = -w$ 的球面像关于复平面对称. 而 w 与 $-w$ 的球面像关于单位球过原点的直径对称, 因此 z 和 w 的球面像是直径对点.

(⇒) 在球极投影下, 对 $(x_1, x_2, x_3) \in \mathbb{S}^2$, 有

$$(x_1, x_2, x_3) \mapsto \frac{x_1 + \mathrm{i}x_2}{1 - x_3}, \quad (-x_1, -x_2, -x_3) \mapsto \frac{-x_1 - \mathrm{i}x_2}{1 + x_3} = \frac{-1}{\frac{x_1 - \mathrm{i}x_2}{1 - x_3}}.$$

因此 z 和 w 的球面像是直径对点当且仅当 $z\overline{w} = -1$. □

习题 1.4.2 设 $z = x + \mathrm{i}y \in \mathbb{C}$, 证明:

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \mathrm{e}^x(\cos y + \mathrm{i}\sin y).$$

证明 注意到

$$\begin{aligned}\lim_{n\to\infty}\left|\left(1+\frac{x+\mathrm{i}y}{n}\right)^n\right| &= \lim_{n\to\infty}\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)^{\frac{n}{2}} = \exp\left[\lim_{n\to\infty}\frac{n}{2}\log\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)\right] \\ &= \exp\left[\lim_{n\to\infty}\frac{n}{2}\left(\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)\right] = \mathrm{e}^x\end{aligned}$$

以及

$$\lim_{n\to\infty}\arg\left(1+\frac{x+\mathrm{i}y}{n}\right)^n = \lim_{n\to\infty} n \arctan\frac{\frac{y}{n}}{1+\frac{x}{n}} = y,$$

便有

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \mathrm{e}^x(\cos y + \mathrm{i}\sin y).$$

□

习题 1.5.3 指出下列点集的内部、边界、闭包和导集:

(1) $\mathbb{N}$.

(2) $E = \left\{\dfrac{1}{k} : k \in \mathbb{N}\right\}$.

(3) $D = \mathbb{B}(1,1) \cup \mathbb{B}(-1,1)$.

(4) $G = \{z \in \mathbb{C} : 1 < |z| \leqslant 2\}$.

(5) $\mathbb{C}$.

解答 (1) 内部 $= \varnothing$, 边界 $= \mathbb{N}$, 闭包 $= \mathbb{N}$, 导集 $= \varnothing$.

(2) 内部 $= \varnothing$, 边界 $=$ 闭包 $= E \cup \{0\}$, 导集 $= \{0\}$.

(3) 内部 $= D$, 边界 $= \{z \in \mathbb{C} : |z-1| = 1$ 或 $|z+1| = 1\}$, 闭包 $=$ 导集 $= \{z \in \mathbb{C} : |z-1| \leqslant 1$ 或 $|z+1| \leqslant 1\}$.

(4) 内部 $= \{z \in \mathbb{C} : 1 < |z| < 2\}$, 边界 $= \{z \in \mathbb{C} : |z| = 1$ 或 $2\}$, 闭包 $=$ 导集 $= \{z \in \mathbb{C} : 1 \leqslant |z| \leqslant 2\}$.

(5) 内部 $= \mathbb{C}$, 边界 $= \varnothing$, 闭包 $\mathbb{C}$, 导集 $= \mathbb{C}$. □

习题 1.5.5 证明：若 D 为开集, 则 $D' = \overline{D} = \partial D \cup D$.

证明 (1) 由于 $\overline{D} = D \cup D'$, 为证 $D' = \overline{D}$, 只需证 $D \subset D'$. 对任意 $x \in D$, 由 D 是开集, 存在 $r > 0$ 使得 $\mathbb{B}(x,r) \subset D$. 于是对任意 $\varepsilon \in (0,r)$ 都有 $\mathbb{B}^\circ(x,\varepsilon) \subset D$, 故 $x \in D'$, $D \subset D'$, 进而 $D' = \overline{D}$.

(2) 由于 D 是开集, $\partial D \cap D = \varnothing$, 而 $\overline{D} = D \cup D'$, 为证 $\overline{D} = \partial D \cup D$, 只需证 $\partial D \subset D'$. 对任意 $x \in \partial D$ 与 $r > 0$, $\mathbb{B}(x,r) \cap D = \mathbb{B}^\circ(x,r) \cap D \neq \varnothing$, 因此 $x \in D'$, 进而 $\partial D \subset D'$, $\overline{D} = \partial D \cup D$. □

习题 1.6.1 满足下列条件的点 z 所组成的点集是什么？如果是域, 说明它是单连通域还是多连通域？

(1) $\operatorname{Re} z = 1$.

(2) $\operatorname{Im} z < -5$.

(3) $|z - \mathrm{i}| + |z + \mathrm{i}| = 5$.

(4) $|z - \mathrm{i}| \leqslant |2 + \mathrm{i}|$.

(5) $\arg(z-1) = \dfrac{\pi}{6}$.

(6) $|z| < 1, \operatorname{Im} z > \dfrac{1}{2}$.

(7) $\left|\dfrac{z-1}{z+1}\right| \leqslant 2$.

(8) $0 < \arg \dfrac{z-\mathrm{i}}{z+\mathrm{i}} < \dfrac{\pi}{4}$.

解答 (1) 直线 $\{z \in \mathbb{C} : \operatorname{Re} z = 1\}$, 非域.

(2) 半平面 $\{z \in \mathbb{C} : \operatorname{Im} z < -5\}$, 单连通域.

(3) 以 $\pm\mathrm{i}$ 为焦点、5 为长轴长的椭圆, 非域.

(4) 以 i 为圆心、$\sqrt{5}$ 为半径的闭圆盘, 非域.

(5) 以 1 为起点 (不含) 且与实轴夹角为 $\dfrac{\pi}{6}$ 的射线, 非域.

(6) 弓形 $\left\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > \frac{1}{2}\right\}$, 单连通域.

(7) $\left\{z \in \mathbb{C} : |z+3| \geqslant 2\sqrt{2}\right\}$, 非域.

(8) $\left\{z \in \mathbb{C} : \operatorname{Re} z < 0 \text{ 且 } |z+1| > \sqrt{2}\right\}$, 单连通域. □

习题 1.6.2 证明: 非空点集 $E \subset \mathbb{R}$ 为连通集, 当且仅当 E 是一个区间.

证明 ($\Rightarrow$) 设 $\varnothing \neq E \subset \mathbb{R}$ 连通. 若 E 不是一个区间, 则存在 $x < z < y$ 满足 $x, y \in E$ 但 $z \notin E$. 于是

$$E = (E \cap (-\infty, z)) \sqcup (E \cap (z, +\infty))$$

是两个非空不交开集的并, 与 E 连通矛盾. 故 E 是区间.

($\Leftarrow$) 设 $E \subset \mathbb{R}$ 为区间. 若 E 不连通, 则存在不交开集 $U, V \subset \mathbb{R}$ 使得

$$U \cap E \neq \varnothing, \quad V \cap E \neq \varnothing, \quad E \subset U \sqcup V.$$

不失一般性, 假设存在 $a < b$ 使得 $a \in U \cap E$ 且 $b \in V \cap E$. 令

$$A = \{x \in U \cap E : x < b\},$$

并记 $c = \sup A$. 则由 A 是开集可知 $c \neq a$, 于是 $a < c \leqslant b$. 特别地, $c \in E$. 但是

- ◇ $c \notin U$: 若 $c \in U$, 则存在 $\varepsilon > 0$ 使得 $b > c + \varepsilon \in U$. 由 E 是区间知 $c + \varepsilon \in U \cap E$, 但这与 $c = \sup A$ 矛盾.
- ◇ $c \notin V$: 若 $c \in V$, 则存在 $\varepsilon > 0$ 使得 $(c - \varepsilon, c] \subset V$. 因为 $c > a$, 所以可取 ε 充分小使得 $(c - \varepsilon, c] \subset E$, 从而 $c - \varepsilon < c$ 也是 A 的上界 (因 $(c - \varepsilon, c] \cap U = \varnothing$), 与 $c = \sup A$ 矛盾.

故 $c \notin U \cup V$, 进而 $c \notin E$, 矛盾. □

习题 1.6.5 证明: 若 D 是有界单连通域, 则 ∂D 连通. 举例说明, 若 D 是无界单连通域, 则 ∂D 可能不连通.

证明 先给出 D 是无界单连通域时的反例: 令 $D = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$, 它是无界单连通域, 但 $\partial D = \{z \in \mathbb{C} : \operatorname{Im} z = \pm 1\}$ 不连通. 下证原命题.

引理 1 若 $D \subset \mathbb{C}$ 是有界单连通域, 则 $\mathbb{C} \setminus D$ 连通.

引理 2 ([Mun] Theorem 63.1(a)) 设 X 是两个开集 U 和 V 之并, 且 $U \cap V$ 可以表示成两个不交开集 A 和 B 之并. 假设有一条 U 中的道路 α 从 A 的一个点 a 到 B 的一个点 b, 并且有一条 V 中的道路 β 从 b 到 a. 记 $f = \alpha * \beta$, 则 f 是一条回路, 且道路同伦类 $[f]$ 生成 $\pi_1(X, a)$ 的一个无限循环子群.

[Mun] J. R. Munkres, *Topology*, 2nd ed., Pearson Education Limited, 2019.

原命题 设 ∂D 不连通, 不妨设 $\partial D = D_1 \cup D_2$, 其中 D_1, D_2 是两个不相交的闭集. 由于 D 有界, D_1, D_2 都是紧致的, 设 $\varepsilon = \frac{1}{3} d(D_1, D_2) > 0$, 构造开集 $A = \bigcup\limits_{z \in D_1} \mathbb{B}(z, \varepsilon)$, $B = \bigcup\limits_{z \in D_2} \mathbb{B}(z, \varepsilon)$, 显然 $D_1 \subset A$, $D_2 \subset B$, 并且仍然有 $A \cap B = \varnothing$. 令 $U = D \cup A \cup B$, $V = (\mathbb{C} \setminus D) \cup A \cup B$, 显然 U 是开集. 对任意 $z \in \partial D$, 都有 $\mathbb{B}(z, \varepsilon) \subset V$, 因此 V 也是开的. 因为 D 连通, 所以 $\overline{D}$ 连通, $U = \overline{D} \cup \bigcup\limits_{z \in \partial D} \mathbb{B}(z, \varepsilon)$,

其中每个开球 $\mathbb{B}(z,\varepsilon)$ 连通, 且和 $\overline{D}$ 至少相交于 z, 故 U 连通. 由引理 1 知 $\mathbb{C}\setminus D$ 连通. 同理, V 连通. 注意到

$$U\cup V=\mathbb{C},\quad U\cap V=A\cup B,\quad U,V \text{ 道路连通 (因它们是连通开集)}.$$

选取 $a\in A$, $b\in B$, 由 U 道路连通, 存在一条 U 中的道路 α 从 a 到 b. 同理存在一条 V 中的道路 β 从 b 到 a. 由引理 2, $f=\alpha*\beta$ 是一条回路, 并且 $[f]$ 生成了 $\pi_1(U\cup V,u)=\pi_1(\mathbb{C},u)$ 的一个无限循环子群. 但因 $\mathbb{C}$ 是单连通的, 其基本群平凡, 没有无限循环子群, 矛盾. 因此 ∂D 是连通的. □

习题 2.2.2 设 $f\in\mathcal{H}(D)$, 并且满足下列条件之一:

(1) $\operatorname{Re} f(z)$ 是常数.

(2) $\operatorname{Im} f(z)$ 是常数.

(3) $|f(z)|$ 是常数.

(4) $\arg f(z)$ 是常数.

(5) $\operatorname{Re} f(z)=[\operatorname{Im} f(z)]^2, z\in D$.

那么 f 是一常数.

证明 (1) 用 u 和 v 记 $f(z)$ 的实部和虚部, 则 $\dfrac{\partial u}{\partial x}\equiv\dfrac{\partial u}{\partial y}\equiv 0$, 由 Cauchy-Riemann 方程, $\dfrac{\partial v}{\partial x}\equiv\dfrac{\partial v}{\partial y}\equiv 0$, 因此 $f'(z)=\dfrac{\partial u}{\partial x}+\mathrm{i}\dfrac{\partial v}{\partial x}\equiv 0$, f 是一常数.

(2) 同 (1) 可得 $\dfrac{\partial u}{\partial x}\equiv\dfrac{\partial u}{\partial y}\equiv\dfrac{\partial v}{\partial x}\equiv\dfrac{\partial v}{\partial y}\equiv 0$, 因此 $f'(z)\equiv 0$, f 是一常数.

(3) 设 $|f(z)|\equiv C$. 若 $C=0$, 则 $f(z)\equiv 0$; 若 $C\neq 0$, 由 $f(z)\overline{f(z)}\equiv C^2$ 得

$$\frac{\partial f}{\partial z}\overline{f(z)}+f(z)\frac{\partial \overline{f}}{\partial z}\equiv\frac{\partial f}{\partial z}\overline{f(z)}\equiv 0.$$

而 $\overline{f(z)}\neq 0$, 因此 $\dfrac{\partial f}{\partial z}=f'(z)=0$, f 是一常数.

(4) 用 u 和 v 记 $f(z)$ 的实部和虚部, 则 $\arg f(z)=\arctan\dfrac{v}{u}$, 且 $u^2+v^2\neq 0$. 由 $\arg f(z)$ 是常数得

$$\begin{cases}\dfrac{\partial}{\partial x}\left(\arctan\dfrac{v}{u}\right)=\dfrac{1}{u^2+v^2}\left(u\dfrac{\partial v}{\partial x}-v\dfrac{\partial u}{\partial x}\right)=0,\\ \dfrac{\partial}{\partial y}\left(\arctan\dfrac{v}{u}\right)=\dfrac{1}{u^2+v^2}\left(u\dfrac{\partial v}{\partial y}-v\dfrac{\partial u}{\partial y}\right)=0\end{cases}\implies\begin{cases}u\dfrac{\partial v}{\partial x}=v\dfrac{\partial u}{\partial x},\\ u\dfrac{\partial v}{\partial y}=v\dfrac{\partial u}{\partial y}.\end{cases}$$

而由 Cauchy-Riemann 方程, $\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial y}=\dfrac{\partial u}{\partial x}$, 代入上式即得

$$\begin{cases}-u\dfrac{\partial u}{\partial y}=v\dfrac{\partial u}{\partial x},\\ u\dfrac{\partial u}{\partial x}=v\dfrac{\partial u}{\partial y}\end{cases}\implies\begin{cases}(u^2+v^2)\dfrac{\partial u}{\partial x}\equiv 0,\\ (u^2+v^2)\dfrac{\partial u}{\partial y}\equiv 0\end{cases}\xRightarrow{u^2+v^2\neq 0}\frac{\partial u}{\partial x}\equiv\frac{\partial u}{\partial y}\equiv 0.$$

由 (1) 即得 $f(z)$ 是一常数.

(5) 用 u 和 v 记 $f(z)$ 的实部和虚部, 则 $u-v^2\equiv 0$, 因此

$$\begin{cases}\dfrac{\partial u}{\partial x}-2v\dfrac{\partial v}{\partial x}\equiv 0,\\ \dfrac{\partial u}{\partial y}-2v\dfrac{\partial v}{\partial y}\equiv 0.\end{cases}$$

由 Cauchy-Riemann 方程, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$, 代入上式即得

$$\begin{cases}\dfrac{\partial v}{\partial y}=2v\dfrac{\partial v}{\partial x},\\ \dfrac{\partial v}{\partial x}+2v\dfrac{\partial v}{\partial y}\equiv 0\end{cases}\implies\begin{cases}\left(1+4v^2\right)\dfrac{\partial v}{\partial x}\equiv 0,\\ \left(1+4v^2\right)\dfrac{\partial v}{\partial y}\equiv 0\end{cases}\implies\frac{\partial v}{\partial x}\equiv\frac{\partial v}{\partial y}\equiv 0.$$

由 (1) 即得 $f(z)$ 是一常数. □

习题 2.2.4 设 $z=r(\cos\theta+\mathrm{i}\sin\theta)$, $f(z)=u(r,\theta)+\mathrm{i}v(r,\theta)$, 证明 Cauchy-Riemann 方程为

$$\begin{cases}\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta},\\ \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}.\end{cases}$$

证明 记 $\begin{cases}x=r\cos\theta,\\ y=r\sin\theta,\end{cases}$ 则 $\begin{pmatrix}\dfrac{\partial u}{\partial r}\\ \dfrac{\partial u}{\partial\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial u}{\partial y}\end{pmatrix}$, 因此

$$\begin{pmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial u}{\partial y}\end{pmatrix}=\begin{pmatrix}\cos\theta & -\dfrac{1}{r}\sin\theta\\ \sin\theta & \dfrac{1}{r}\cos\theta\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial r}\\ \dfrac{\partial u}{\partial\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial u}{\partial\theta}\\ \sin\theta\dfrac{\partial u}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial u}{\partial\theta}\end{pmatrix}.$$

同理可得

$$\begin{pmatrix}\dfrac{\partial v}{\partial x}\\ \dfrac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}\cos\theta\dfrac{\partial v}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial v}{\partial\theta}\\ \sin\theta\dfrac{\partial v}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial v}{\partial\theta}\end{pmatrix}.$$

此时 Cauchy-Riemann 方程为

$$\begin{cases}\cos\theta\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial u}{\partial\theta}=\sin\theta\dfrac{\partial v}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial v}{\partial\theta},\\ \sin\theta\dfrac{\partial u}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial u}{\partial\theta}=\dfrac{1}{r}\sin\theta\dfrac{\partial v}{\partial\theta}-\cos\theta\dfrac{\partial v}{\partial r}.\end{cases}$$

整理即得

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\dfrac{\partial v}{\partial\theta}\\ \dfrac{\partial v}{\partial r}+\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\implies\begin{cases}\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta},\\ \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}.\end{cases}$$

□

习题 2.2.7 设 D 是 $\mathbb{C}$ 中的域, $f \in \mathcal{C}^2(D)$. 证明: 对每个 $z \in D$, 有

$$\frac{\partial^2 f}{\partial z \partial \bar{z}}(z) = \frac{\partial^2 f}{\partial \bar{z} \partial z}(z).$$

证明 由于 $f \in \mathcal{C}^2(D)$, 其二阶偏导数具有对称性, $\dfrac{\partial^2}{\partial z \partial \bar{z}} = \dfrac{1}{4}\Delta = \dfrac{\partial^2}{\partial \bar{z} \partial z}$. □

习题 2.2.11 设 D 是域, $f : D \to \mathbb{C} \setminus (-\infty, 0]$ 是非常数的全纯函数, 则 $\log|f(z)|$ 和 $\arg f(z)$ 是 D 上的调和函数, 而 $|f(z)|$ 不是 D 上的调和函数.

证明 由

$$\begin{aligned}\Delta \log|f(z)| &= \frac{1}{2}\Delta \log|f(z)|^2 = 2\frac{\partial^2}{\partial z \partial \bar{z}} \log\left(f(z)\overline{f(z)}\right) \\ &= 2\frac{\partial}{\partial z}\left(\frac{f(z)\overline{f'(z)}}{f(z)\overline{f(z)}}\right) = 2\frac{\partial}{\partial z}\left(\frac{\overline{f'(z)}}{\overline{f(z)}}\right) \xlongequal{\text{C-R 方程}} 0\end{aligned}$$

知 $\log|f(z)|$ 是 D 上的调和函数. 由

$$\mathrm{e}^{2\mathrm{i} \arg f(z)} = \frac{f(z)}{\overline{f(z)}}$$

可得

$$2\mathrm{i}\mathrm{e}^{2\mathrm{i} \arg f(z)}\frac{\partial}{\partial z} \arg f(z) = \frac{f'(z)}{\overline{f(z)}} \implies \frac{\partial}{\partial z} \arg f(z) = \frac{f'(z)}{2\mathrm{i} f(z)},$$

因此

$$\Delta \arg f(z) = 4\frac{\partial^2}{\partial \bar{z} \partial z} \arg f(z) = \frac{\partial}{\partial \bar{z}}\left(\frac{f'(z)}{2\mathrm{i} f(z)}\right) = 0,$$

即 $\arg f(z)$ 是 D 上的调和函数. 而

$$\frac{\partial}{\partial z}|f(z)| = \frac{f'(z)\overline{f(z)}}{2\sqrt{f(z)\overline{f(z)}}},$$

进而

$$\Delta|f(z)| = 4\frac{\partial^2}{\partial \bar{z} \partial z}|f(z)| = 2f'(z) \cdot \frac{\overline{f'(z)}|f(z)| - \frac{1}{2}|f(z)|\overline{f'(z)}}{|f(z)|^2} = \frac{|f'(z)|^2}{|f(z)|},$$

由 $f(z)$ 非常数, $|f'(z)|$ 不恒为 0, 因此 $|f(z)|$ 不是 D 上的调和函数. □

习题 2.2.13 设 u 是域 D 上的实值调和函数, $|\nabla u| \neq 0$, φ 是 $u(D)$ 上的实函数. 证明: $\varphi \circ u$ 是 D 上的调和函数当且仅当 φ 是线性函数.

证明 记 $\psi = \varphi \circ u$, 则 $\Delta \psi = \varphi''(u)|\nabla u|^2 + \varphi'(u)\Delta u = \varphi''(u)|\nabla u|^2$. 故 $\Delta \psi \equiv 0 \iff \varphi''(u) \equiv 0$. □

习题 2.3.1 求映射 $w = \dfrac{z - \mathrm{i}}{z + \mathrm{i}}$ 在 $z_1 = -1$ 和 $z_2 = \mathrm{i}$ 处的转动角和伸缩率.

解答 由于 $\dfrac{\partial w}{\partial z} = \dfrac{2\mathrm{i}}{(z + \mathrm{i})^2}$, $w'(z_1) = -1$, $w'(z_2) = -\dfrac{\mathrm{i}}{2}$, 映射 w 在 z_1 处的转动角为 π, 伸缩率为 1; 在 z_2 处的转动角为 $-\dfrac{\pi}{2}$, 伸缩率为 $\dfrac{1}{2}$. □

习题 2.3.2 设 f 是域 D 上的全纯函数, 且 $f'(z)$ 在 D 上不取零值. 试证:

(1) 对每一个 $u_0 + \mathrm{i}v_0 \in f(D)$, 曲线 $\operatorname{Re} f(z) = u_0$ 和曲线 $\operatorname{Im} f(z) = v_0$ 正交.

(2) 对每一个 $r_0\mathrm{e}^{\mathrm{i}\theta_0} \in f(D) \setminus \{0\}, -\pi < \theta_0 \leqslant \pi$, 曲线 $|f(z)| = r_0$ 与曲线 $\arg f(z) = \theta_0$ 正交.

证明 (1) 用 u 和 v 记 $f(z)$ 的实部和虚部, 则曲线 $u(x,y)=u_0$ 在 (x,y) 处的法向量为 $\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$, 曲线 $v(x,y)=v_0$ 在 (x,y) 处的法向量为 $\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)\xlongequal{\text{C-R 方程}}\left(-\frac{\partial u}{\partial y},\frac{\partial u}{\partial x}\right)$. 因此在这两条曲线交点处两法向量正交, 即这两条曲线正交.

(2) 设 $f(z)=R(r,\theta)\mathrm{e}^{\mathrm{i}\Theta(r,\theta)}$.

① 对 $\log f(z)=\log R(r,\theta)+\mathrm{i}\Theta(r,\theta)$ 运用习题 2.2.4 即得极坐标系下的 Cauchy-Riemann 方程

$$\begin{cases}\dfrac{\partial R}{\partial r}=\dfrac{R}{r}\dfrac{\partial\Theta}{\partial\theta},\\ \dfrac{\partial R}{\partial\theta}=-Rr\dfrac{\partial\Theta}{\partial r}.\end{cases}$$

② 曲线 $R(r,\theta)=r_0$ 在 (r,θ) 处的法向量为 $\frac{\partial R}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial R}{\partial\theta}\mathbf{e}_\theta$, 曲线 $\Theta(r,\theta)=\theta_0$ 在 (r,θ) 处的法向量为 $\frac{\partial\Theta}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial\Theta}{\partial\theta}\mathbf{e}_\theta\xlongequal{\text{C-R 方程}}-\frac{1}{Rr}\frac{\partial R}{\partial\theta}\mathbf{e}_r+\frac{1}{R}\frac{\partial R}{\partial r}\mathbf{e}_\theta$. 因此在这两条曲线交点处两法向量正交, 即这两条曲线正交. □

习题 2.3.3 设 $f\in\mathcal{H}(\mathbb{B}(0,1)\cup\{1\})$, 且 $f(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$, $f(1)=1$. 证明: $f'(1)\geqslant 0$.

证明 由于 $f(z)$ 在 $z=1$ 处全纯,

$$f(z)=f(1)+f'(1)(z-1)+o(|z-1|)=1+f'(1)+o(|z-1|),\quad z\to 1.$$

由题设, 当 $|z|<1$ 时 $|f(z)|<1$, 因此

$$|1+f'(1)+o(|z-1|)|<1,\quad \mathbb{B}(0,1)\ni z\to 1.$$

展开即得

$$\operatorname{Re}(f'(1)(z-1))+o(|z-1|)<0,\quad \mathbb{B}(0,1)\ni z\to 1.$$

令 $z-1=r\mathrm{e}^{\mathrm{i}\theta}$, $\theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, 上式化为

$$\operatorname{Re}\big(f'(1)r\mathrm{e}^{\mathrm{i}\theta}\big)+o(r)<0\iff\operatorname{Re}\big(f'(1)\mathrm{e}^{\mathrm{i}\theta}\big)+o(1)<0,\quad r\to 0^+.$$

于是

$$\operatorname{Re}\big(f'(1)\mathrm{e}^{\mathrm{i}\theta}\big)\leqslant 0,\quad \forall\theta\in\left(\tfrac{\pi}{2},\tfrac{3\pi}{2}\right).$$

令 $f'(1)=|f'(1)|\mathrm{e}^{\mathrm{i}\arg f'(1)}$. 若 $|f'(1)|\neq 0$, 则

$$\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\left(\arg f'(1)+\theta\right)}\right)\leqslant 0,\quad \forall\theta\in\left(\tfrac{\pi}{2},\tfrac{3\pi}{2}\right).$$

因此

$$\arg f'(1)+\theta\in\left[\tfrac{\pi}{2},\tfrac{3\pi}{2}\right],\quad \forall\theta\in\left(\tfrac{\pi}{2},\tfrac{3\pi}{2}\right),$$

由此可见 $\arg f'(1)=0$, 从而 $f'(1)=|f'(1)|>0$. 故 $f'(1)\geqslant 0$. □

习题 2.4.2 求 $\left|\mathrm{e}^{z^2}\right|$ 和 $\arg\mathrm{e}^{z^2}$.

解答 $\left|\mathrm{e}^{z^2}\right|=\mathrm{e}^{(\operatorname{Re}z)^2-(\operatorname{Im}z)^2}$, $\arg\mathrm{e}^{z^2}=2\operatorname{Re}z\operatorname{Im}z$. □

习题 2.4.4 设 f 是整函数, $f(0)=1$. 证明:

(1) 若 $f'(z)=f(z)$ 对每个 $z\in\mathbb{C}$ 成立, 则 $f(z)\equiv \mathrm{e}^z$.

(2) 若对每个 $z,w\in\mathbb{C}$, 有 $f(z+w)=f(z)f(w)$, 且 $f'(0)=1$, 则 $f(z)\equiv\mathrm{e}^z$.

证明 (1) 由

$$\frac{\partial}{\partial z}\left(\frac{f(z)}{\mathrm{e}^z}\right)=\frac{f'(z)-f(z)}{\mathrm{e}^z}\equiv 0,\quad \left.\frac{f(z)}{\mathrm{e}^z}\right|_{z=0}=1$$

即知 $f(z)\equiv\mathrm{e}^z$.

(2) 由于

$$\frac{f(z+w)-f(z)}{w}=f(z)\cdot\frac{f(w)-f(0)}{w-0}\xRightarrow[f'(0)=1]{w\to 0}f'(z)\equiv f(z),$$

由 (1) 即知 $f(z)=\mathrm{e}^z$. □

习题 2.4.15 称 $\varphi(z)=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$ 为 Rokovsky 函数. 证明下面四个域都是 φ 的单叶性域:

(1) 上半平面 $\{z\in\mathbb{C}:\operatorname{Im}z>0\}$.

(2) 下半平面 $\{z\in\mathbb{C}:\operatorname{Im}z<0\}$.

(3) 无心单位圆盘 $\{z\in\mathbb{C}:0<|z|<1\}$.

(4) 单位圆盘的外部 $\{z\in\mathbb{C}:|z|>1\}$.

证明 设 $z_1,z_2\in\mathbb{C}$ 使得 $\varphi(z_1)=\varphi(z_2)$, 则 $(z_1z_2-1)(z_1-z_2)=0$, 因此只要域 D 中任意两点不满足 $z_1z_2=1$, D 就是 $\varphi(z)$ 的单叶性域.

(1) 对任意 $z_1,z_2\in\{z\in\mathbb{C}:\operatorname{Im}z>0\}$, 由 $\arg z_1,\arg z_2\in(0,\pi)$ 得 $\arg(z_1z_2)\in(0,2\pi)$, 因此 $z_1z_2\neq 1$.

(2) 通过 $z_1z_2=1\iff\overline{z_1z_2}=1$ 转化为 (1).

(3) 对任意 $z_1,z_2\in\{z\in\mathbb{C}:0<|z|<1\}$, 由 $|z_1|,|z_2|<1$ 得 $|z_1z_2|<1$, 因此 $z_1z_2\neq 1$.

(4) 通过 $z_1z_2=1\iff\dfrac{1}{z_1}\dfrac{1}{z_2}=1$ 转化为 (3). □

习题 2.4.16 求习题 2.4.15 中的四个域在映射 $\varphi(z)=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$ 下的像.

解答 设 $z=r\mathrm{e}^{\mathrm{i}\theta}$, $\varphi(z)=u+\mathrm{i}v$, 则

$$u=\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta,\quad v=\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta.$$

因此 φ 将圆周 $|z|=r_0\neq 0$ 映为曲线

$$u=\frac{1}{2}\left(r_0+\frac{1}{r_0}\right)\cos\theta,\quad v=\frac{1}{2}\left(r_0-\frac{1}{r_0}\right)\sin\theta.$$

当 $r_0\neq 1$ 时, 这是半轴长为 $a=\dfrac{1}{2}\left(r_0+\dfrac{1}{r_0}\right)$, $b=\dfrac{1}{2}\left|r_0-\dfrac{1}{r_0}\right|$, 且由 $a^2-b^2\equiv 1$ 知 $z=\pm 1$ 为所有椭圆的公共焦点. 当 $r_0\to 1$ 时, $a\to 1, b\to 0$, 椭圆压缩成实轴上的线段 $[-1,1]$; 当 $r_0\to 0^+$ 或 $r_0\to+\infty$ 时, $a,b\to+\infty$, 椭圆扩张为圆周. 故

- ◇ 无心单位圆盘 $\{z\in\mathbb{C}:0<|z|<1\}\xrightarrow{\varphi}\mathbb{C}\setminus[-1,1]$.
- ◇ 单位圆盘的外部 $\{z\in\mathbb{C}:|z|>1\}\xrightarrow{\varphi}\mathbb{C}\setminus[-1,1]$.

再考虑射线 $\arg z=\theta_0$ $(\theta\in[0,2\pi))$, 它在 φ 下的像为

$$u=\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta_0,\quad v=\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta_0.$$

当 $\theta_0=0$ 时, 这是射线 $\{u:u\geqslant 1\}$; 当 $\theta_0=\pi$ 时, 这是射线 $\{u:u\leqslant -1\}$; 当 $\theta_0=\dfrac{\pi}{2}$ 或 $\dfrac{3\pi}{2}$ 时, 这是虚轴; 当 θ_0 不取上述值时, 这是双曲线

$$\frac{u^2}{\cos^2\theta_0}-\frac{v^2}{\sin^2\theta_0}=1,$$

且由 $\cos^2\theta_0+\sin^2\theta_0\equiv 1$ 知 $z=\pm 1$ 为所有双曲线的公共焦点. 故

- ◇ 上半平面 $\{z\in\mathbb{C}:\operatorname{Im}z>0\}\xrightarrow{\varphi}\mathbb{C}\setminus((-\infty,-1]\cup[1,+\infty))$.
- ◇ 下半平面 $\{z\in\mathbb{C}:\operatorname{Im}z>0\}\xrightarrow{\varphi}\mathbb{C}\setminus((-\infty,-1]\cup[1,+\infty))$. □

习题 2.4.18 证明: $w=\cos z$ 将半条形域 $\{z\in\mathbb{C}:0<\operatorname{Re}z<2\pi,\operatorname{Im}z>0\}$ 一一地映为 $\mathbb{C}\setminus[-1,+\infty)$.

证明 记 $\mu(z)=\mathrm{i}z,\eta(z)=\mathrm{e}^z,\varphi(z)=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$, 则 $w=\varphi\circ\eta\circ\mu$, 且有

$$\begin{array}{ccc}\{z\in\mathbb{C}:0<\operatorname{Re}z<2\pi,\operatorname{Im}z>0\} & \xrightarrow[1:1]{\mu} & \{z\in\mathbb{C}:\operatorname{Re}z<0,0<\operatorname{Im}z<2\pi\}\\ & & \Big\downarrow{\scriptstyle 1:1}\,\eta\\ \mathbb{C}\setminus[-1,+\infty) & \xleftarrow[1:1]{\varphi} & \mathbb{B}(0,1)\setminus[0,1)\end{array}$$

其中第一个箭头为双射是显然的, 第二个箭头为双射可由 $\{z\in\mathbb{C}:\operatorname{Re}z<0,0<\operatorname{Im}z<2\pi\}$ 是 η 的单叶域得到, 第三个箭头为双射证明如下: 由习题 2.4.15, 无心单位圆盘是 φ 的单叶域, 进而 $\mathbb{B}(0,1)\setminus[0,1)$ 也是 φ 的单叶域; 再由习题 2.4.16, φ 将无心单位圆盘映为 $\mathbb{C}\setminus[-1,1]$, 因此 φ 将 $\mathbb{B}(0,1)\setminus[0,1)$ 映为

$$(\mathbb{C}\setminus[-1,1])\setminus\varphi([0,1))=\mathbb{C}\setminus([-1,1]\cup(1,+\infty))=\mathbb{C}\setminus[-1,+\infty),$$

由此得到第三个箭头, 且其为双射. 由双射的复合即得所欲证. □

习题 2.4.19 证明: $w=\sin z$ 将半条形域 $\left\{z\in\mathbb{C}:-\dfrac{\pi}{2}<\operatorname{Re}z<\dfrac{\pi}{2},\operatorname{Im}z>0\right\}$ 一一地映为上半平面.

证明 由于 $\sin z=\cos\left(z-\dfrac{\pi}{2}\right)$, 只需考虑 $\{z\in\mathbb{C}:-\pi<\operatorname{Re}z<0,\operatorname{Im}z>0\}$ 在函数 $w=\cos z$ 下的像. 记 $\mu(z)=\mathrm{i}z,\eta(z)=\mathrm{e}^z,\varphi(z)=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$, 则 $w=\varphi\circ\eta\circ\mu$, 且有

$$\begin{array}{ccc}\{z\in\mathbb{C}:-\pi<\operatorname{Re}z<0,\operatorname{Im}z>0\} & \xrightarrow[1:1]{\mu} & \{z\in\mathbb{C}:\operatorname{Re}z<0,-\pi<\operatorname{Im}z<0\}\\ & & \Big\downarrow{\scriptstyle 1:1}\,\eta\\ \{z\in\mathbb{C}:\operatorname{Im}z>0\} & \xleftarrow[1:1]{\varphi} & \{z\in\mathbb{C}:|z|<1\text{ 且 }\operatorname{Im}z<0\}\end{array}$$

其中第一个箭头为双射是显然的, 第二个箭头为双射可由 $\{z\in\mathbb{C}:\operatorname{Re}z<0,-\pi<\operatorname{Im}z<0\}$ 是 η 的单叶域得到, 第三个箭头为双射证明如下: 由习题 2.4.15, 无心单位圆盘是 φ 的单叶域, 进而

$\{z\in\mathbb{C}:\operatorname{Re}z<0,-\pi<\operatorname{Im}z<0\}$ 也是 φ 的单叶域; 再由习题 2.4.16 中的讨论可见, φ 将单位圆盘内部的半径为 r_0 下半圆周映为半长轴长 $\frac{1}{2}\left(r_0+\frac{1}{r_0}\right)$、半短轴长 $\frac{1}{2}\left|r_0-\frac{1}{r_0}\right|$ 的上半椭圆, 因此

$$\{z\in\mathbb{C}:\operatorname{Re}z<0,-\pi<\operatorname{Im}z<0\}\xrightarrow{\varphi}\{z\in\mathbb{C}:\operatorname{Im}z>0\},$$

且这是双射. □

习题 2.4.21 当 z 按逆时针方向沿圆周 $\{z\in\mathbb{C}:|z|=2\}$ 旋转一圈后, 计算下列函数辐角的增量:

(1) $(z-1)^{\frac{1}{2}}$.

(2) $\left(1+z^4\right)^{\frac{1}{3}}$.

(3) $\left(z^2+2z-3\right)^{\frac{1}{4}}$.

(4) $\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}$.

(5) $\left(\frac{z^2-1}{z^2+5}\right)^{\frac{1}{7}}$.

解答 记 $C=\{z\in\mathbb{C}:|z|=2\}$. 对有理函数 $R(z)=\prod_{j=1}^{m}(z-a_j)^{n_j}$ 与 $F(z)=R(z)^{\frac{1}{n}}$, 若 $C\cap\{a_j\}_{j=1}^m=\varnothing$, 记 $\Lambda=\{j:a_j$ 在 C 内部$\}$, 则有

$$\Delta_C\operatorname{Arg}R(z)=\sum_{j=1}^{m}n_j\Delta_C\operatorname{Arg}(z-a_j)=2\pi\sum_{j\in\Lambda}n_j\implies\Delta_C\operatorname{Arg}F(z)=\frac{2\pi}{n}\sum_{j\in\Lambda}n_j.$$

(1) 由于 1 在 C 的内部, $\Delta_C\operatorname{Arg}(z-1)^{\frac{1}{2}}=\frac{2\pi}{2}\cdot 1=\pi$.

(2) 由 $1+z^4$ 的根 z 均满足 $|z|^4=|-1|^4=1$, 其 4 个根均位于 C 的内部, $\Delta_C\left(1+z^4\right)^{\frac{1}{3}}=\frac{2\pi}{3}\cdot 4=\frac{8\pi}{3}$.

(3) 由于 $z^2+2z-3=(z+3)(z-1)$, 1 在 C 的内部, 而 -3 在 C 的外部, $\Delta_C\left(z^2+2z-3\right)^{\frac{1}{4}}=\frac{2\pi}{4}\cdot 1=\frac{\pi}{2}$.

(4) 由于 ± 1 均在 C 的内部, $\Delta_C\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}=\frac{2\pi}{2}\cdot(1-1)=0$.

(5) 由于 ± 1 在 C 的内部, 而 $\pm\sqrt{5}\mathrm{i}$ 在 C 的外部, $\Delta_C\left(\frac{z^2-1}{z^2+5}\right)^{\frac{1}{7}}=\frac{2\pi}{7}\cdot 2=\frac{4\pi}{7}$. □

习题 2.4.22 设 $f(z)=\frac{z^{p-1}}{(1-z)^p}$, $0<p<1$. 证明: f 能在域 $D=\mathbb{C}\setminus[0,1]$ 上选出单值的全纯分支.

证明 由于 $f(z)=\frac{1}{z}\left(\frac{z}{1-z}\right)^p=\frac{1}{z}\exp\left(p\operatorname{Log}\frac{z}{1-z}\right)$, 只需证 $\operatorname{Log}\frac{z}{1-z}$ 能在 $D=\mathbb{C}\setminus[0,1]$ 上选出单值的全纯分支. 当 $z\notin[0,1]$ 时, $\frac{z}{1-z}\notin[0,+\infty)$, 而 $\operatorname{Log}z$ 在 $\mathbb{C}\setminus[0,+\infty)$ 上可选出单值全纯分支, 得证. □

习题 2.4.26 设 D 是 z 平面上去掉线段 $[-1,\mathrm{i}]$, $[1,\mathrm{i}]$ 和射线 $z=\mathrm{i}t$ $(1\leqslant t<+\infty)$ 后所得的域, 证明函数 $\operatorname{Log}\left(1-z^2\right)$ 能在 D 上分出单值全纯分支. 设 f 是满足 $f(0)=0$ 的那个分支, 试计算 $f(2)$ 的值.

证明 对任意不经过 ± 1 的简单闭曲线,

$$\Delta_C \operatorname{Log}\big(1-z^2\big) = \Delta_C \operatorname{Log}(1+z) + \Delta_C \operatorname{Log}(1-z).$$

- 若 C 仅包含点 1 且沿逆时针方向, 则 $\Delta_C \operatorname{Log}\big(1-z^2\big) = \Delta_C \operatorname{Log}(1-z) = \mathrm{i}\Delta_C \operatorname{Arg}(1-z) = 2\pi\mathrm{i}$.
- 若 C 仅包含点 -1 且沿逆时针方向, 则 $\Delta_C \operatorname{Log}\big(1-z^2\big) = \Delta_C \operatorname{Log}(1+z) = \mathrm{i}\Delta_C \operatorname{Arg}(1+z) = 2\pi\mathrm{i}$.
- 若 C 同时包含 ± 1 且沿逆时针方向, 则 $\Delta_C \operatorname{Log}\big(1-z^2\big) = \Delta_C \operatorname{Log}(1-z) + \Delta_C \operatorname{Log}(1+z) = 4\pi\mathrm{i}$.
- 若 C 不包含 ± 1, 则 $\Delta_C \operatorname{Log}\big(1-z^2\big) = 0$.

由于 D 中任一简单闭曲线无法仅包含 1 或 -1, 也无法同时包含 ± 1, 由上述讨论即知 $\operatorname{Log}\big(1-z^2\big)$ 能在 D 上分出单值全纯分支. 对于满足 $f(0)=0$ 的分支 f, 当 z 沿 D 中简单曲线从 0 变动到 2 时,

$$\begin{aligned} f(2)-f(0) &= \Delta_\gamma \operatorname{Log}\big(1-z^2\big) = \big(\log\big|1-2^2\big| - \log 1\big) + \mathrm{i}[\Delta_\gamma \operatorname{Arg}(1+z) + \Delta_\gamma \operatorname{Arg}(1-z)] \\ &= \mathrm{i}(0+\pi) = \log 3 + \pi\mathrm{i}. \end{aligned}$$

故 $f(2) = \log 3 + \pi\mathrm{i}$. □

习题 2.4.27 证明函数 $\sqrt[4]{(1-z)^3(1+z)}$ 能在 $\mathbb{C}\setminus[-1,1]$ 上选出一个单值全纯分支 f, 满足 $f(\mathrm{i}) = \sqrt{2}\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}$. 试计算 $f(-\mathrm{i})$ 的值.

证明 承接习题 2.4.21 解答开头的讨论, 我们还有

$$\Delta_C F(z) = |R(z_0)|^{\frac{1}{n}} \mathrm{e}^{\frac{\mathrm{i}}{n}\operatorname{Arg} R(z_0)} \Big[\mathrm{e}^{\frac{\mathrm{i}}{n}\Delta_C \operatorname{Arg} R(z)} - 1\Big],$$

其中 z_0 为环绕曲线 C 时的起点. 因此

$$\Delta_C F(z) = 0 \iff \mathrm{e}^{\frac{\mathrm{i}}{n}\Delta_C \operatorname{Arg} R(z)} = 1 \iff \Delta_C \operatorname{Arg} R(z) = 2kn\pi \iff \sum_{j\in\Lambda} n_j = kn, \quad k\in\mathbb{Z}.$$

本题中, 对于 $R(z) = (1-z)^3(1+z)$ 与 $F(z) = \big[(1-z)^3(1+z)\big]^{\frac{1}{4}}$,

- 由于 3 不是 4 的整数倍, 因此 1 是 $F(z)$ 的枝点.
- 由于 1 不是 4 的整数倍, 因此 -1 是 $F(z)$ 的枝点.
- 由于 $3+1=4$ 是 4 的整数倍, 因此 ∞ 不是 $F(z)$ 的枝点.

因此对 $\mathbb{C}\setminus[-1,1]$ 上的任一简单闭曲线 γ, 要么 γ 同时包含 ± 1 两点, 要么 γ 不包含 ± 1 两点, 在这两种情况下均有 $\Delta_\gamma F(z) = 0$. 又 $(1-\mathrm{i})^3(1+\mathrm{i}) = -4\mathrm{i} = \Big(\sqrt{2}\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}\Big)^4$, 于是能在 $\mathbb{C}\setminus[-1,1]$ 上选出一个满足 $f(\mathrm{i}) = \sqrt{2}\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}$ 的单值全纯分支 f. 现取 E 为以 ± 1 为焦点、$\pm\mathrm{i}$ 为上下顶点的椭圆的左半部分, 则

$$\begin{gathered} f(-\mathrm{i}) - f(\mathrm{i}) = \Delta_E F(z) = \sqrt{2}\mathrm{e}^{\frac{\mathrm{i}}{4}\operatorname{Arg} R(\mathrm{i})}\Big(\mathrm{e}^{\frac{\mathrm{i}}{4}\Delta_E \operatorname{Arg} R(z)} - 1\Big), \\ \Delta_E \operatorname{Arg} R(z) = \frac{\pi}{2}\cdot 3 + \frac{3\pi}{2} = 3\pi. \end{gathered}$$

因此

$$f(-\mathrm{i}) = \sqrt{2}\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}} + \sqrt{2}\mathrm{e}^{\frac{\mathrm{i}}{4}\operatorname{Arg}(-4\mathrm{i})}\Big(\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}} - 1\Big) = \sqrt{2}\mathrm{e}^{\frac{5\pi}{8}\mathrm{i}}.$$

□

习题 2.5.2 求出把上半平面映为单位圆盘的分式线性变换, 使得 $-1, 0, 1$ 分别映为 $1, \mathrm{i}, -1$.

解答 设所求的分式线性变换将 z 映为 w, 则 $\dfrac{z-0}{z-1} : \dfrac{-1-0}{-1-1} = \dfrac{w-\mathrm{i}}{w-(-1)} : \dfrac{1-\mathrm{i}}{1-(-1)}$, 解得 $w = \dfrac{z-\mathrm{i}}{\mathrm{i}z-1}$.
检验: 对 $x \in \mathbb{R}$ 与 $y > 0$, 有 $\left|\dfrac{(x+\mathrm{i}y)-\mathrm{i}}{\mathrm{i}(x+\mathrm{i}y)-1}\right|^2 = \dfrac{x^2+(y-1)^2}{x^2+(y+1)^2} < 1$. □

习题 2.5.3 设 $a, b, c, d \in \mathbb{R}$, 则分式线性变换 $w = \dfrac{az+b}{cz+d}$ 把上半平面映为上半平面 $\iff ad - bc > 0$.

证明 由于 $a, b, c, d \in \mathbb{R}$, $w = \dfrac{az+b}{cz+d}$ 必将 $\mathbb{R}$ 映为 $\mathbb{R}$. 又欲证两边均蕴含 $ad - bc \neq 0$, 故不妨假设之.

($\Rightarrow$) 若 $ad - bc < 0$, 则 $w' = \dfrac{ad-bc}{(cz+d)^2} < 0$. 当 z 在 $\mathbb{R}$ 上由 $-\infty$ 趋向 $+\infty$ 时, w 由 $+\infty$ 趋向 $-\infty$, 根据全纯函数的保角性, w 把上半平面映为下半平面, 矛盾. 故 $ad - bc > 0$.

($\Leftarrow$) 由 $w' = \dfrac{ad-bc}{(cz+d)^2} > 0$, 当 z 在 $\mathbb{R}$ 上由 $-\infty$ 趋向 $+\infty$ 时, w 也由 $-\infty$ 趋向 $+\infty$, 根据全纯函数的保角性, w 把上半平面映为上半平面. □

习题 2.5.4 试求把单位圆盘的外部 $\{z : |z| > 1\}$ 映为右半平面 $\{w : \operatorname{Re} w > 0\}$ 的分式线性变换, 使得

(1) $1, -\mathrm{i}, -1$ 分别变为 $\mathrm{i}, 0, -\mathrm{i}$.

(2) $-\mathrm{i}, \mathrm{i}, 1$ 分别变为 $\mathrm{i}, 0, -\mathrm{i}$.

证明 设所求的分式线性变换将 z 映为 w.

(1) $\dfrac{z-(-\mathrm{i})}{z-(-1)} : \dfrac{1-(-\mathrm{i})}{1-(-1)} = \dfrac{w-0}{w-(-\mathrm{i})} : \dfrac{\mathrm{i}-0}{\mathrm{i}-(-\mathrm{i})} \implies w = \dfrac{z+\mathrm{i}}{z-\mathrm{i}}$. 检验: 对满足 $x^2 + y^2 > 1$ 的 $x, y \in \mathbb{R}$, 有 $\operatorname{Re}\dfrac{(x+\mathrm{i}y)+\mathrm{i}}{(x+\mathrm{i}y)-\mathrm{i}} = \dfrac{x^2+y^2-1}{x^2+(y-1)^2} > 0$.

(2) $\dfrac{z-\mathrm{i}}{z-1} : \dfrac{-\mathrm{i}-\mathrm{i}}{-\mathrm{i}-1} = \dfrac{w-0}{w-(-\mathrm{i})} : \dfrac{\mathrm{i}-0}{\mathrm{i}-(-\mathrm{i})} \implies w = \dfrac{z-\mathrm{i}}{(2-\mathrm{i})z+(2\mathrm{i}-1)}$. 检验: 对满足 $x^2 + y^2 > 1$ 的 $x, y \in \mathbb{R}$, 有 $\operatorname{Re}\dfrac{(x+\mathrm{i}y)-\mathrm{i}}{(2-\mathrm{i})(x+\mathrm{i}y)+(2\mathrm{i}-1)} = \dfrac{2\big(x^2+y^2-1\big)}{(2x+y-1)^2+(2y-x+2)^2} > 0$. □

习题 2.5.9 证明: z_1, z_2 关于圆周

$$az\bar{z} + \bar{\beta}z + \beta\bar{z} + d = 0$$

对称的充要条件是

$$az_1\overline{z_2} + \bar{\beta}z_1 + \beta\overline{z_2} + d = 0.$$

证明 **(直线)** 此时 $a = 0$. 若 z_1, z_2 关于所给直线对称, 则 $z_2 - z_1 \perp \mathrm{i}\beta$, 即 $\operatorname{Re}(\mathrm{i}\beta\overline{z_2 - z_1}) = 0$, 展开得

$$\mathrm{i}\beta(\overline{z_2} - \overline{z_1}) - \mathrm{i}\bar{\beta}(z_2 - z_1) = 0 \iff \beta\overline{z_2} + \bar{\beta}z_1 = \beta\overline{z_1} + \bar{\beta}z_2.$$

而 $\dfrac{z_1+z_2}{2}$ 满足所给直线方程:

$$\bar{\beta}\frac{z_1+z_2}{2} + \beta\frac{\overline{z_1}+\overline{z_2}}{2} + d = 0.$$

联立以上两式即得

$$\beta\overline{z_1} + \bar{\beta}z_2 + d = 0.$$

反之, 若 z_1, z_2 满足上式, 对上式取共轭得

$$\bar{\beta} z_1 + \beta \overline{z_2} + d = 0,$$

两式相加得

$$\bar{\beta} \frac{z_1 + z_2}{2} + \beta \frac{\overline{z_1} + \overline{z_2}}{2} + d = 0,$$

两式作差得

$$\beta(\overline{z_2} - \overline{z_1}) - \bar{\beta}(z_2 - z_1) = 0,$$

再乘 $\frac{\mathrm{i}}{2}$ 即得 $\mathrm{Re}(\mathrm{i}\beta\overline{z_2 - z_1}) = 0$. 故 z_1, z_2 关于此直线对称.

(圆周) 记圆周的圆心为 z_0、半径为 R. 若 z_1, z_2 关于此圆周对称, 则

$$z_2 - z_0 = \frac{R^2}{\overline{z_1} - \overline{z_0}}.$$

这是因为对上式取模与辐角可得

$$\begin{cases} |z_2 - z_0||z_1 - z_0| = R^2, \\ \mathrm{Arg}(z_2 - z_0) = -\mathrm{Arg}(\overline{z_1} - \overline{z_0}) = \mathrm{Arg}(z_1 - z_0). \end{cases}$$

代入所给圆周方程的等价形式

$$\left| z + \frac{\beta}{a} \right| = \frac{\sqrt{|\beta|^2 - ad}}{|a|}$$

即得

$$z_2 + \frac{\beta}{a} = \frac{\frac{|\beta|^2 - ad}{a^2}}{\overline{z_1} + \frac{\bar{\beta}}{a}},$$

化简即

$$a z_1 \overline{z_2} + \bar{\beta} z_1 + \beta \overline{z_2} + d = 0.$$

反之, 若 z_1, z_2 满足上式, 将上述过程反向即得 z_1, z_2 关于所给圆周对称. □

习题 2.5.10 设 $T(z) = \dfrac{az + b}{cz + d}$ 是一个分式线性变换, 如果记

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

那么

$$T^{-1}(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

证明 $T^{-1}(z) = \dfrac{-dz + b}{cz - a}$, 而由题, $\begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, 因此

$$(c\alpha + d\gamma) z^2 + (c\beta + d\delta - a\alpha - b\gamma) z - (a\beta + b\delta) = 0 \iff \frac{-dz + b}{cz - a} = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

□

习题 2.5.11 设 $T_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$, $T_2(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$ 是两个分式线性变换, 如果记

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

那么

$$(T_1 \circ T_2)(z) = \frac{az+b}{cz+d}.$$

证明 $(T_1 \circ T_2)(z) = \dfrac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(a_2c_1 + c_2d_1)z + (b_2c_1 + d_1d_2)}$, 而由题, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ a_2c_1 + c_2d_1 & b_2c_1 + d_1d_2 \end{pmatrix}$, 于是 $(T_1 \circ T_2)(z) = \dfrac{az+b}{cz+d}$. □

习题 2.5.16 求一单叶全纯映射, 把半条形域 $\left\{z : -\dfrac{\pi}{2} < \operatorname{Re} z < \dfrac{\pi}{2}, \operatorname{Im} z > 0\right\}$ 映为上半平面, 且把 $\dfrac{\pi}{2}, -\dfrac{\pi}{2}, 0$ 分别映为 $1, -1, 0$.

解答 由习题 2.4.19 知 $w = \sin z$ 满足题意. 亦可如下分解求之, 复合结果仍为 $\sin z$.

$$\begin{array}{ccc}
\left\{z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}, \operatorname{Im} z > 0\right\} & \xrightarrow{z \mapsto \mathrm{i}z} & \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}, \operatorname{Re} z < 0\right\} \\
 & & \downarrow z \mapsto \mathrm{e}^z \\
\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\} & \xleftarrow{z \mapsto \mathrm{i}z} & \{z \in \mathbb{C} : |z| < 1, \operatorname{Re} z > 0\} \\
(1) \downarrow z \mapsto \frac{z+1}{z-1} & & \\
\{z \in \mathbb{C} : \operatorname{Re} z < 0, \operatorname{Im} z < 0\} & \xrightarrow{z \mapsto z^2} & \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \\
 & & (2) \downarrow z \mapsto -\frac{z+1}{z-1} \\
 & & \{z \in \mathbb{C} : \operatorname{Im} z > 0\}
\end{array}$$

其中用到的两个分式线性变换如下:

(1) $w_1(z) = \dfrac{z+1}{z-1}$ 将上半单位圆盘映为第三象限 (由于二者在 Riemann 球上为全等的新月形, 结合保圆性及保角性可知这样的分式线性变换的确存在), 且使 $-1 \mapsto 0, 1 \mapsto \infty, \mathrm{i} \mapsto -\mathrm{i}$.

(2) $w_2(z) = -\dfrac{z+1}{z-1}$ 将上半平面映为上半平面, 且使 $0 \mapsto 1, \infty \mapsto -1, -1 \mapsto 0$. □

习题 2.5.17 求一单叶全纯映射, 把除去线段 $[a, a+h\mathrm{i}]$ 的条形域 $\{z : 0 < \operatorname{Im} z < 1\}$ 映为条形域 $\{w : 0 < \operatorname{Im} w < 1\}$, 其中 $a \in \mathbb{R}$, $0 < h < 1$.

解答 分解如下：

$$\{z:0<\operatorname{Im}z<1\}\setminus[a,a+h\mathrm{i}]\xrightarrow{z\mapsto\pi(z+a)}\{z\in\mathbb{C}:0<\operatorname{Im}z<\pi\}\setminus[0,h\pi\mathrm{i}]$$

$$\xrightarrow{z\mapsto\mathrm{e}^z}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\setminus\{z\in\mathbb{C}:|z|=1,0\leqslant\arg z\leqslant h\pi\}$$

$$\xrightarrow{z\mapsto\frac{z-1}{z+1}}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\setminus\left[0,\frac{1-\cos(h\pi)}{\sin(h\pi)}\mathrm{i}\right]$$

$$\xrightarrow{z\mapsto z^2}\mathbb{C}\setminus\left[-\left(\frac{1-\cos(h\pi)}{\sin(h\pi)}\right)^2,+\infty\right)\xrightarrow{z\mapsto z+\left(\frac{1-\cos(h\pi)}{\sin(h\pi)}\right)^2}\mathbb{C}\setminus[0,+\infty)$$

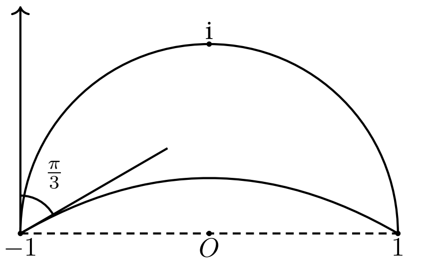
$$\xrightarrow{z\mapsto\sqrt{z}}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\xrightarrow{z\mapsto\frac{1}{\pi}\log z}\{z\in\mathbb{C}:0<\operatorname{Im}z<1\}$$

复合结果为

$$w=\frac{1}{2\pi}\log\left[\left(\frac{\mathrm{e}^{\pi(z+a)}-1}{\mathrm{e}^{\pi(z+a)}+1}\right)^2+\left(\frac{1-\cos(h\pi)}{\sin(h\pi)}\right)^2\right].$$

□

习题 2.5.18 求一单叶全纯映射，把图示的月牙形域映为 $\mathbb{B}(0,1)$.



题 2.5.18 图

解答 记图示月牙形域为 D, 则有

$$D\xrightarrow{z\mapsto\frac{z+1}{z-1}}\left\{z\in\mathbb{C}:\frac{7\pi}{6}<\arg z<\frac{3\pi}{2}\right\}\xrightarrow{z\mapsto\log z}\left\{z\in\mathbb{C}:\frac{7\pi}{6}<\operatorname{Im}z<\frac{3\pi}{2}\right\}$$

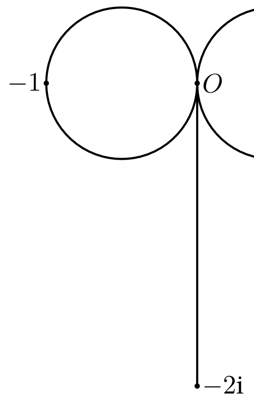
$$\xrightarrow{z\mapsto 6z-7\pi}\{z\in\mathbb{C}:0<\operatorname{Im}z<2\pi\}\xrightarrow{z\mapsto\mathrm{e}^z}\mathbb{C}\setminus[0,+\infty)\xrightarrow{z\mapsto\sqrt{z}}\{z\in\mathbb{C}:\operatorname{Im}z>0\}\xrightarrow{z\mapsto\frac{z-\mathrm{i}}{z+\mathrm{i}}}\mathbb{B}(0,1)$$

第一个箭头 $z\mapsto\dfrac{z+1}{z-1}$ 将两圆弧映为共起点的两射线，注意到当 z 在 $\mathbb{R}$ 上由 -1 到 1 时，$w=1+\dfrac{2}{z-1}$ 在 $\mathbb{R}$ 上由 0 到 $-\infty$, 因此由保角性可确定负半实轴到两射线的角度分别为 $\dfrac{\pi}{6}$ 和 $\dfrac{\pi}{2}$. 复合结果为

$$w=\frac{\sqrt{\mathrm{e}^{6\log\frac{z+1}{z-1}-7\pi}}-\mathrm{i}}{\sqrt{\mathrm{e}^{6\log\frac{z+1}{z-1}-7\pi}}+\mathrm{i}}=\frac{(z+1)^3-\mathrm{i}\mathrm{e}^{\frac{7\pi}{2}}(z-1)^3}{(z+1)^3+\mathrm{i}\mathrm{e}^{\frac{7\pi}{2}}(z-1)^3}.$$

□

习题 2.5.20 求一单叶全纯映射, 把图示 $\mathbb{B}\left(-\frac{1}{2},\frac{1}{2}\right)$ 和 $\mathbb{B}\left(\frac{1}{2},\frac{1}{2}\right)$ 的外部除去线段 $[-2\mathrm{i},0]$ 所成的域映为上半平面.



题 2.5.20 图

解答 记图示区域为 D, 则有

$$D \xrightarrow{z\mapsto \frac{1}{z}} \{z\in\mathbb{C}: -1<\operatorname{Re} z<1\}\setminus\left\{z\in\mathbb{C}: \operatorname{Re} z=0 \text{ 且 } \operatorname{Im} z\geqslant \tfrac{1}{2}\right\}$$

$$\xrightarrow{z\mapsto \pi\mathrm{i}z+\frac{\pi}{2}+\pi\mathrm{i}} \{z\in\mathbb{C}: 0<\operatorname{Im} z<2\pi\}\setminus\{z\in\mathbb{C}: \operatorname{Im} z=\pi \text{ 且 } \operatorname{Re} z\leqslant 0\}$$

$$\xrightarrow{z\mapsto \mathrm{e}^z} \mathbb{C}\setminus[-1,+\infty) \xrightarrow{z\mapsto z+1} \mathbb{C}\setminus[0,+\infty) \xrightarrow{z\mapsto\sqrt{z}} \{z\in\mathbb{C}: \operatorname{Im} z>0\}$$

复合结果为

$$w=\sqrt{\mathrm{e}^{\frac{\pi\mathrm{i}}{z}+\frac{\pi}{2}+\pi\mathrm{i}}+1}=\sqrt{1-\mathrm{e}^{\frac{\pi\mathrm{i}}{z}+\frac{\pi}{2}}}.$$

□

习题 2.5.21 设 $0<r<a$, 求一单叶全纯映射, 把域 $\{z\in\mathbb{C}: \operatorname{Re} z>0, |z-a|>r\}$ 映为同心圆环 $\{w\in\mathbb{C}: \rho<|w|<1\}$.

解答 虚轴与圆周 $|z-a|=r$ 的公共对称点显然在实轴上, 设其为 $\pm x$ $(0<x<a)$, 则 $(a-x)(a+x)=r^2$, 解得 $x=\sqrt{a^2-r^2}$. 因此分式线性变换

$$w=k\cdot\frac{z+\sqrt{a^2-r^2}}{z-\sqrt{a^2-r^2}},\quad k\in\mathbb{C}$$

将所给域映为同心于原点的圆环. 此时 $0\mapsto -k$, $a-r\mapsto -k\cdot\dfrac{a+\sqrt{a^2-r^2}}{r}$. 取

$$k=\frac{r}{a+\sqrt{a^2-r^2}}=\frac{a-\sqrt{a^2-r^2}}{r},$$

则 w 将所给域映为同心圆环 $\{w\in\mathbb{C}: \rho<|w|<1\}$, 其中 $\rho=k$. □

习题 3.1.2 计算积分 $\displaystyle\int_{|z|=1}\frac{\mathrm{d}z}{z+2}$, 并证明 $\displaystyle\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta=0$.

解答 由于 $\frac{1}{z+2}$ 在 $\mathbb{B}(0,1)$ 上全纯, 在 $\overline{\mathbb{B}(0,1)}$ 上连续, $\int\limits_{|z|=1} \frac{\mathrm{d}z}{z+2} = 0$. 另一方面, 由

$$0 = \int\limits_{|z|=1} \frac{\mathrm{d}z}{z+2} = \int_0^{2\pi} \frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}+2} = \mathrm{i}\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}+2}\,\mathrm{d}\theta$$

可得

$$\begin{aligned} 0 &= \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}+2}\,\mathrm{d}\theta = \int_0^{2\pi} \frac{(\cos\theta + \mathrm{i}\sin\theta)(2+\cos\theta - \mathrm{i}\sin\theta)}{(2+\cos\theta+\mathrm{i}\sin\theta)(2+\cos\theta-\mathrm{i}\sin\theta)}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{2\cos\theta + 1 + 2\mathrm{i}\sin\theta}{5+4\cos\theta}\,\mathrm{d}\theta = \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta + 2\mathrm{i}\int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta}\,\mathrm{d}\theta, \end{aligned}$$

而

$$\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta = 2\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}, \quad \int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta}\,\mathrm{d}\theta = 0,$$

因此

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,\mathrm{d}\theta = 0.$$

□

习题 3.1.4 如果多项式 $Q(z)$ 比多项式 $P(z)$ 高两次, 试证:

$$\lim_{R\to\infty} \int\limits_{|z|=R} \frac{P(z)}{Q(z)}\,\mathrm{d}z = 0.$$

证明 设 $\lim\limits_{|z|\to\infty}\left|\frac{z^2P(z)}{Q(z)}\right| = M$, 则存在 $R_0 > 0$, 使得当 $R > R_0$ 时, $\left|\frac{z^2P(z)}{Q(z)}\right| \leqslant 2M$, 此时

$$\left|\int\limits_{|z|=R} \frac{P(z)}{Q(z)}\,\mathrm{d}z\right| \leqslant \int\limits_{|z|=R} \left|\frac{P(z)}{Q(z)}\right||\mathrm{d}z| \leqslant \int\limits_{|z|=R} \frac{2M}{|z|^2}|\mathrm{d}z| = \frac{4\pi M}{R} \to 0, \quad R\to\infty.$$

□

习题 3.1.5 计算积分 $\int\limits_{|z|=r} z^n\bar{z}^k\,\mathrm{d}z$, 其中 $n, k \in \mathbb{Z}$.

解答 $\int\limits_{|z|=r} z^n\bar{z}^k\,\mathrm{d}z = \int_0^{2\pi} \left(r\mathrm{e}^{\mathrm{i}\theta}\right)^n \left(r\mathrm{e}^{-\mathrm{i}\theta}\right)^k \mathrm{d}r\mathrm{e}^{\mathrm{i}\theta} = \mathrm{i}r^{n+k+1}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-k+1)\theta}\,\mathrm{d}\theta = \begin{cases} 0, & n+1\neq k, \\ 2\pi\mathrm{i}r^{n+k+1}, & n+1 = k. \end{cases}$ □

习题 3.2.1 计算积分:

(1) $\int\limits_{|z|=r} \frac{|\mathrm{d}z|}{|z-a|^2}, |a| \neq r.$

(2) $\int\limits_{|z|=2} \frac{2z-1}{z(z-1)}\,\mathrm{d}z.$

(3) $\int\limits_{|z|=5} \frac{z\,\mathrm{d}z}{z^4-1}.$

(4) $\displaystyle\int\limits_{|z|=2a} \frac{\mathrm{e}^z}{z^2+a^2}\,\mathrm{d}z, a>0.$

解答 (1) $\displaystyle\int\limits_{|z|=r} \frac{|\mathrm{d}z|}{|z-a|^2} = \int_0^{2\pi} \frac{r\,\mathrm{d}\theta}{|r\mathrm{e}^{\mathrm{i}\theta}-a|^2} = \int_0^{2\pi} \frac{r\,\mathrm{d}\theta}{(r\mathrm{e}^{\mathrm{i}\theta}-a)(r\mathrm{e}^{-\mathrm{i}\theta}-\bar{a})} = \int_0^{2\pi} \frac{r^2\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{(r\mathrm{e}^{\mathrm{i}\theta}-a)(r^2-\bar{a}r\mathrm{e}^{\mathrm{i}\theta})} =$ $\displaystyle\frac{r}{\mathrm{i}}\int\limits_{|z|=r} \frac{\mathrm{d}z}{(z-a)(r^2-\bar{a}z)} = \frac{r}{\mathrm{i}(r^2-|a|^2)}\int\limits_{|z|=r}\left(\frac{1}{z-a}+\frac{1}{\frac{r^2}{\bar{a}}-z}\right)\mathrm{d}z.$

◇ 若 $|a|<r$, 则 $\displaystyle\int\limits_{|z|=r}\frac{\mathrm{d}z}{z-a} = 2\pi\mathrm{i}, \int\limits_{|z|=r}\frac{\mathrm{d}z}{\frac{r^2}{\bar{a}}-z} = 0.$

◇ 若 $|a|>r$, 则 $\displaystyle\int\limits_{|z|=r}\frac{\mathrm{d}z}{z-a} = 0, \int\limits_{|z|=r}\frac{\mathrm{d}z}{\frac{r^2}{\bar{a}}-z} = -2\pi\mathrm{i}.$

故 $\displaystyle\int\limits_{|z|=r}\frac{|\mathrm{d}z|}{|z-a|^2} = \frac{2\pi r}{|r^2-|a|^2|}.$

(2) $\displaystyle\int\limits_{|z|=2}\frac{2z-1}{z(z-1)}\,\mathrm{d}z = \int\limits_{|z|=2}\left(\frac{1}{z}+\frac{1}{z-1}\right)\mathrm{d}z = 4\pi\mathrm{i}.$

(3) $\displaystyle\int\limits_{|z|=5}\frac{z\,\mathrm{d}z}{z^4-1} = \frac{1}{2}\int\limits_{|z|=5}\frac{\mathrm{d}z^2}{(z^2-1)(z^2+1)} = \frac{1}{4}\int\limits_{|z|=5}\left(\frac{1}{z^2-1}-\frac{1}{z^2+1}\right)\mathrm{d}z^2 = 0.$

(4) 由 Cauchy 积分公式,

$$\int\limits_{|z|=2a}\frac{\mathrm{e}^z}{z^2+a^2}\,\mathrm{d}z = \frac{1}{2a\mathrm{i}}\int\limits_{|z|=2a}\left(\frac{\mathrm{e}^z}{z-a\mathrm{i}}-\frac{\mathrm{e}^z}{z+a\mathrm{i}}\right)\mathrm{d}z = \frac{1}{2a\mathrm{i}}\left(2\pi\mathrm{i}\mathrm{e}^{a\mathrm{i}}-2\pi\mathrm{i}\mathrm{e}^{-a\mathrm{i}}\right) = \frac{2\pi\mathrm{i}\sin a}{a}.$$

□

习题 3.2.2 设 f 在 $\{z: r<|z|<\infty\}$ 中全纯, 且 $\lim\limits_{z\to\infty} zf(z) = A$. 证明:

$$\int\limits_{|z|=R} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}A,$$

其中 $R>r$.

证明 对于 $R'>R$, 有

$$\left|\int\limits_{|z|=R} f(z)\,\mathrm{d}z - 2\pi\mathrm{i}A\right| = \left|\int\limits_{|z|=R'}\left(f(z)\,\mathrm{d}z-\frac{A}{z}\right)\mathrm{d}z\right| \leqslant \int\limits_{|z|=R'}\frac{|zf(z)-A|}{R'}|\mathrm{d}z|$$
$$\leqslant 2\pi\cdot\sup_{|z|=R'}|zf(z)-A| \to 0, \quad R'\to\infty.$$

□

习题 3.4.1 计算下列积分:

(1) $\displaystyle\int\limits_{|z-1|=1}\frac{\sin z}{z^2-1}\,\mathrm{d}z.$

(2) $\displaystyle\int\limits_{|z|=2} \frac{\mathrm{d}z}{1+z^2}$.

(3) $\displaystyle\int\limits_{4x^2+y^2=2y} \frac{\mathrm{e}^{\pi z}}{(1+z^2)^2}\,\mathrm{d}z$.

(4) $\displaystyle\int\limits_{|z|=\frac{3}{2}} \frac{\mathrm{d}z}{(z^2+1)(z^2+4)}$.

(5) $\displaystyle\int\limits_{|z|=2} \frac{\mathrm{d}z}{z^3(z-1)^3(z-3)^5}$.

(6) $\displaystyle\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)}$, 其中 n 为正整数, a,b 不在圆周 $|z|=R$ 上.

解答 (1) $\displaystyle\int\limits_{|z-1|=1} \frac{\sin z}{z^2-1}\,\mathrm{d}z = \int\limits_{|z-1|=1} \frac{\frac{\sin z}{z+1}}{z-1}\,\mathrm{d}z = 2\pi\mathrm{i}\cdot\frac{\sin z}{z+1}\bigg|_{z=1} = \pi\mathrm{i}\sin 1.$

(2) 记 $\varepsilon=\dfrac{1}{2}, \gamma_1=\{z:|z-\mathrm{i}|=\varepsilon\}, \gamma_2=\{z:|z+\mathrm{i}|=\varepsilon\}$, 则

$$\int\limits_{|z|=2} \frac{\mathrm{d}z}{1+z^2} = \int\limits_{\gamma_1} \frac{\frac{\mathrm{d}z}{z+\mathrm{i}}}{z-\mathrm{i}} + \int\limits_{\gamma_2} \frac{\frac{\mathrm{d}z}{z-\mathrm{i}}}{z-(-\mathrm{i})} = 2\pi\mathrm{i}\left(\frac{1}{\mathrm{i}+\mathrm{i}}+\frac{1}{-\mathrm{i}-\mathrm{i}}\right) = 0.$$

(3) 记 $E=\{(x,y):4x^2+y^2=2y\}=\{(x,y):4x^2+(y-1)^2=1\}$, 则

$$\int\limits_{4x^2+y^2=2y} \frac{\mathrm{e}^{\pi z}}{(1+z^2)^2}\,\mathrm{d}z = \int\limits_{E} \frac{\frac{\mathrm{e}^{\pi z}}{(z+\mathrm{i})^2}\,\mathrm{d}z}{(z-\mathrm{i})^2} = \frac{2\pi\mathrm{i}}{1!}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{e}^{\pi z}}{(z+\mathrm{i})^2}\right)\bigg|_{z=\mathrm{i}} = 2\pi\mathrm{i}\cdot\frac{\mathrm{e}^{\pi z}(\pi z+\pi\mathrm{i}-2)}{(z+\mathrm{i})^3}\bigg|_{z=\mathrm{i}}$$
$$= \frac{\pi(\pi\mathrm{i}-1)}{2}.$$

(4) 记 $\varepsilon=\dfrac{1}{4}, \gamma_1=\{z:|z-\mathrm{i}|=\varepsilon\}, \gamma_2=\{z:|z+\mathrm{i}|=\varepsilon\}$, 则

$$\int\limits_{|z|=\frac{3}{2}} \frac{\mathrm{d}z}{(z^2+1)(z^2+4)} = \int\limits_{\gamma_1} \frac{\frac{\mathrm{d}z}{(z+\mathrm{i})(z^2+4)}}{z-\mathrm{i}} + \int\limits_{\gamma_2} \frac{\frac{\mathrm{d}z}{(z-\mathrm{i})(z^2+4)}}{z-(-\mathrm{i})} = 2\pi\mathrm{i}\left(\frac{1}{6\mathrm{i}}+\frac{1}{-6\mathrm{i}}\right) = 0.$$

(5) 记 $\varepsilon=\dfrac{1}{4}, \gamma_1=\{z:|z|=\varepsilon\}, \gamma_2=\{z:|z-1|=\varepsilon\}$, 则

$$\int\limits_{|z|=2} \frac{\mathrm{d}z}{z^3(z-1)^3(z-3)^5} = \int\limits_{\gamma_1} \frac{\frac{\mathrm{d}z}{(z-1)^3(z-3)^5}}{(z-0)^3} + \int\limits_{\gamma_2} \frac{\frac{\mathrm{d}z}{z^3(z-3)^5}}{(z-1)^3}$$
$$= \frac{2\pi\mathrm{i}}{2!}\cdot\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{(z-1)^3(z-3)^5}\right)\bigg|_{z=0} + \frac{2\pi\mathrm{i}}{2!}\cdot\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{z^3(z-3)^5}\right)\bigg|_{z=1}$$
$$= \pi\mathrm{i}\left(\frac{76}{3^6}-\frac{9}{2^6}\right).$$

(6) ① 若 a, b 均在圆周 $|z| = R$ 外, 则由 Cauchy 定理, $\displaystyle\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)} = 0.$

② 若 a 在圆周 $|z| = R$ 外, b 在圆周 $|z| = R$ 内, 记 $\varepsilon = \dfrac{R-|b|}{2}, \gamma = \{z : |z-b| = \varepsilon\}$, 则

$$\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)} = \int\limits_{\gamma} \frac{\frac{\mathrm{d}z}{(z-a)^n}}{z-b} = \frac{2\pi\mathrm{i}}{(b-a)^n}.$$

③ 若 a 在圆周 $|z| = R$ 内, b 在圆周 $|z| = R$ 外, 记 $\varepsilon = \dfrac{R-|a|}{2}, \gamma = \{z : |z-a| = \varepsilon\}$, 则

$$\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)} = \int\limits_{\gamma} \frac{\frac{\mathrm{d}z}{z-b}}{(z-a)^n} = \frac{2\pi\mathrm{i}}{(n-1)!} \cdot \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left(\frac{1}{z-b}\right)\Bigg|_{z=a} = -\frac{2\pi\mathrm{i}}{(b-a)^n}.$$

④ 若 a, b 均在圆周 $|z| = R$ 内, 记 $\gamma_1 = \{z : |z-a| = \varepsilon\}, \gamma_2 = \{z : |z-b| = \varepsilon\}$, 其中 $\varepsilon < \min\{R-|a|, R-|b|\}$ 充分小以使 γ_1, γ_2 各自所围区域不交. 于是

$$\int\limits_{|z|=R} \frac{\mathrm{d}z}{(z-a)^n(z-b)} = 0 = \int\limits_{\gamma_1} \frac{\frac{\mathrm{d}z}{z-b}}{(z-a)^n} + \int\limits_{\gamma_2} \frac{\frac{\mathrm{d}z}{z-b}}{(z-a)^n} = ③ + ② = 0.$$

□

习题 3.4.4 称

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(z^2-1\right)^n$$

是 Legendre 多项式. 证明:

(1) Legendre 多项式有如下的积分表示:

$$P_n(z) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\gamma} \frac{\left(\zeta^2-1\right)^n}{2^n(\zeta-z)^{n+1}} \,\mathrm{d}\zeta,$$

其中 γ 是任意内部包含 z 的可求长简单闭曲线.

(2) 如果取

$$\gamma = \left\{\zeta \in \mathbb{C} : |\zeta - x| = \sqrt{x^2-1}\right\} \quad (1 < x < +\infty),$$

那么有如下的 Laplace 公式:

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left(x + \sqrt{x^2-1}\cos\theta\right)^n \mathrm{d}\theta.$$

证明 (1) 由 Cauchy 积分公式,

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(z^2-1\right)^n = \frac{n!}{2\pi\mathrm{i}} \int\limits_{\gamma} \frac{\left(\zeta^2-1\right)^n}{(\zeta-z)^{n+1}} \,\mathrm{d}\zeta,$$

整理即得欲证积分表示.

(2) 由 (1) 所得积分表示,

$$P_n(x)=\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}\frac{\left(\zeta^2-1\right)^n}{2^n(\zeta-x)^{n+1}}\,\mathrm{d}\zeta=\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\left(x+\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}\right)^2-1}{2\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}}\right]^n\mathrm{d}\theta.$$

而

$$\int_\pi^{2\pi}\left[\frac{\left(x+\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}\right)^2-1}{2\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}}\right]^n\mathrm{d}\theta\xlongequal{\beta=2\pi-\theta}\int_0^{\pi}\left[\frac{\left(x+\sqrt{x^2-1}\mathrm{e}^{-\mathrm{i}\beta}\right)^2-1}{2\sqrt{x^2-1}\mathrm{e}^{-\mathrm{i}\beta}}\right]^n\mathrm{d}\beta,$$

因此

$$\begin{aligned}P_n(x)&=\frac{1}{\pi}\int_0^{\pi}\mathrm{Re}\left\{\left[\frac{\left(x+\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}\right)^2-1}{2\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}}\right]^n\right\}\mathrm{d}\theta\\&=\frac{1}{\pi}\int_0^{\pi}\mathrm{Re}\left\{\left[\frac{\left(x^2-1\right)+\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}\left(\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}+2x\right)}{2\sqrt{x^2-1}\mathrm{e}^{\mathrm{i}\theta}}\right]^n\right\}\mathrm{d}\theta\\&=\frac{1}{\pi}\int_0^{\pi}\mathrm{Re}\left\{\left[\frac{\sqrt{x^2-1}\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)+2x}{2}\right]^n\right\}\mathrm{d}\theta\\&=\frac{1}{\pi}\int_0^{\pi}\left(x+\sqrt{x^2-1}\cos\theta\right)^n\mathrm{d}\theta.\end{aligned}$$

□

习题 3.4.5 设 $f\in\mathcal{H}(\mathbb{B}(0,1))\cap\mathcal{C}\left(\overline{\mathbb{B}(0,1)}\right)$. 证明:

(1) $\displaystyle\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\cos^2\left(\tfrac{\theta}{2}\right)\mathrm{d}\theta=2f(0)+f'(0)$.

(2) $\displaystyle\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\sin^2\left(\tfrac{\theta}{2}\right)\mathrm{d}\theta=2f(0)-f'(0)$.

证明 由 Cauchy 积分公式,

$$f(0)=\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=1}\frac{f(z)}{z}\,\mathrm{d}z=\frac{1}{2\pi}\int_0^{2\pi}\frac{f\left(\mathrm{e}^{\mathrm{i}\theta}\right)}{\mathrm{e}^{\mathrm{i}\theta}}\cdot\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=\frac{1}{2\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta,$$

$$f'(0)=\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=1}\frac{f(z)}{z^2}\,\mathrm{d}z=\frac{1}{2\pi}\int_0^{2\pi}\frac{f\left(\mathrm{e}^{\mathrm{i}\theta}\right)}{\mathrm{e}^{2\mathrm{i}\theta}}\cdot\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=\frac{1}{2\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}\theta.$$

由 Cauchy 定理,

$$\int\limits_{|z|=1}f(z)\,\mathrm{d}z=\mathrm{i}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=0\implies\frac{1}{2\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=0.$$

因此

$$\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\cos^2\left(\tfrac{\theta}{2}\right)\mathrm{d}\theta=\frac{1}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\left(1+\frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}}{2}\right)\mathrm{d}\theta=2f(0)+f'(0),$$

进而

$$\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\sin^2\left(\tfrac{\theta}{2}\right)\mathrm{d}\theta=\frac{2}{\pi}\int_0^{2\pi}f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\left[1-\cos^2\left(\tfrac{\theta}{2}\right)\right]\mathrm{d}\theta=4f(0)-\left[2f(0)+f'(0)\right]=2f(0)-f'(0).$$

□

习题 3.4.8 (Schwarz 积分公式) 设 $f \in \mathcal{H}(\mathbb{B}(0,R)) \cap \mathcal{C}\left(\overline{\mathbb{B}(0,R)}\right)$, $f = u + \mathrm{i}v$. 证明：f 可用实部表示为

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R\mathrm{e}^{\mathrm{i}\theta}+z}{R\mathrm{e}^{\mathrm{i}\theta}-z} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{d}\theta + \mathrm{i}v(0).$$

证明 对于 $z \in \mathbb{B}(0,R)$, 由 Cauchy 积分公式,

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int_{|z|=R} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)R\mathrm{e}^{\mathrm{i}\theta}}{R\mathrm{e}^{\mathrm{i}\theta}-z}\,\mathrm{d}\theta.$$

记 z 关于圆周 $|z| = R$ 的对称点为 $z^* = \dfrac{R^2}{\bar{z}}$, 则由 Cauchy 定理,

$$\int_{|z|=R} \frac{f(\zeta)}{\zeta - z^*}\,\mathrm{d}\zeta = 0 \implies \frac{1}{2\pi}\int_0^{2\pi} \frac{f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)R\mathrm{e}^{\mathrm{i}\theta}\bar{z}}{R\mathrm{e}^{\mathrm{i}\theta}\bar{z}-R^2}\,\mathrm{d}\theta = 0.$$

将以上两式作差即得

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\left[\frac{R\mathrm{e}^{\mathrm{i}\theta}}{R\mathrm{e}^{\mathrm{i}\theta}-z} - \frac{\bar{z}}{\bar{z}-R\mathrm{e}^{-\mathrm{i}\theta}}\right]\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\frac{R^2-|z|^2}{\left|R\mathrm{e}^{\mathrm{i}\theta}-z\right|^2}\,\mathrm{d}\theta.$$

两端取实部即得

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\frac{R^2-|z|^2}{\left|R\mathrm{e}^{\mathrm{i}\theta}-z\right|^2}\,\mathrm{d}\theta.$$

注意到

$$\frac{R^2-|z|^2}{\left|R\mathrm{e}^{\mathrm{i}\theta}-z\right|^2} = \operatorname{Re}\left(\frac{R\mathrm{e}^{\mathrm{i}\theta}+z}{R\mathrm{e}^{\mathrm{i}\theta}-z}\right),$$

令

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\frac{R\mathrm{e}^{\mathrm{i}\theta}+z}{R\mathrm{e}^{\mathrm{i}\theta}-z}\,\mathrm{d}\theta,$$

则 $\operatorname{Re} g(z) = u(z)$. 由于 $g(z) \in \mathcal{H}(\mathbb{B}(0,R))$, 令 $h(z) = f(z) - g(z)$, 则 $h(z) \in \mathcal{H}(\mathbb{B}(0,R))$, 且 $\operatorname{Re} h(z) \equiv 0$. 由习题 2.2.2 即知 $h(z) \equiv C$ 为常数. 由

$$g(0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta = \operatorname{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta\right\} \xlongequal{\text{平均值公式}} \operatorname{Re} f(0) = u(0)$$

即知

$$C = f(0) - g(0) = u(0) + \mathrm{i}v(0) - u(0) = \mathrm{i}v(0).$$

故

$$f(z) = g(z) + \mathrm{i}v(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R\mathrm{e}^{\mathrm{i}\theta}+z}{R\mathrm{e}^{\mathrm{i}\theta}-z} u\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta + \mathrm{i}v(0).$$

□

习题 3.5.1 设 f 是有界整函数, z_1, z_2 是 $\mathbb{B}(0,r)$ 中任意两点. 证明：

$$\int_{|z|=r} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z = 0.$$

并由此得出 Liouville 定理.

证明 由 Cauchy 积分公式,

$$\int\limits_{|z|=r} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z = \frac{1}{z_1-z_2}\int\limits_{|z|=r}\left(\frac{f(z)}{z-z_1}-\frac{f(z)}{z-z_2}\right)\mathrm{d}z = 2\pi\mathrm{i}\cdot\frac{f(z_1)-f(z_2)}{z_1-z_2}.$$

由于 f 有界, 存在 $M>0$ 使得 $|f(z)|\leqslant M$. 又 $f\in\mathcal{H}(\mathbb{C})$, 由 Cauchy 定理与长大不等式, 对 $R>r$, 有

$$\left|\int\limits_{|z|=r} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z\right| = \left|\int\limits_{|z|=R} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z\right| \leqslant \frac{2\pi RM}{(R-|z_1|)(R-|z_2|)} \to 0,\quad R\to+\infty.$$

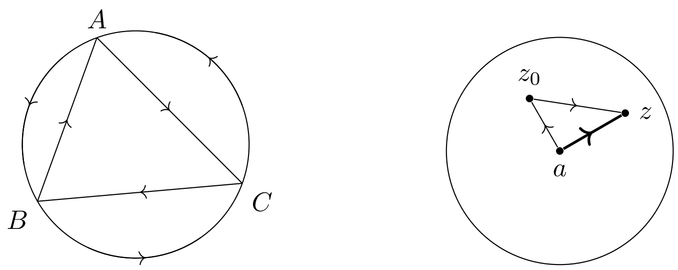
故 $\displaystyle\int\limits_{|z|=r} \frac{f(z)}{(z-z_1)(z-z_2)}\,\mathrm{d}z = 0$, 进而 $f(z_1)=f(z_2)$, 由 z_1, z_2 的任意性即证 Liouville 定理. □

习题 3.5.4 设 f 是整函数, 如果 $f(\mathbb{C})\subset\{z\in\mathbb{C}:\mathrm{Im}\,z>0\}$, 证明 f 是一个常值函数.

证明 令 $g(z)=\dfrac{f(z)-\mathrm{i}}{f(z)+\mathrm{i}}$, 由题设即得 $g(z)\in\mathcal{H}(\mathbb{C})$ 且 $|g(z)|\leqslant 1$. 根据 Liouville 定理, $g(z)$ 为常值函数, 从而 $f(z)$ 亦为常值函数. □

习题 3.5.8 设 f 是域 D 上的连续函数, 如果对于任意边界和内部都位于 D 中的弓形域 G, 总有 $\displaystyle\int\limits_{\partial G} f(z)\,\mathrm{d}z=0$, 那么 f 是 D 上的全纯函数. 如果把弓形域换成圆盘, 结论是否仍然成立?

证明 (1) 沿弓形域积分为 0 蕴含沿圆盘积分为 0, 进而沿任意外切圆在 D 中的三角形积分为 0. 而 D 中任意三角形均可被剖分为若干个外切圆在 D 中的三角形, 因此沿 D 中任意三角形积分为 0.



为证 f 在 D 上全纯, 只需证 f 在 D 中每个开球上全纯, 因此可不妨设 $D=\mathbb{B}(a,R)$. 任取 $z\in D$, 设 $F(z)=\displaystyle\int\limits_{[a,z]} f(w)\,\mathrm{d}w$. 固定 $z_0\in G$, 由沿三角形积分为 0 可得

$$F(z)=\int\limits_{[a,z_0]} f(w)\,\mathrm{d}w+\int\limits_{[z_0,z]} f(w)\,\mathrm{d}w \implies \frac{F(z)-F(z_0)}{z-z_0}=\frac{1}{z-z_0}\int\limits_{[z_0,z]} f(w)\,\mathrm{d}w.$$

因此

$$\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)=\frac{1}{z-z_0}\int\limits_{[z,z_0]}[f(w)-f(z_0)]\,\mathrm{d}w,$$

进而由长大不等式,

$$\left|\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)\right|\leqslant\max_{w\in[z,z_0]}|f(w)-f(z_0)|.$$

由于 $f \in \mathcal{C}(D)$, 对任意 $\varepsilon > 0$, 存在 $\delta > 0$, 当 $z \in \mathbb{B}(z_0, \delta) \cap D$ 时, 就有 $|f(z) - f(z_0)| < \varepsilon$. 此时

$$\max_{w \in [z, z_0]} |f(w) - f(z_0)| < \varepsilon,$$

故

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$

于是 $F(z)$ 在 D 上全纯, 从而 $f(z) = F'(z)$ 在 D 上全纯.

(2) 若把弓形域换成圆盘, 结论仍成立.

① 先考虑 $f = u + \mathrm{i}v \in \mathcal{C}^1(D)$ 的情形. 对任意 $\mathbb{B}(z_0, r) \subset D$, 有

$$0 = \int_{\partial \mathbb{B}(z_0, r)} f(z) \,\mathrm{d}z = \int_{\partial \mathbb{B}(z_0, r)} (u \,\mathrm{d}x - v \,\mathrm{d}y) + \mathrm{i} \int_{\partial \mathbb{B}(z_0, r)} (u \,\mathrm{d}y + v \,\mathrm{d}x).$$

由 Green 公式可得

$$0 = -\int_{\partial \mathbb{B}(z_0, r)} (u \,\mathrm{d}x - v \,\mathrm{d}y) = \iint_{\mathbb{B}(z_0, r)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \mathrm{d}x \,\mathrm{d}y,$$

$$0 = \int_{\partial \mathbb{B}(z_0, r)} (u \,\mathrm{d}y + v \,\mathrm{d}x) = \iint_{\mathbb{B}(z_0, r)} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \mathrm{d}x \,\mathrm{d}y.$$

将以上两式两边同除以 πr^2, 并令 $r \to 0^+$, 由 $u, v \in \mathcal{C}^1(D)$ 即得

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0,$$

这是 Cauchy-Riemann 方程, 故 f 在 D 上全纯.

② 现考虑一般的 $f \in \mathcal{C}(D)$. 设 $\phi(z)$ 为 $\mathbb{C}$ 上的实值函数, 且满足

- ◇ $\phi(z) \geqslant 0$.
- ◇ $\iint_{\mathbb{C}} \phi(z) \,\mathrm{d}x \,\mathrm{d}y = 1$.
- ◇ $\phi \in \mathcal{C}^1(\mathbb{C})$.
- ◇ $\operatorname{supp}(\phi) \subset \overline{\mathbb{B}(0, 1)}$.

对 $\varepsilon > 0$, 定义 $\phi_\varepsilon(z) = \dfrac{\phi\left(\frac{z}{\varepsilon}\right)}{\varepsilon^2}$, 则 $\phi_\varepsilon(z)$ 同样满足上述前三点性质, 且 $\operatorname{supp}(\phi_\varepsilon) \subset \overline{\mathbb{B}(0, \varepsilon)}$. 设

$$f_\varepsilon(z) = \iint_{\mathbb{C}} f(z - \zeta) \phi_\varepsilon(\zeta) \,\mathrm{d}\xi \,\mathrm{d}\eta, \quad \zeta = \xi + \mathrm{i}\eta,$$

则当 $\varepsilon \to 0^+$ 时, $f_\varepsilon(z)$ 局部一致收敛到 $f(z)$, 且对任意 $\mathbb{B}(z_0, r) \subset D$, 有

$$\begin{aligned}
\int_{\partial \mathbb{B}(z_0, r)} f_\varepsilon(z) \,\mathrm{d}z &= \int_{\partial \mathbb{B}(z_0, r)} \iint_{\mathbb{C}} f(z - \zeta) \phi_\varepsilon(\zeta) \,\mathrm{d}\xi \,\mathrm{d}\eta \,\mathrm{d}z \\
&= \iint_{\mathbb{C}} \left\{ \int_{\partial \mathbb{B}(z_0, r)} f(z - \zeta) \,\mathrm{d}z \right\} \phi_\varepsilon(\zeta) \,\mathrm{d}\xi \,\mathrm{d}\eta
\end{aligned}$$

$$
\begin{aligned}
&= \iint_{\mathbb{C}} \left\{ \int\limits_{\partial\mathbb{B}(z_0-\zeta,r)} f(z)\,\mathrm{d}z \right\} \phi_\varepsilon(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= 0.
\end{aligned}
$$

由①即知 $f_\varepsilon(z) \in \mathcal{H}(D)$. 由于 $f(z)$ 是 $f_\varepsilon(z)$ 的局部一致极限, $f(z) \in \mathcal{H}(D)$. □

习题 4.2.2 求下列幂级数的收敛半径:

(3) $\displaystyle\sum_{n=0}^{\infty}[3+(-1)^n]^n z^n$.

(4) $\displaystyle\sum_{n=0}^{\infty}\frac{n^n}{n!}z^n$.

解答 (3) $\displaystyle\limsup_{n\to\infty}\sqrt[n]{[3+(-1)^n]^n} = \lim_{n\to\infty}\sqrt[n]{4^n} = 4 \implies$ 收敛半径 $R = \dfrac{1}{4}$.

(4) $\displaystyle\limsup_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} = \lim_{n\to\infty}\frac{n}{(2\pi n)^{\frac{1}{2n}}\frac{n}{\mathrm{e}}} = \mathrm{e} \implies$ 收敛半径 $R = \dfrac{1}{\mathrm{e}}$. □

习题 4.2.4 设正数列 $\{a_n\}$ 单调收敛于 0. 证明:

(1) $\displaystyle\sum_{n=0}^{\infty}a_n z^n$ 的收敛半径 $R \geqslant 1$.

(2) $\displaystyle\sum_{n=0}^{\infty}a_n z^n$ 在 $\partial\mathbb{B}(0,1)\setminus\{1\}$ 上处处收敛.

证明 (1) 由于 $a_n \downarrow 0$, 存在正整数 N, 当 $n > N$ 时, $a_n < 1$, 从而 $\sqrt[n]{a_n} < 1$, 因此 $\displaystyle\limsup_{n\to\infty}\sqrt[n]{a_n} \leqslant 1$, 收敛半径 $R \geqslant 1$.

(2) 当 $z \in \partial\mathbb{B}(0,1)\setminus\{1\}$ 时, $\displaystyle\left|\sum_{k=0}^{n}z^k\right| = \left|\frac{1-z^{n+1}}{1-z}\right| \leqslant \frac{2}{|1-z|}$, 而 $a_n \downarrow 0$, 由 Dirichlet 判别法得证. □

习题 4.2.7 证明: 若 $f(z) = \displaystyle\sum_{n=0}^{\infty}a_n z^n$ 是 $\mathbb{B}(0,1)$ 上的有界全纯函数, 则 $\displaystyle\sum_{n=0}^{\infty}|a_n|^2 < +\infty$.

证明 设 $|f(z)| \leqslant M, \forall z \in \mathbb{B}(0,1)$. 对 $r \in (0,1)$, 有

$$
\begin{aligned}
2\pi M^2 &\geqslant \int_0^{2\pi}\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2\mathrm{d}\theta = \int_0^{2\pi}\sum_{n=0}^{\infty}a_n r^n \mathrm{e}^{\mathrm{i}n\theta}\sum_{m=0}^{\infty}\overline{a_m}r^n\mathrm{e}^{-\mathrm{i}m\theta}\,\mathrm{d}\theta \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n\overline{a_m}r^{n+m}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta = 2\pi\sum_{n=0}^{\infty}|a_n|^2 r^{2n},
\end{aligned}
$$

因此

$$
\sum_{n=0}^{\infty}|a_n|^2 r^{2n} \leqslant M^2 \implies \sum_{n=0}^{m}|a_n|^2 r^{2n} \leqslant M^2, \quad \forall m \geqslant n \implies \sum_{n=0}^{m}|a_n|^2 R^{2n} \leqslant M^2, \quad \forall m \geqslant n.
$$

故

$$
\sum_{n=0}^{\infty}|a_n|^2 < +\infty.
$$

□

习题 4.3.1 设 D 是域, $a\in D$, 函数 $f\in\mathcal{H}(D\setminus\{a\})$. 证明: 若 $\lim\limits_{z\to a}(z-a)f(z)=0$, 则 $f\in\mathcal{H}(D)$.

证明 设 $\varphi(z)=\begin{cases}(z-a)f(z), & z\in D\setminus\{a\},\\ 0, & z=a.\end{cases}$ 则 $\varphi\in\mathcal{C}(D)\cap\mathcal{H}(D\setminus\{a\})$. 任取 D 中可求长简单闭曲线 γ, 且 γ 所围区域在 D 中, 则不论 a 与 γ 的位置关系, 均有 $\displaystyle\int_\gamma \varphi(z)\,\mathrm{d}z=0$ (当 a 在 γ 所围区域中时, 可添加过 a 的曲线). 由 Morera 定理得 $\varphi\in\mathcal{H}(D)$. 于是, 当补充定义 $f(a)=\varphi'(a)$ 后便有

$$\lim_{z\to a}f(z)=\lim_{z\to a}\frac{\varphi(z)-\varphi(a)}{z-a}=\varphi'(a)=f(a),$$

因此 $f\in\mathcal{C}(D)\cap\mathcal{H}(D\setminus\{a\})$, 同前可得 $f\in\mathcal{H}(D)$. □

习题 4.3.5 是否存在 $f\in\mathcal{H}(\mathbb{B}(0,1))$, 使得下述条件之一成立:

(2) $f\left(\frac{1}{2n}\right)=0, f\left(\frac{1}{2n-1}\right)=1, n=1,2,3,\cdots$.

(3) $f\left(\frac{1}{n}\right)=f\left(-\frac{1}{n}\right)=\frac{1}{n^2}, n=2,3,4\cdots$.

解答 (2) 不存在. 令 $n\to\infty$, 由 f 在 $z=0$ 处连续即得矛盾.

(3) 不存在. 因为由唯一性定理, $f\left(\frac{1}{n}\right)=\frac{1}{n^2}$ 要求 $f(z)=z^2$, 但这与 $f\left(-\frac{1}{n}\right)=\frac{1}{n^2}$ 矛盾. □

习题 4.3.6 设 $f(z)=\displaystyle\sum_{n=0}^{\infty}a_nz^n$ 的收敛半径 $R>0$, $0<r<R$, $A(r)=\max\limits_{|z|=r}\operatorname{Re}f(z)$. 证明:

(1) $a_nr^n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\left[\operatorname{Re}f\left(re^{\mathrm{i}\theta}\right)\right]\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta, \forall n\in\mathbb{N}$.

(2) $|a_n|r^n\leqslant 2A(r)-2\operatorname{Re}f(0), \forall n\in\mathbb{N}$.

证明 (1) 由 Cauchy 积分公式,

$$a_n=\frac{1}{2\pi\mathrm{i}}\sum_{m=0}^{\infty}\int_{|z|=r}\frac{a_m}{z^{m+1}}\,\mathrm{d}z=\frac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\frac{f\left(re^{\mathrm{i}\theta}\right)}{r^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}}r\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=\frac{1}{2\pi r^n}\int_0^{2\pi}f\left(re^{\mathrm{i}\theta}\right)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta.$$

而

$$0=\int_{|z|=r}f(z)z^{n-1}\,\mathrm{d}z=\mathrm{i}r^n\int_0^{2\pi}f\left(re^{\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\theta \implies \frac{1}{2\pi r^n}\int_0^{2\pi}\overline{f\left(re^{\mathrm{i}\theta}\right)}\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta=0,$$

因此

$$a_n=\frac{1}{\pi r^n}\int_0^{2\pi}\left[\operatorname{Re}f\left(re^{\mathrm{i}\theta}\right)\right]\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta.$$

(2) 利用 $\displaystyle\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta=0$ 可得

$$\begin{aligned}|a_n|r^n&=\frac{1}{\pi}\left|\int_0^{2\pi}\left[\operatorname{Re}f\left(re^{\mathrm{i}\theta}\right)-A(r)\right]\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta\right|\\&\leqslant\frac{1}{\pi}\int_0^{2\pi}\left|\operatorname{Re}f\left(re^{\mathrm{i}\theta}\right)-A(r)\right|\mathrm{d}\theta\\&=\frac{1}{\pi}\int_0^{2\pi}\left[A(r)-\operatorname{Re}f\left(re^{\mathrm{i}\theta}\right)\right]\mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&= 2A(r) - \frac{1}{\pi}\operatorname{Re}\left\{\int_0^{2\pi} f\big(re^{i\theta}\big)\,d\theta\right\} \\
&= 2A(r) - 2\operatorname{Re} f(0),
\end{aligned}
$$

其中最后一个等式用到了平均值公式. □

习题 4.3.14 设 D 是域, $a \in D$, $f \in \mathcal{H}(D)$, 并且 $\sum\limits_{n=0}^{\infty} f^{(n)}(a)$ 收敛. 证明:

(1) f 是整函数.

(2) $\sum\limits_{n=0}^{\infty} f^{(n)}(z)$ 在 $\mathbb{C}$ 上内闭一致收敛.

证明 (1) 由于 $f \in \mathcal{H}(D)$, 存在 $\varepsilon > 0$, 使得在 $\mathbb{B}(a,\varepsilon)$ 上有展开式

$$
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.
$$

由 $\sum\limits_{n=0}^{\infty} f^{(n)}(a)$ 收敛可知 $\lim\limits_{n\to\infty}\left|f^{(n)}(a)\right| = 0$, 因此

$$
\limsup_{n\to\infty} \sqrt[n]{\left|\frac{f^{(n)}(a)}{n!}\right|} = 0 \implies \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \text{ 的收敛半径为 } +\infty.
$$

设 $S(z) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(z-a)^n, z \in \mathbb{C}$, 则 $S(z)$ 是 $f(z)$ 在 $\mathbb{C}$ 上的解析延拓 (由零点孤立性知延拓唯一). 故 f 可延拓为整函数.

(2) 由于 $\sum\limits_{n=0}^{\infty} f^{(n)}(a)$ 收敛, 对任意 $\varepsilon > 0$, 存在正整数 N, 使得

$$
\left|f^{(p+1)}(a) + \cdots + f^{(q)}(a)\right| < \varepsilon, \quad \forall q > p > N.
$$

对任意紧集 $K \subset \mathbb{C}$, 记 $M = \max\limits_{z\in K}\left\{e^{|z-a|}\right\}$, 则

$$
\begin{aligned}
\left|\sum_{k=p+1}^{q} S^{(k)}(z)\right| &= \left|\sum_{k=p+1}^{q}\sum_{n=0}^{\infty} \frac{f^{(n+k)}(a)}{n!}(z-a)^n\right| = \left|\sum_{n=0}^{\infty} \frac{f^{(n+p+1)}(a) + \cdots + f^{(n+q)}(a)}{n!}(z-a)^n\right| \\
&\leqslant \varepsilon \sum_{n=0}^{\infty} \frac{|z-a|^n}{n!} = \varepsilon e^{|z-a|} \leqslant M\varepsilon.
\end{aligned}
$$

因此 $\sum\limits_{n=0}^{\infty} S^{(n)}(z)$ 在 K 上一致收敛. 再由 K 的任意性即得 $\sum\limits_{n=0}^{\infty} S^{(n)}(z)$ 在 $\mathbb{C}$ 上内闭一致收敛. □

习题 4.4.6 设 $0 < r < 1$. 证明: 当 n 充分大时, 多项式 $1 + 2z + 3z^2 + \cdots + nz^{n-1}$ 在 $\mathbb{B}(0,r)$ 中没有根.

证明 由于级数 $\sum\limits_{k=0}^{\infty}(k+1)z^k$ 的收敛半径为 1, 当 $|z| < 1$ 时, $\sum\limits_{k=0}^{\infty}(k+1)z^k = \left(\sum\limits_{k=0}^{\infty} z^{k+1}\right)' = \dfrac{1}{(1-z)^2}$. 由

于此级数在 $\mathbb{B}(0,r)$ 中内闭一致收敛, 由 Hurwitz 定理, 当 n 充分大时, 部分和 $\sum_{k=0}^{n}(k+1)z^k$ 在 $\mathbb{B}(0,r)$ 中的零点个数与 $\dfrac{1}{(1-z)^2}$ 相同, 即无零点. □

习题 4.4.7 设 $r>0$. 证明：当 n 充分大时, 多项式 $1+z+\dfrac{1}{2!}z^2+\cdots+\dfrac{1}{n!}z^n$ 在 $\mathbb{B}(0,r)$ 中没有根.

证明 由于级数 $\sum_{k=0}^{\infty}\dfrac{z^k}{k!}$ 在 $\mathbb{B}(0,r)$ 中内闭一致收敛到 e^z, 由 Hurwitz 定理, 当 n 充分大时, 部分和 $\sum_{k=0}^{n}\dfrac{z^k}{k!}$ 在 $\mathbb{B}(0,r)$ 中的零点个数与 e^z 相同, 即无零点. □

习题 4.4.8 设 $f(z)\in\mathcal{H}\left(\overline{\mathbb{B}(0,1)}\right)$, 且 $f'(z)$ 在 $\partial\mathbb{B}(0,1)$ 上无零点. 证明：当 n 充分大时, $F_n(z)=n\left[f\left(z+\frac{1}{n}\right)-f(z)\right]$ 与 $f'(z)$ 在 $\mathbb{B}(0,1)$ 中的零点个数相等.

证明 对任意 $0<r<1$, 由于 $f'(z)\in\mathcal{C}\left(\overline{\mathbb{B}(0,r)}\right)$, $F_n(z)$ 在 $\overline{\mathbb{B}(0,r)}$ 上一致收敛到 $f'(z)$, 即 $F_n(z)$ 在 $\mathbb{B}(0,1)$ 中内闭一致收敛到 $f'(z)$. 又 $f'(z)$ 在 $\partial\mathbb{B}(0,1)$ 上无零点, 由 Hurwitz 定理, 当 n 充分大时, $F_n(z)$ 在 $\mathbb{B}(0,1)$ 中的零点个数与 $f'(z)$ 相同. □

习题 4.4.11 求下列全纯函数在 $\mathbb{B}(0,1)$ 中的零点个数：

(1) $z^9-2z^6+z^2-8z-2$.

(2) $2z^5-z^3+3z^2-z+8$.

(3) $z^7-5z^4+z^2-2$.

(4) e^z-4z^n+1.

解答 记每问中的函数为 $f(z)$, $\gamma=\partial\mathbb{B}(0,1)$.

(1) 设 $g(z)=-8z$, 则当 $z\in\gamma$ 时, $|f(z)-g(z)|=\left|z^9-2z^6+z^2-2\right|\leqslant|z|^9+2|z|^6+|z|^2+2=6<8=|g(z)|$, 由 Rouché 定理知 f 和 g 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 1 个.

(2) 设 $g(z)=8$, 则当 $z\in\gamma$ 时, $|f(z)-g(z)|=\left|2z^5-z^3+3z^2-z\right|\leqslant 2|z|^5+|z|^3+3|z|^2+|z|=7<8=|g(z)|$, 由 Rouché 定理知 f 和 g 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 0 个.

(3) 设 $g(z)=-5z^4$, 则当 $z\in\gamma$ 时, $|f(z)-g(z)|=\left|z^7+z^2-2\right|\leqslant|z|^7+|z|^2+2=4<5=|g(z)|$, 由 Rouché 定理知 f 和 g 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 4 个.

(4) 设 $g(z)=-4z^n$, 则当 $z\in\gamma$ 时, $|f(z)-g(z)|=|\mathrm{e}^z-1|\leqslant\mathrm{e}^{|z|}+1=\mathrm{e}+1<4=|g(z)|$, 由 Rouché 定理知 f 和 g 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 n 个. □

习题 4.4.12 若 $f\in\mathcal{H}(\mathbb{B}(0,1))\cap\mathcal{C}\left(\overline{\mathbb{B}(0,1)}\right)$, $f\left(\overline{\mathbb{B}(0,1)}\right)\subset\mathbb{B}(0,1)$, 则 $f(z)$ 在 $\mathbb{B}(0,1)$ 中有唯一的不动点.

证明 令 $g(z)=f(z)-z$, $h(z)=-z$, 则当 $z\in\partial\mathbb{B}(0,1)$ 时, $|g(z)-h(z)|=|f(z)|<1=|h(z)|$, 由 Rouché 定理知 g 和 h 在 $\mathbb{B}(0,1)$ 中的零点个数相同, 为 1 个, 即 $f(z)$ 在 $\mathbb{B}(0,1)$ 中有唯一的不动点. □

习题 4.4.13 设 $a_1,a_2,\cdots,a_n\in\mathbb{B}(0,1)$, $f(z)=\prod_{k=1}^{n}\dfrac{a_k-z}{1-\overline{a_k}z}$. 证明：

(1) 若 $b\in\mathbb{B}(0,1)$, 则 $f(z)=b$ 在 $\mathbb{B}(0,1)$ 中恰有 n 个根.

(2) 若 $b\in\mathbb{B}(\infty,1)$, 则 $f(z)=b$ 在 $\mathbb{B}(\infty,1)$ 中恰有 n 个根.

证明 (1) 注意到 Blaschke 因子 $\dfrac{a-z}{1-\bar{a}z}$ ($|a|<1$) 有如下性质:

$$\left|\frac{a-z}{1-\bar{a}z}\right|<1 \iff |z|<1, \quad \left|\frac{a-z}{1-\bar{a}z}\right|=1 \iff |z|=1, \quad \left|\frac{a-z}{1-\bar{a}z}\right|>1 \iff |z|>1.$$

而 $f(z)=b \iff \prod_{k=1}^{n}(a_k-z)=b\prod_{k=1}^{n}(1-\overline{a_k}z)$ (这是 n 次方程, 因为 $|b\overline{a_1\cdots a_n}|<1$) 在 $\mathbb{C}$ 上恰有 n 个根. 此时 $|f(z)|=|b|<1$, 因此 $|z|<1$ (否则, 每项 $\left|\dfrac{a_k-z}{1-\overline{a_k}z}\right|\geqslant 1$, 进而 $|f(z)|\geqslant 1$), 即 $f(z)=b$ 在 $\mathbb{B}(0,1)$ 中恰有 n 个根.

(2) 即证 $f\left(\frac{1}{\bar{z}}\right)=b$ 在 $\mathbb{B}(0,1)$ 中恰有 n 个根, 这等价于证明 $\dfrac{1}{\overline{f\left(\frac{1}{\bar{z}}\right)}}=\dfrac{1}{\bar{b}}$ 在 $\mathbb{B}(0,1)$ 中恰有 n 个根. 而

$$\frac{1}{\overline{f\left(\frac{1}{\bar{z}}\right)}}=\prod_{k=1}^{n}\frac{1-a_k\frac{1}{z}}{\overline{a_k}-\frac{1}{z}}=\prod_{k=1}^{n}\frac{a_k-z}{1-\overline{a_k}z}=f(z),$$

而此时 $\left|\frac{1}{\bar{b}}\right|<1$, 因此由 (1) 即得证. □

习题 4.5.4 设 $f\in\mathcal{H}(\mathbb{B}(0,R))$. 证明: $M(r)=\max\limits_{|z|=r}|f(z)|$ 是 $[0,R)$ 上的增函数.

证明 不妨设 f 非常数. 由最大模原理, $M(r)=\max\limits_{|z|\leqslant r}|f(z)|$, 由此可见 $M(r)$ 为 $[0,R)$ 上的增函数. □

习题 4.5.5 利用最大模原理证明代数学基本定理.

证明 设 $P(z)\in\mathbb{C}[z]$, $\deg P=n$ ($n\geqslant 1$). 假设 $P(z)$ 在 $\mathbb{C}$ 中没有零点. 取 $R>0$ 使得当 $|z|\geqslant R$ 时有 $|P(z)|>|P(0)|$, 则 $|P(z)|$ 在闭圆盘 $\overline{\mathbb{B}(0,R)}$ 上的最小值在内部取到. 由于 $P(z)$ 在 $\mathbb{B}(0,R)$ 中无零点, 由最大模原理, $\left|\dfrac{1}{P(z)}\right|$ 在 $\mathbb{B}(0,R)$ 内取不到最大值, 即 $|P(z)|$ 在 $\mathbb{B}(0,R)$ 内取不到最小值, 矛盾. □

习题 4.5.10 设 $f\in\mathcal{H}(\mathbb{B}(0,R))$, $f(\mathbb{B}(0,R))\subset\mathbb{B}(0,M)$, $f(0)=0$. 证明:

(1) $|f(z)|\leqslant\dfrac{M}{R}|z|, |f'(0)|\leqslant\dfrac{M}{R}, \forall z\in\mathbb{B}(0,R)\setminus\{0\}$.

(2) 等号成立当且仅当 $f(z)=\dfrac{M}{R}\mathrm{e}^{\mathrm{i}\theta}z$ ($\theta\in\mathbb{R}$).

证明 考虑函数

$$g:\mathbb{B}(0,1)\to\mathbb{B}(0,1), \quad z\mapsto\frac{1}{M}f(Rz).$$

由于 $g\in\mathcal{H}(\mathbb{B}(0,1))$, $g(0)=0$, 由 Schwarz 引理可得

$$|g(z)|\leqslant|z|, \quad |g'(0)|\leqslant 1, \quad \forall z\in\mathbb{B}(0,1),$$

也即

$$|f(z)|\leqslant\frac{M}{R}|z|, \quad |f'(0)|\leqslant\frac{M}{R}, \quad \forall z\in\mathbb{B}(0,R).$$

等号成立当且仅当 $g(z)=\mathrm{e}^{\mathrm{i}\theta}z$ ($\theta\in\mathbb{R}$) 即 $f(z)=\dfrac{M}{R}\mathrm{e}^{\mathrm{i}\theta}z$ ($\theta\in\mathbb{R}$). □

习题 4.5.11 设 $f\in\mathcal{H}(\mathbb{B}(0,1))$, $f(0)=0$, 并且存在 $A>0$, 使得 $\operatorname{Re}f(z)\leqslant A, \forall z\in\mathbb{B}(0,1)$. 证明:

$$|f(z)|\leqslant\frac{2A|z|}{1-|z|}, \quad \forall z\in\mathbb{B}(0,1).$$

证明 设 $g(z)=\dfrac{z}{z-2A}$, 则 g 是从 $\{z\in\mathbb{C}:\operatorname{Re}z<A\}$ 到 $\mathbb{B}(0,1)$ 的共形变换 (分解如下), 且 $g(0)=0$.

$$\{z\in\mathbb{C}:\operatorname{Re}z<A\}\xrightarrow[0\mapsto -A]{z\mapsto z-A}\{z\in\mathbb{C}:\operatorname{Re}z<0\}\xrightarrow[-A\mapsto A\mathrm{i}]{z\mapsto -\mathrm{i}z}\mathbb{H}\xrightarrow[A\mathrm{i}\mapsto 0]{z\mapsto\frac{z-A\mathrm{i}}{z+A\mathrm{i}}}\mathbb{B}(0,1)$$

考虑 $h(z)=g\circ f(z)=\dfrac{f(z)}{f(z)-2A}$, 则 $h(0)=0$ 且 $|h(z)|\leqslant 1$, 由 Schwarz 引理可得 $|h(z)|\leqslant|z|$, 因此

$$\frac{|f(z)|}{|f(z)|+2A}\leqslant\frac{|f(z)|}{|f(z)-2A|}\leqslant|z|\implies|f(z)|\leqslant\frac{2A|z|}{1-|z|},\quad\forall z\in\mathbb{B}(0,1).$$ □

习题 4.5.12 (Carathéodory 不等式) 设 $f\in\mathcal{H}(\mathbb{B}(0,R))\cap\mathcal{C}\left(\overline{\mathbb{B}(0,R)}\right)$, $M(r)=\max\limits_{|z|=r}|f(z)|$, $A(r)=\max\limits_{|z|=r}\operatorname{Re}f(z)$ $(0\leqslant r\leqslant R)$. 证明:

$$M(r)\leqslant\frac{2r}{R-r}A(R)+\frac{R+r}{R-r}|f(0)|,\quad\forall r\in[0,R).$$

证明 设 $g(z)=f(Rz)-f(0)$, 则 $g(z)\in\mathcal{H}(\mathbb{B}(0,1))$ 且 $g(0)=0$. 对 $\mathbb{B}(0,1)$ 上的调和函数 $\operatorname{Re}g(z)$ 使用最大模原理可得

$$\max_{|z|\leqslant 1}\operatorname{Re}g(z)=\max_{|z|=1}\operatorname{Re}g(z)=A(R)-\operatorname{Re}f(0).$$

由习题 4.5.11 即得

$$|g(z)|\leqslant\frac{2[A(R)-\operatorname{Re}f(0)]\cdot|z|}{1-|z|}\leqslant\frac{2[A(R)+|f(0)|]\cdot|z|}{1-|z|},\quad\forall z\in\mathbb{B}(0,1).$$

由 $f(z)=g\left(\frac{z}{R}\right)+f(0)$ 即得

$$\begin{aligned}|f(z)|&\leqslant\left|g\left(\tfrac{z}{R}\right)\right|+|f(0)|\leqslant\frac{2[A(R)+|f(0)|]\cdot\left|\frac{z}{R}\right|}{1-\left|\frac{z}{R}\right|}+|f(0)|=\frac{2[A(R)+|f(0)|]\cdot|z|}{R-|z|}+|f(0)|\\&=\frac{2|z|}{R-|z|}A(R)+\frac{R+|z|}{R-|z|}|f(0)|,\quad\forall z\in\mathbb{B}(0,R).\end{aligned}$$

故

$$M(r)=\max_{|z|=r}|f(z)|\leqslant\frac{2r}{R-r}A(R)+\frac{R+r}{R-r}|f(0)|,\quad\forall r\in[0,R).$$ □

习题 4.5.18 设 $f\in\mathcal{H}(\mathbb{B}(0,1))$, $f(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$. 证明:

$$\frac{|f(0)|-|z|}{1-|f(0)||z|}\leqslant|f(z)|\leqslant\frac{|f(0)|+|z|}{1+|f(0)||z|}.$$

证明 记 $b=f(0)$, 对 $a\in\mathbb{B}(0,1)$, 记 $\phi_a(z)=\dfrac{z-a}{1-\bar{a}z}$, 则由 Schwarz-Pick 定理,

$$|\varphi_b(f(z))|\leqslant|\varphi_0(z)|\quad\text{即}\quad\left|\frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}\right|\leqslant|z|,\quad z\in\mathbb{B}(0,1).$$

另一方面, 由习题 1.1.6 (3),

$$\frac{||f(z)|-|f(0)||}{1-|f(0)||f(z)|} \leqslant \left|\frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}\right|.$$

由上述两个不等式即得

$$\frac{||f(z)|-|f(0)||}{1-|f(0)||f(z)|} \leqslant |z|,$$

也即

$$\begin{cases} |z|-|f(0)||f(z)||z| \geqslant |f(z)|-|f(0)|, \\ |z|-|f(0)||f(z)||z| \geqslant |f(0)|-|f(z)|. \end{cases}$$

整理即得

$$\frac{|f(0)|-|z|}{1-|f(0)||z|} \leqslant |f(z)| \leqslant \frac{|f(0)|+|z|}{1+|f(0)||z|}.$$

□

习题 4.5.19 设 $f \in \mathcal{H}(\mathbb{B}(0,1))$, $f(\mathbb{B}(0,1)) \subset \mathbb{B}(0,M)$. 证明:

$$M|f'(0)| \leqslant M^2 - |f(0)|^2.$$

证明 记 $a = \frac{f(0)}{M}$, $g(z) = \frac{a-z}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{B}(0,1))$. 考虑 $h(z) = g\left(\frac{f(z)}{M}\right)$, 则 h 是从 $\mathbb{B}(0,1)$ 到 $\mathbb{B}(0,1)$ 的共形变换, 且 $h(0)=0$. 由 Schwarz 引理, $|h'(0)| \leqslant 1$. 注意到 $g^{-1} = g$, 因此 $M \cdot g \circ h = f$,

$$|f'(0)| = M|g'(0)| \cdot |h'(0)| \leqslant M|g'(0)| = M\big||a|^2-1\big| = M\big(1-|a|^2\big) = \frac{M^2-|f(0)|^2}{M},$$

得所欲证. □

习题 4.5.20 设 $f \in \mathcal{H}(\mathbb{B}(0,1))$, $f(0)=0$, $f(\mathbb{B}(0,1)) \subset \mathbb{B}(0,1)$. 证明: 若存在 $z_1, z_2 \in \mathbb{B}(0,1)$, 使得 $z_1 \neq z_2$, $|z_1| = |z_2|$, $f(z_1) = f(z_2)$, 则

$$|f(z_1)| = |f(z_2)| \leqslant |z_1|^2 = |z_2|^2.$$

证明 令

$$F(z) = \frac{f(z_1)-f(z)}{1-\overline{f(z_1)}f(z)} \cdot \frac{1-\overline{z_1}z}{z_1-z} \cdot \frac{1-\overline{z_2}z}{z_2-z}.$$

注意到 z_1, z_2 均为 $F(z)$ 的可去奇点, 因此 $F(z) \in \mathcal{H}(\mathbb{B}(0,1))$, 由最大模原理, 及 $f(\mathbb{B}(0,1)) \subset \mathbb{B}(0,1)$, 有

$$\max_{|z|\leqslant 1}|F(z)| = \max_{|z|=1}|F(z)| = 1.$$

特别地,

$$|F(0)| = \left|\frac{f(z_1)}{z_1 z_2}\right| \leqslant 1 \implies |f(z_1)| = |f(z_2)| \leqslant |z_1 z_2| = |z_1|^2 = |z_2|^2.$$

□

习题 4.5.21 设 $f \in \mathcal{H}(\mathbb{B}(0,1))$, $f(0)=0$, $f(\mathbb{B}(0,1)) \subset \mathbb{B}(0,1)$. 证明:

$$|z|\frac{|f'(0)|-|z|}{1-|f'(0)||z|} \leqslant |f(z)| \leqslant |z|\frac{|f'(0)|+|z|}{1+|f'(0)||z|}.$$

证明 令 $g(z)=\begin{cases}\dfrac{f(z)}{z}, & 0<|z|<1,\\ f'(0), & z=0.\end{cases}$ 由 Schwarz 引理知 $|g(z)|\leqslant 1$. 对 $g(z)$ 用习题 4.5.18 结论即可. □

习题 4.5.30 设 $f\in\mathcal{H}(\mathbb{B}(0,1))$, $f(0)=0$, 并且 $|\operatorname{Re} f(z)|<1,\forall z\in\mathbb{B}(0,1)$. 证明:

(1) $|\operatorname{Re} f(z)|\leqslant \dfrac{4}{\pi}\arctan|z|,\forall z\in\mathbb{B}(0,1)$.

(2) $|\operatorname{Im} f(z)|\leqslant \dfrac{2}{\pi}\log\left(\dfrac{1+|z|}{1-|z|}\right),\forall z\in\mathbb{B}(0,1)$.

证明 先构造共形变换 $g:\{z\in\mathbb{C}:|\operatorname{Re} f(z)|<1\}\to\mathbb{B}(0,1)$ 使得 $g(0)=0$, 分解如下:

$$\{z\in\mathbb{C}:|\operatorname{Re} z|<1\}\xrightarrow[0\mapsto 0]{z\mapsto\frac{\pi\mathrm{i}}{2}z}\{z\in\mathbb{C}:|\operatorname{Im} z|<\tfrac{\pi}{2}\}\xrightarrow[0\mapsto 1]{z\mapsto \mathrm{e}^z}\{z\in\mathbb{C}:\operatorname{Re} z>0\}\xrightarrow[1\mapsto\mathrm{i}]{z\mapsto\mathrm{i}z}\{z\in\mathbb{C}:\operatorname{Im} z>0\}\xrightarrow[\mathrm{i}\mapsto 0]{z\mapsto\frac{z-\mathrm{i}}{z+\mathrm{i}}}\mathbb{B}(0,1)$$

复合结果为 $g(z)=\dfrac{\mathrm{e}^{\frac{\pi\mathrm{i}}{2}z}-1}{\mathrm{e}^{\frac{\pi\mathrm{i}}{2}z}+1}$. 考虑 $h(z)=g\circ f(z)=\dfrac{\mathrm{e}^{\frac{\pi\mathrm{i}}{2}f(z)}-1}{\mathrm{e}^{\frac{\pi\mathrm{i}}{2}f(z)}+1}$, 则 $h:\mathbb{B}(0,1)\to\mathbb{B}(0,1)$ 且 $h(0)=0$, 由 Schwarz 引理可得 $|h(z)|\leqslant|z|$. 而由 $f(0)=0$ 可解得

$$f(z)=\frac{2}{\pi\mathrm{i}}\log\frac{1+h(z)}{1-h(z)}\implies\begin{cases}\operatorname{Re} f(z)=\dfrac{2}{\pi}\arg\left(\dfrac{1+h(z)}{1-h(z)}\right),\\ \operatorname{Im} f(z)=-\dfrac{2}{\pi}\log\left|\dfrac{1+h(z)}{1-h(z)}\right|.\end{cases}$$

因此由

$$\log\left(\frac{1-|z|}{1+|z|}\right)\leqslant\log\left|\frac{1+h(z)}{1-h(z)}\right|\leqslant\log\left(\frac{1+|z|}{1-|z|}\right)$$

即得结论 (2). 由 $|\operatorname{Re} f(z)|<1$ 可得

$$\left|\arg\left(\frac{1+h(z)}{1-h(z)}\right)\right|<\frac{\pi}{2}.$$

因此

$$\frac{1+h(z)}{1-h(z)}=\frac{1+|h(z)|^2+2\mathrm{i}\operatorname{Im} h(z)}{|1-h(z)|^2}\implies\arg\left(\frac{1+h(z)}{1-h(z)}\right)=\arctan\left(\frac{2\operatorname{Im} h(z)}{1-|h(z)|^2}\right),$$

进而

$$|\operatorname{Re} f(z)|=\frac{2}{\pi}\left|\arctan\left(\frac{2\operatorname{Im} h(z)}{1-|h(z)|^2}\right)\right|\leqslant\frac{2}{\pi}\arctan\left(\frac{2|z|}{1-|z|^2}\right)\overset{\star}{=}\frac{2}{\pi}\cdot 2\arctan|z|,$$

$\star$ 处用到了正切函数的二倍角公式及 $|z|<1$ 时 $\arctan|z|\in\left(0,\frac{\pi}{4}\right)$. 故结论 (1) 得证. □

习题 4.5.31 设 $f\in\mathcal{H}(\mathbb{B}(0,1)\cup\{1\})$, $f(0)=0$, $f(1)=1$, $f(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$. 证明: $f'(1)\geqslant 1$.

证明 设 $g(z)=\begin{cases}\dfrac{f(z)}{z}, & z\neq 0,\\ f'(0), & z=0.\end{cases}$ 由习题 4.3.1 即知 $g\in\mathcal{H}(\mathbb{B}(0,1))$. 由最大模原理,

$$\max_{|z|\leqslant 1}|g(z)|=\max_{|z|=1}|g(z)|=\max_{|z|=1}\left|\frac{f(z)}{z}\right|=\max_{|z|=1}|f(z)|=\max_{|z|\leqslant 1}|f(z)|,$$

因此 $g(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$ 且 $g(1)=1$. 由习题 2.3.3 即得 $g'(1)=f'(1)-f(1)\geqslant 0$, 故 $f'(1)\geqslant 1$. □

习题 4.5.32 设 P 是一个 k 次多项式, 在单位圆周上满足 $\left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant 1$. 证明：对任意单位圆盘外的 z, 有 $|P(z)| \leqslant |z|^k$.

证明 设 $f(z) = \dfrac{P(z)}{z^k}$, 则 $f \in \mathcal{H}\left(\overline{\mathbb{B}(0,1)}^{\mathrm{c}}\right)$. 由最大模原理, $\max\limits_{|z|\geqslant 1}|f(z)| = \max\limits_{|z|=1}|f(z)| \leqslant 1$, 得所欲证. □

习题 5.2.2 下列初等全纯函数有哪些奇点？指出其类别：

(2) $\dfrac{\mathrm{e}^{\frac{1}{1-z}}}{\mathrm{e}^z - 1}$.

(4) $\tan z$.

(6) $\mathrm{e}^{\cot\frac{1}{z}}$.

解答 (2) 1 阶极点：$2k\pi\mathrm{i}$ $(k \in \mathbb{Z})$; 本性奇点：1; 非孤立奇点：∞.

(4) $\tan z = -\mathrm{i}\dfrac{\mathrm{e}^{2\mathrm{i}z} - 1}{\mathrm{e}^{2\mathrm{i}z} + 1}$. 1 阶极点：$\left(k + \frac{1}{2}\right)\pi$ $(k \in \mathbb{Z})$; 非孤立奇点：∞.

(6) $\mathrm{e}^{\cot\frac{1}{z}} = \exp\left(\mathrm{i}\dfrac{\mathrm{e}^{\frac{2\mathrm{i}}{z}} + 1}{\mathrm{e}^{\frac{2\mathrm{i}}{z}} - 1}\right)$. 本性奇点：$\dfrac{1}{k\pi}$ $(k \in \mathbb{Z})$, ∞; 非孤立奇点：0. □

习题 5.2.3 若 z_0 是全纯函数 $f : \mathbb{B}(z_0, r) \setminus \{z_0\} \to \mathbb{C} \setminus \{0\}$ 的本性奇点, 则 z_0 也是 $\dfrac{1}{f(z)}$ 的本性奇点.

证明 由 $f(z) \neq 0, \forall z \in \mathbb{B}(z_0, r) \setminus \{z_0\}$ 知 z_0 是 $\dfrac{1}{f(z)}$ 的孤立奇点. 由于 z_0 是 $f(z)$ 的本性奇点, 对任意 $A \in \overline{\mathbb{C}}$, 在任意 $\mathbb{B}(z_0, \delta) \setminus \{z_0\} \subset \mathbb{B}(z_0, r)$ 中存在一列互异的 $z_n \to z_0$ 使得 $f(z_n) \to \dfrac{1}{A}$, 进而 $\dfrac{1}{f(z_n)} \to A$, 即 z_0 是 $\dfrac{1}{f(z)}$ 的本性奇点. □

习题 5.2.4 设 $R(z)$ 是有理函数, $z_1, z_2, \cdots, z_n$ 是 $R(z)$ 在 $\overline{\mathbb{C}}$ 上的全部不同的极点. 证明：若 z_0 是全纯函数 $f : \mathbb{B}(z_0, r) \setminus \{z_0\} \to \overline{\mathbb{C}} \setminus \{z_1, z_2, \cdots, z_n\}$ 的本性奇点, 则 z_0 也是 $R(f(z))$ 的本性奇点.

证明 由于 z_0 是 $f(z)$ 的本性奇点, 取互异的 $A, B \in \mathbb{C} \setminus \{z_1, z_2, \cdots, z_n\}$ 满足 $R(A) \neq R(B)$, 则存在两列点列 $a_n \to z_0$ 与 $b_n \to z_0$ 使得 $f(a_n) \to A$ 且 $f(b_n) \to B$. 此时 $R(f(a_n)) \to A$ 而 $R(f(b_n)) \to B$, 二者不等, 因此 z_0 是 $R(f(z))$ 的本性奇点. □

习题 5.2.8 设 f 在 $\mathbb{B}(0, R) \setminus \{0\}$ 上全纯. 若 $\operatorname{Re} f(z) > 0, \forall z \in \mathbb{B}(0, R) \setminus \{0\}$, 则 0 是 f 的可去奇点.

证明 由 $\operatorname{Re} f(z) > 0, \forall z \in \mathbb{B}(0, R) \setminus \{0\}$ 可见 0 不是 f 的本性奇点. 故只需证 0 不是 f 的极点. 用反证法, 若 0 是 f 的极点, 设 $g(z) = \dfrac{1}{f(z)}$, 则 $g(0) = 0$. 而对 $z \in \mathbb{B}(0, R) \setminus \{0\}$, 由 $\operatorname{Re} f(z) > 0$ 可知 $\operatorname{Re} g(z) > 0$, 由平均值公式, 当 $r \in (0, R)$ 时,

$$0 = \operatorname{Re} g(0) = \operatorname{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} g\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta\right\} = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} g\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta > 0,$$

矛盾. 故 0 不是 f 的极点, 从而 0 是 f 的可去奇点. □

习题 5.4.1 证明：留数定理与 Cauchy 积分公式等价.

定理 1 (留数定理) 设 γ 是可求长 Jordan 曲线, 函数 $f(z)$ 在 γ 内部 D 中除去 $z_1, z_2, \cdots, z_n$ 外全纯, 且在 $\overline{D} \setminus \{z_1, z_2, \cdots, z_n\}$ 上连续, 则

$$\int\limits_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{n}\operatorname{Res}(f, z_k).$$

定理 2 (Cauchy 积分公式) 设区域 D 是可求长 Jordan 曲线 γ 的内部, $f(z)\in\mathcal{H}(D)\cap\mathcal{C}(\overline{D})$, 则

(1) 在 D 内 $f(z)=\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_\gamma\dfrac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta$.

(2) $f(z)$ 在 D 内有各阶导数, 且在 D 内 $f^{(n)}(z)=\dfrac{n!}{2\pi\mathrm{i}}\displaystyle\int_\gamma\dfrac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta\ (n=1,2,\cdots)$.

证明 $\boxed{\textbf{(1)}\Rightarrow\textbf{(2)}}$ 对 $n\geqslant 0$, $\zeta=z$ 是 $\dfrac{f(\zeta)}{(\zeta-z)^{n+1}}$ 的 $n+1$ 阶极点, 由留数定理,

$$\int_\gamma\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta=2\pi\mathrm{i}\,\mathrm{Res}\Big(\tfrac{f(\zeta)}{(\zeta-z)^{n+1}},z\Big)=\frac{2\pi\mathrm{i}}{n!}\lim_{\zeta\to z}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[(\zeta-z)^{n+1}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\right]=\frac{2\pi\mathrm{i}f^{(n)}(z)}{n!}.$$

$\boxed{\textbf{(2)}\Rightarrow\textbf{(1)}}$ 由多连通域的 Cauchy 定理, 不妨设 $f(z)$ 在 D 中只有 1 个奇点 a, 并设 f 在 a 的邻域内有 Laurent 展开 $f(z)=\displaystyle\sum_{n=-\infty}^{+\infty}c_n(z-a)^n$. 由 Cauchy 积分公式,

$$\int_\gamma f(z)\,\mathrm{d}z=\int_\gamma\sum_{n=-\infty}^{+\infty}c_n(z-a)^n\,\mathrm{d}z=\sum_{n=-\infty}^{+\infty}\int_\gamma c_n(z-a)^n\,\mathrm{d}z=2\pi\mathrm{i}c_{-1}=2\pi\mathrm{i}\,\mathrm{Res}(f,a).$$

□

习题 5.4.2 若 a 是 $f\in\mathcal{H}(\mathbb{B}(a,R)\setminus\{a\})$ 的可去奇点, 其中 $a\neq\infty$, 则显然 $\mathrm{Res}(f,a)=0$. 举例说明, 若 ∞ 是 $f\in\mathcal{H}(\mathbb{B}(\infty,R))$ 的可去奇点, 则 $\mathrm{Res}(f,\infty)$ 可能不等于 0.

解答 设 $f(z)=1+\dfrac{1}{z}$, 则 ∞ 是 $f(z)\in\mathcal{H}(\mathbb{B}(\infty,R))$ $(R>0)$ 的可去奇点, 但

$$\mathrm{Res}(f,\infty)=-\frac{1}{2\pi\mathrm{i}}\int_{|z|=1}\left(1+\frac{1}{z}\right)\mathrm{d}z=-1.$$

□

习题 5.4.3 设 $f\in\mathcal{H}(\mathbb{B}(\infty,R))$. 证明:

(1) 若 ∞ 是 f 的可去奇点, 则 $\mathrm{Res}(f,\infty)=\displaystyle\lim_{z\to\infty}z^2f'(z)$.

(2) 若 ∞ 是 f 的 m 阶极点, 则 $\mathrm{Res}(f,\infty)=\dfrac{(-1)^m}{(m+1)!}\displaystyle\lim_{z\to\infty}z^{m+2}f^{(m+1)}(z)$.

证明 (1) 若 ∞ 是 f 的可去奇点, 可设

$$f(z)=\sum_{n=0}^{\infty}\frac{c_n}{z^n},\quad z\in\mathbb{B}(\infty,R).$$

于是

$$\mathrm{Res}(f,\infty)\xlongequal{\rho>R}-\frac{1}{2\pi\mathrm{i}}\int_{|z|=\rho}\sum_{n=0}^{\infty}\frac{c_n}{z^n}\,\mathrm{d}z=-c_1=\lim_{z\to\infty}\sum_{n=1}^{\infty}\frac{-nc_n}{z^{n-1}}=\lim_{z\to\infty}z^2f'(z).$$

(2) 若 ∞ 是 f 的 m 阶极点, 可设

$$f(z)=\sum_{n=-m}^{\infty}\frac{c_n}{z^n},\quad z\in\mathbb{B}(\infty,R).$$

于是

$$\operatorname{Res}(f,\infty) \xlongequal{\rho>R} -\frac{1}{2\pi \mathrm{i}}\int_{|z|=\rho}\sum_{n=-m}^{\infty}\frac{c_n}{z^n}\,\mathrm{d}z = -c_1.$$

而

$$f^{(m+1)}(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\left(\sum_{n=1}^{\infty}\frac{c_n}{z^n}\right) = (-1)^{m+1}\sum_{n=1}^{\infty}\frac{(n+m)!}{(n-1)!}\cdot\frac{c_n}{z^{n+m+1}},$$

因此

$$\frac{(-1)^m}{(m+1)!}\lim_{z\to\infty} z^{m+2}f^{(m+1)}(z) = -\frac{1}{(m+1)!}\lim_{z\to\infty}\sum_{n=1}^{\infty}\frac{(n+m)!}{(n-1)!}\cdot\frac{c_n}{z^{n-1}} = -c_1 = \operatorname{Res}(f,\infty).$$

□

习题 5.4.4 设 $f,g\in\mathcal{H}(\mathbb{B}(a,r))$, $f(a)\neq 0$, a 是 g 的 2 阶零点, 计算 $\operatorname{Res}\left(\frac{f}{g},a\right)$.

解答 设

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n,\quad g(z)=\sum_{n=0}^{\infty}b_n(z-a)^n,\quad z\in\mathbb{B}(a,r),$$

其中

$$a_n=\frac{f^{(n)}(a)}{n!}, a_0\neq 0,\quad b_n=\frac{g^{(n)}(a)}{n!}, b_0=b_1=0, b_2\neq 0.$$

设 $h(z)=\dfrac{g(z)}{(z-a)^2}=\sum\limits_{n=2}^{\infty}b_n(z-a)^{n-2}$. 由于 $h(a)=b_2\neq 0$, 在 a 的邻域内 $g(z)\neq 0$, 此时

$$\frac{f(z)}{g(z)}=\frac{f(z)}{(z-a)^2h(z)}$$

以 a 为 2 阶极点, 因此

$$\begin{aligned}\operatorname{Res}\left(\frac{f}{g},a\right)&=\lim_{z\to a}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-a)^2\frac{f(z)}{g(z)}\right]=\lim_{z\to a}\left(\frac{f(z)}{h(z)}\right)'\\&=\lim_{z\to a}\frac{f'(z)h(z)-f(z)h'(z)}{h^2(z)}=\frac{f'(a)h(a)-f(a)h'(a)}{h^2(a)},\end{aligned}$$

代入 $h(a)=b_2=\dfrac{g''(a)}{2}$ 与 $h'(a)=b_3=\dfrac{g'''(a)}{6}$ 即得

$$\operatorname{Res}\left(\frac{f}{g},a\right)=\frac{f'(a)\frac{g''(a)}{2}-f(a)\frac{g'''(a)}{6}}{\left(\frac{g''(a)}{2}\right)^2}=\frac{6f'(a)g''(a)-2f(a)g'''(a)}{3[g''(a)]^2}.$$

□

习题 5.4.8 指出下列初等函数在 $\overline{\mathbb{C}}$ 中的全部孤立奇点, 并求出这些初等函数在它们各自孤立奇点处的留数:

(1) $\dfrac{1}{z^3-z^5}$.

(2) $\dfrac{z^3+z^2+2}{z(z^2-1)^2}$.

(3) $\dfrac{z^2+z-1}{z^2(z-1)}$.

(4) $\dfrac{z^{n-1}}{z^n+a^n}$ $(a\neq 0, n\in\mathbb{N})$.

(5) $\dfrac{1}{\sin z}$.

(6) $\sin\dfrac{z}{z+1}$.

(7) $\dfrac{\mathrm{e}^z}{z(z-1)}$.

(8) $\dfrac{\mathrm{e}^{\pi z}}{z^2+1}$.

解答 将每问中的函数记为 $f(z)$.

(1) 孤立奇点为 $0, 1, -1, \infty$.

① 由 $\dfrac{1}{z^3-z^5}=\dfrac{1}{z^3(1-z^2)}=\dfrac{1}{z^3}\left(1+z^2+z^4+z^6+\cdots\right)$ 知 $\operatorname{Res}(f,0)=1$.

② 1 为 1 阶极点, 因此 $\operatorname{Res}(f,1)=\lim\limits_{z\to 1}\dfrac{z-1}{z^3-z^5}=\lim\limits_{z\to 1}\dfrac{-1}{z^3(1+z)}=-\dfrac{1}{2}$.

③ -1 为 1 阶极点, 因此 $\operatorname{Res}(f,-1)=\lim\limits_{z\to -1}\dfrac{z+1}{z^3-z^5}=\lim\limits_{z\to -1}\dfrac{1}{z^3(1-z)}=-\dfrac{1}{2}$.

④ $\operatorname{Res}(f,\infty)=-\left(1-\dfrac{1}{2}-\dfrac{1}{2}\right)=0$.

(2) 孤立奇点为 $0, 1, -1, \infty$.

① 0 为 1 阶极点, 因此 $\operatorname{Res}(f,0)=\lim\limits_{z\to 0}\dfrac{z^3+z^2+2}{\left(z^2-1\right)^2}=2$.

② 1 为 2 阶极点, 因此 $\operatorname{Res}(f,1)=\dfrac{1}{1!}\lim\limits_{z\to 1}\left(\dfrac{z^3+z^2+2}{z(z+1)^2}\right)'=\lim\limits_{z\to 1}\dfrac{z^4+2z^3-5z^2-8z-2}{z^2(z+1)^4}=-\dfrac{3}{4}$.

③ -1 为 2 阶极点, 因此 $\operatorname{Res}(f,1)=\dfrac{1}{1!}\lim\limits_{z\to -1}\left(\dfrac{z^3+z^2+2}{z(z-1)^2}\right)'=-\dfrac{5}{4}$.

④ $\operatorname{Res}(f,\infty)=-\left(2-\dfrac{3}{4}-\dfrac{5}{4}\right)=0$.

(3) 孤立奇点为 $0, 1, \infty$.

① 0 为 2 阶极点, 因此 $\operatorname{Res}(f,0)=\dfrac{1}{1!}\lim\limits_{z\to 0}\left(\dfrac{z^2+z-1}{z-1}\right)'=0$.

② 1 为 1 阶极点, 因此 $\operatorname{Res}(f,1)=\lim\limits_{z\to 1}\dfrac{z^2+z-1}{z^2}=1$.

③ $\operatorname{Res}(f,\infty)=-(0+1)=-1$.

(4) 孤立奇点为 $a(-1)^{\frac{1}{n}}=a\mathrm{e}^{\frac{\mathrm{i}(2k+1)\pi}{n}}$ $(k=0,1,\cdots,n-1)$ 及 ∞. 由于 $z_k=a\mathrm{e}^{\frac{\mathrm{i}(2k+1)\pi}{n}}$ 是 1 阶极点,

$$\operatorname{Res}(f,z_k)=\lim_{z\to z_k}\frac{z^{n-1}(z-z_k)}{z^n+a^n}=\lim_{z\to z_k}\frac{z^{n-1}}{\frac{z^n+a^n}{z-z_k}}=\frac{z_k^{n-1}}{(z^n+a^n)'|_{z=z_k}}=\frac{1}{n}.$$

由于 ∞ 为可去奇点, $\operatorname{Res}(f,\infty)=-\sum\limits_{k=0}^{n-1}\operatorname{Res}(f,z_k)=-1$.

(5) 孤立奇点为 $k\pi$ $(k\in\mathbb{Z})$. 由于 $k\pi$ 为 1 阶极点, $\operatorname{Res}(f,k\pi)=\lim\limits_{z\to k\pi}\dfrac{z-k\pi}{\sin z}=(-1)^k$.

(6) 孤立奇点为 $-1, \infty$.

① 由于 ∞ 是 f 的可去奇点, 由习题 5.4.3 (1),

$$\operatorname{Res}(f, \infty) = \lim_{z\to\infty} z^2 f'(z) = \lim_{z\to\infty} \frac{z^2}{(1+z)^2}\cos\left(\frac{z}{1+z}\right) = \cos 1.$$

② $\operatorname{Res}(f, -1) = -\operatorname{Res}(f, \infty) = -\cos 1$.

(7) 孤立奇点为 $0, 1, \infty$.

① 0 为 1 阶极点, 因此 $\operatorname{Res}(f, 0) = \lim\limits_{z\to 0} \dfrac{\mathrm{e}^z}{z-1} = -1$.

② 1 为 1 阶极点, 因此 $\operatorname{Res}(f, 1) = \lim\limits_{z\to 1} \dfrac{\mathrm{e}^z}{z} = \mathrm{e}$.

③ $\operatorname{Res}(f, \infty) = -(-1+\mathrm{e}) = 1 - \mathrm{e}$.

(8) 孤立奇点为 $\mathrm{i}, -\mathrm{i}, \infty$.

① i 为 1 阶极点, 因此 $\operatorname{Res}(f, \mathrm{i}) = \lim\limits_{z\to \mathrm{i}} \dfrac{\mathrm{e}^{\pi z}}{(z+\mathrm{i})} = \dfrac{\mathrm{i}}{2}$.

② $-\mathrm{i}$ 为 1 阶极点, 因此 $\operatorname{Res}(f, -\mathrm{i}) = \lim\limits_{z\to -\mathrm{i}} \dfrac{\mathrm{e}^{\pi z}}{z-\mathrm{i}} = -\dfrac{\mathrm{i}}{2}$.

③ $\operatorname{Res}(f, \infty) = -\left(\dfrac{\mathrm{i}}{2} - \dfrac{\mathrm{i}}{2}\right) = 0$. □

习题 5.4.9 设 $f, g \in \mathcal{H}(\mathbb{B}(0,R)) \cap \mathcal{C}\left(\overline{\mathbb{B}(0,R)}\right)$, g 在 $\partial\mathbb{B}(0,R)$ 上无零点, g 在 $\mathbb{B}(0,R)$ 中的全部零点 $z_1, z_2, \cdots, z_n$ 都是 1 阶零点, 求

$$\frac{1}{2\pi\mathrm{i}} \int_{|z|=R} \frac{f(z)}{zg(z)}\,\mathrm{d}z.$$

解答 (1) 若 $z_1, z_2, \cdots, z_n \neq 0$.

① 若 $f(z_k) \neq 0$, 则 z_k 为 $\dfrac{f(z)}{zg(z)}$ 的 1 阶极点, $\operatorname{Res}\left(\dfrac{f(z)}{zg(z)}, z_k\right) = \lim\limits_{z\to z_k} \dfrac{f(z)}{\frac{zg(z)}{z-z_k}} = \dfrac{f(z_k)}{z_k g'(z_k)}$.

② 若 $f(z_k) = 0$, 则 z_k 为 $\dfrac{f(z)}{zg(z)}$ 的可去奇点, $\operatorname{Res}\left(\dfrac{f(z)}{zg(z)}, z_k\right) = 0$.

③ 对于充分小的 ε, 由 Cauchy 积分公式, $\dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_{|z|=\varepsilon} \dfrac{f(z)}{zg(z)}\,\mathrm{d}z = \dfrac{f(0)}{g(0)}$.

故由留数定理, $\dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_{|z|=R} \dfrac{f(z)}{zg(z)}\,\mathrm{d}z = \dfrac{f(0)}{g(0)} + \sum_{k=1}^{n} \dfrac{f(z_k)}{z_k g'(z_k)}$.

(2) 若 $z_1, z_2, \cdots, z_n$ 中有 0, 不妨设 $z_n = 0$.

① 若 0 是 $f(z)$ 的 m 阶零点 $(m \geqslant 2)$, 则 0 是 $\dfrac{f(z)}{zg(z)}$ 的可去奇点, $\operatorname{Res}\left(\dfrac{f(z)}{zg(z)}, 0\right) = 0$.

② 若 0 是 $f(z)$ 的 1 阶零点, 则 0 是 $\dfrac{f(z)}{zg(z)}$ 的 1 阶极点, $\operatorname{Res}\left(\dfrac{f(z)}{zg(z)}, 0\right) = \lim\limits_{z\to 0} \dfrac{f(z)}{g(z)} = \dfrac{f'(0)}{g'(0)}$.

③ 若 $f(0) \neq 0$, 由于 0 是 $zg(z)$ 的 2 阶零点, 由习题 5.4.4,

$$\operatorname{Res}\left(\frac{f(z)}{zg(z)}, 0\right) = \left.\frac{6f'(z)(zg(z))'' - 2f(z)(zg(z))'''}{3[(zg(z))'']^2}\right|_{z=0} = \frac{6f'(0)\cdot 2g'(0) - 2f(0)\cdot 3g''(0)}{12[g'(0)]^2}$$

$$= \frac{f'(0)}{g'(0)} - \frac{f(0)g''(0)}{2[g'(0)]^2}.$$

故由留数定理,

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=R} \frac{f(z)}{zg(z)}\,\mathrm{d}z = \begin{cases} \displaystyle\sum_{k=1}^{n-1} \frac{f(z_k)}{z_k g'(z_k)}, & 0\ \text{是}\ f(z)\ \text{的}\ m\ \text{阶零点},\ m \geqslant 2, \\ \displaystyle\frac{f'(0)}{g'(0)} - \frac{f(0)g''(0)}{2[g'(0)]^2} + \sum_{k=1}^{n-1} \frac{f(z_k)}{z_k g'(z_k)}, & \text{其他}. \end{cases}$$

□

习题 5.4.12 设 D 是由有限条可求长简单闭曲线围成的域, $f(z)$ 在 D 上亚纯, 在 D 中的全部彼此不同的极点为 $w_1, w_2, \cdots, w_m$, 其相应的 Laurent 展开式的主要部分为 $f_1(z), f_2(z), \cdots, f_m(z)$, 并且在 $\overline{D} \setminus \{w_1, w_2, \cdots, w_m\}$ 上连续. 证明：对于任意 $z \in D$, 有

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = f(z) - \sum_{j=1}^{m} f_j(z).$$

证明 设 $f_j(z) = \sum\limits_{k=1}^{\infty} \dfrac{c_{-k}}{(z-w_j)^k}$. 由留数定理, $\dfrac{1}{2\pi \mathrm{i}} \displaystyle\int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{j=1}^{m} \operatorname{Res}\left(\frac{f(\zeta)}{\zeta - z}, w_j\right) + \operatorname{Res}\left(\frac{f(\zeta)}{\zeta - z}, z\right)$.

(1) 若 $z \notin \{w_1, w_2, \cdots, w_m\}$, 则 z 是 $\dfrac{f(\zeta)}{\zeta - z}$ 的 1 阶极点, 从而

$$\operatorname{Res}\left(\frac{f(\zeta)}{\zeta - z}, z\right) = \lim_{\zeta \to z} (\zeta - z) \frac{f(\zeta)}{\zeta - z} = f(z),$$
$$\operatorname{Res}\left(\frac{f(\zeta)}{\zeta - z}, w_j\right) = \operatorname{Res}\left(\frac{f_j(\zeta)}{\zeta - z}, w_j\right) = \frac{c_{-1}}{w_j - z} = -f_j(z).$$

因此 $\dfrac{1}{2\pi \mathrm{i}} \displaystyle\int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = f(z) - \sum_{j=1}^{m} f_j(z)$.

(2) 若 $z \in \{w_1, w_2, \cdots, w_m\}$, 不妨设 $w_m = z$, 则

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta &= \sum_{j=1}^{m-1} \operatorname{Res}\left(\frac{f(\zeta)}{\zeta - z}, w_j\right) + \operatorname{Res}\left(\frac{f(\zeta)}{\zeta - z}, z\right) \\ &\xlongequal{(1)} -\sum_{j=1}^{m-1} f_j(z) + \underbrace{\operatorname{Res}\left(\frac{f(\zeta) - f_m(\zeta)}{\zeta - z}, z\right)}_{z\ \text{是其 1 阶极点}} + \operatorname{Res}\left(\frac{f_m(\zeta)}{\zeta - z}, z\right) \\ &= -\sum_{j=1}^{m-1} f_j(z) + \lim_{\zeta \to z} (\zeta - z) \frac{f(\zeta) - f_m(\zeta)}{\zeta - z} + 0 \\ &= f(z) - \sum_{j=1}^{m} f_j(z). \end{aligned}$$

□

习题 5.5.1 利用留数定理和 Cauchy 积分公式计算下列积分：

(1) $\displaystyle\int_0^{+\infty} \frac{x^2+1}{x^4+1}\,\mathrm{d}x$.

(4) $\displaystyle\int_0^{2\pi} \frac{1}{a+b\cos\theta}\,\mathrm{d}\theta\ (0<b<a)$.

(9) $\displaystyle\int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x$.

(17) $\displaystyle\int_{-1}^{1} \frac{\sqrt[4]{(1-x)^3(1+x)}}{1+x^2}\,\mathrm{d}x$.

(21) $\displaystyle\int_0^{+\infty} \frac{\log x}{x^2-1}\,\mathrm{d}x$.

(24) $\displaystyle\int_0^{+\infty} \frac{\sin x}{\mathrm{e}^x-1}\,\mathrm{d}x$.

(28) $\displaystyle\int_0^{+\infty} \mathrm{e}^{-ax^2}\cos\bigl(bx^2\bigr)\,\mathrm{d}x\ (a>0)$.

(29) $\displaystyle\int_0^{\frac{\pi}{2}} \log\sin\theta\,\mathrm{d}\theta$.

解答 (1) 由于 $\gcd\bigl(x^2+1, x^4+1\bigr)=1$, x^4+1 无实根, 在上半平面中有根 $a_1=\zeta_8, a_2=\zeta_8^3$, 且 $\deg\bigl(x^4+1\bigr)-\deg\bigl(x^2+1\bigr)=2$, 因此

$$\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{k=1}^{2}\operatorname{Res}\left(\frac{z^2+1}{z^4+1}, a_k\right),$$

其中

$$\operatorname{Res}\left(\frac{z^2+1}{z^4+1}, a_1\right) = \lim_{z\to\zeta_8}\frac{\bigl(z^2+1\bigr)(z-\zeta_8)}{z^4+1} = \lim_{z\to\zeta_8}\frac{z^2+1}{\frac{z^4+1}{z-\zeta_8}} = \left.\frac{z^2+1}{\bigl(z^4+1\bigr)'}\right|_{z=\zeta_8} = -\frac{\mathrm{i}}{2\sqrt{2}},$$

$$\operatorname{Res}\left(\frac{z^2+1}{z^4+1}, a_2\right) = \lim_{z\to\zeta_8^3}\frac{\bigl(z^2+1\bigr)(z-\zeta_8^3)}{z^4+1} = \lim_{z\to\zeta_8^3}\frac{z^2+1}{\frac{z^4+1}{z-\zeta_8^3}} = \left.\frac{z^2+1}{\bigl(z^4+1\bigr)'}\right|_{z=\zeta_8^3} = -\frac{\mathrm{i}}{2\sqrt{2}}.$$

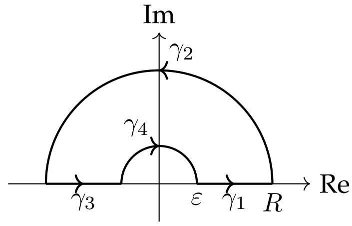
故

$$\int_0^{+\infty} \frac{x^2+1}{x^4+1}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

(4) 令 $z=\mathrm{e}^{\mathrm{i}\theta}$, 则 $\cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$, $\mathrm{d}z = \mathrm{i}z\,\mathrm{d}\theta$, 从而

$$\begin{aligned}\int_0^{2\pi} \frac{1}{a+b\cos\theta}\,\mathrm{d}\theta &= \int_{|z|=1} \frac{\mathrm{d}z}{\mathrm{i}z\bigl[a+\frac{b}{2}\bigl(z+\frac{1}{z}\bigr)\bigr]} = \frac{2}{b\mathrm{i}}\int_{|z|=1} \frac{\mathrm{d}z}{z^2+\frac{2a}{b}z+1} \\ &= \frac{2}{b\mathrm{i}}\cdot 2\pi\mathrm{i}\operatorname{Res}\left(\frac{1}{z^2+\frac{2a}{b}z+1}, \frac{-a+\sqrt{a^2-b^2}}{b}\right) \\ &= \frac{4\pi}{b}\cdot\frac{1}{\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)} \\ &= \frac{2\pi}{\sqrt{a^2-b^2}}.\end{aligned}$$

(9) 选取如图积分路径.



设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ 所围区域为 D. 由 $\dfrac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\in\mathcal{H}(D)$ 知 $\displaystyle\int_{\partial D}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z=0$. 我们有

$$
\begin{aligned}
\operatorname{Re}\left\{\int_{\gamma_1}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z\right\}&=\int_{\gamma_1}\operatorname{Re}\left\{\frac{\cos 2x+\mathrm{i}\sin 2x-1}{x^2}\right\}\mathrm{d}x\xrightarrow[R\to+\infty]{\varepsilon\to0^+}-2\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x,\\
\operatorname{Re}\left\{\int_{\gamma_3}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z\right\}&=\int_{\gamma_3}\operatorname{Re}\left\{\frac{\cos 2x+\mathrm{i}\sin 2x-1}{x^2}\right\}\mathrm{d}x\xrightarrow[R\to+\infty]{\varepsilon\to0^+}-2\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x,\\
\left|\int_{\gamma_2}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z\right|&\leqslant\int_0^{\pi}\frac{\left|\mathrm{e}^{2\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}}\right|+1}{R}\,\mathrm{d}\theta=\int_0^{\pi}\frac{\mathrm{e}^{-2R\sin\theta}+1}{R}\,\mathrm{d}\theta\leqslant\frac{2\pi}{R}\xrightarrow{R\to+\infty}0.
\end{aligned}
$$

以及

$$
\int_{\gamma_4}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z=\int_{\gamma_4}\sum_{k=1}^{\infty}\frac{(2\mathrm{i}z)^k}{k!}\cdot z^{-2}\,\mathrm{d}z=\int_{\gamma_4}\frac{2\mathrm{i}}{z}\,\mathrm{d}z+\int_{\gamma_4}\sum_{k=0}^{\infty}\frac{(2\mathrm{i})^{k+2}z^k}{(k+2)!}\,\mathrm{d}z,
$$

其中

$$
\int_{\gamma_4}\frac{2\mathrm{i}}{z}\,\mathrm{d}z=\int_{\pi}^{0}\frac{2\mathrm{i}}{\varepsilon\mathrm{e}^{\mathrm{i}\theta}}\cdot\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta=2\pi,
$$

$$
\left|\int_{\gamma_4}\sum_{k=0}^{\infty}\frac{(2\mathrm{i})^{k+2}z^k}{(k+2)!}\,\mathrm{d}z\right|\overset{\varepsilon<1}{\leqslant}\pi\varepsilon\cdot\underbrace{\max_{|z|\leqslant1}\left|\sum_{k=0}^{\infty}\frac{(2\mathrm{i})^{k+2}z^k}{(k+2)!}\right|}_{<+\infty}\xrightarrow{\varepsilon\to0^+}0.
$$

故令 $\varepsilon\to0^+, R\to+\infty$ 就得到

$$
0=\operatorname{Re}\left\{\int_{\partial D}\frac{\mathrm{e}^{2\mathrm{i}z}-1}{z^2}\,\mathrm{d}z\right\}=-4\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x+2\pi,
$$

即

$$
\int_0^{+\infty}\left(\frac{\sin x}{x}\right)^2\mathrm{d}x=\frac{\pi}{2}.
$$

(17) 令 $f(z)=\dfrac{1}{1+z^2}, r=\dfrac{1}{4}, s=\dfrac{3}{4}$, 则 $r+s=1\in\mathbb{Z}$, $f(z)$ 在 $\mathbb{C}$ 中仅有极点 $a_1=\mathrm{i}, a_2=-\mathrm{i}$, 且 $\displaystyle\lim_{z\to\infty}z^{r+s+1}f(z)=\lim_{z\to\infty}\frac{z^2}{1+z^2}=1$, 由定理 5.5.14,

$$
\int_{-1}^{1}(x+1)^r(1-x)^s f(x)\,\mathrm{d}x=-\frac{\pi}{\sin s\pi}+\frac{\pi}{\mathrm{e}^{-s\pi\mathrm{i}}\sin s\pi}\sum_{k=1}^{2}\operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2},a_k\right).
$$

而

$$\operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2}, \mathrm{i}\right) = \lim_{z\to\mathrm{i}} \frac{\sqrt[4]{(1-z)^3(1+z)}}{z+\mathrm{i}} = \frac{1}{2\mathrm{i}} \lim_{z\to\mathrm{i}} \sqrt[4]{(1-z)^3(1+z)},$$

$$\operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2}, -\mathrm{i}\right) = \lim_{z\to-\mathrm{i}} \frac{\sqrt[4]{(1-z)^3(1+z)}}{z-\mathrm{i}} = \frac{1}{-2\mathrm{i}} \lim_{z\to-\mathrm{i}} \sqrt[4]{(1-z)^3(1+z)},$$

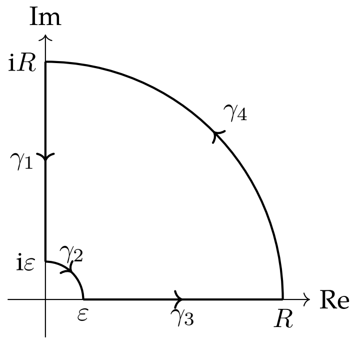
由习题 2.4.27 即得

$$\begin{aligned}&\operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2}, \mathrm{i}\right) + \operatorname{Res}\left(\frac{\sqrt[4]{(1-z)^3(1+z)}}{1+z^2}, -\mathrm{i}\right)\\ =&\frac{1}{2\mathrm{i}}\left(\lim_{z\to\mathrm{i}} \sqrt[4]{(1-z)^3(1+z)} - \lim_{z\to-\mathrm{i}} \sqrt[4]{(1-z)^3(1+z)}\right) = \frac{\sqrt{2}}{2\mathrm{i}}\left(\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}} - \mathrm{e}^{\frac{5\pi}{8}\mathrm{i}}\right).\end{aligned}$$

故

$$\int_{-1}^{1} \frac{\sqrt[4]{(1-x)^3(1+x)}}{1+x^2}\,\mathrm{d}x = -\frac{\pi}{\sin\frac{3\pi}{4}} + \frac{\pi}{\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}} \cdot \frac{1}{\sqrt{2}\mathrm{i}}\left(\mathrm{e}^{-\frac{\pi}{8}\mathrm{i}} - \mathrm{e}^{\frac{5\pi}{8}\mathrm{i}}\right) = \left(\sqrt{2+\sqrt{2}} - \sqrt{2}\right)\pi.$$

(21) 选取如图积分路径.



设 $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ 所围区域为 D. 由 $\dfrac{\log z}{z^2-1} \in \mathcal{H}(D)$ 知 $\displaystyle\int_{\partial D} \frac{\log z}{z^2-1}\,\mathrm{d}z = 0$ (注意 1 是 $\dfrac{\log z}{z^2-1}$ 的可去奇点).

① 在 γ_1 上,

$$\begin{aligned}\operatorname{Re}\left\{\int_{\gamma_1} \frac{\log z}{z^2-1}\,\mathrm{d}z\right\} &= -\int_{\varepsilon}^{R} \operatorname{Im}\left\{\frac{\log(\mathrm{i}t)}{t^2+1}\right\}\mathrm{d}t = -\int_{\varepsilon}^{R} \frac{\frac{\pi}{2}}{t^2+1}\,\mathrm{d}t \xrightarrow[R\to+\infty]{\varepsilon\to0^+} -\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{t^2+1}\\ &= -\frac{\pi^2}{4}.\end{aligned}$$

② 在 γ_2 上, 由于 $\displaystyle\lim_{z\to0} \frac{z\log z}{z^2-1} = 0$, 若记 $M(\varepsilon) = \displaystyle\max_{\gamma_2(\varepsilon)}\left|\frac{z\log z}{z^2-1}\right|$, 则 $\displaystyle\lim_{\varepsilon\to0^+} M(\varepsilon) = 0$. 当 $z = \varepsilon\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z = \mathrm{i}z\,\mathrm{d}\theta$, 因此

$$\left|\int_{\gamma_2} \frac{\log z}{z^2-1}\,\mathrm{d}z\right| = \left|\int_{\gamma_2} \frac{\frac{z\log z}{z^2-1}}{z}\,\mathrm{d}z\right| \leqslant \int_0^{\frac{\pi}{2}} M(\varepsilon) = \frac{\pi}{2}M(\varepsilon) \xrightarrow{\varepsilon\to0^+} 0.$$

③ 在 γ_3 上,

$$\int\limits_{\gamma_3} \frac{\log z}{z^2-1}\,\mathrm{d}z = \int_{\varepsilon}^{R} \frac{\log x}{x^2-1}\,\mathrm{d}x \xrightarrow[R\to+\infty]{\varepsilon\to0^+} \int_0^{+\infty} \frac{\log x}{x^2-1}\,\mathrm{d}x.$$

④ 在 γ_4 上, 由于 $\lim\limits_{z\to\infty} \dfrac{\log z}{z^2-1} = 0$, 若记 $M(R) = \max\limits_{\gamma_4(R)} \left|\dfrac{\log z}{z^2-1}\right|$, 则 $\lim\limits_{R\to+\infty} M(R) = 0$. 当 $z = R\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z = \mathrm{i}z\,\mathrm{d}\theta$, 因此

$$\left|\int\limits_{\gamma_4} \frac{\log z}{z^2-1}\,\mathrm{d}z\right| \leqslant \int_0^{\frac{\pi}{2}} M(R)\,\mathrm{d}\theta = \frac{\pi}{2}M(R) \xrightarrow{R\to+\infty} 0.$$

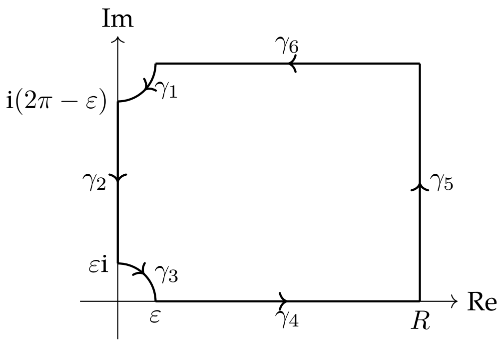
故

$$0 = \mathrm{Re}\left\{\int\limits_{\gamma} \frac{\log z}{z^2-1}\,\mathrm{d}z\right\} = -\frac{\pi^2}{4} + \int_0^{+\infty} \frac{\log x}{x^2-1}\,\mathrm{d}x,$$

即

$$\int_0^{+\infty} \frac{\log x}{x^2-1}\,\mathrm{d}x = \frac{\pi^2}{4}.$$

(24) 选取如图积分路径.



设 $\gamma = \gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4\cup\gamma_5\cup\gamma_6$ 所围区域为 D. 由 $\dfrac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1} \in \mathcal{H}(D)$ 知 $\int\limits_{\partial D} \dfrac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z = 0$. 我们有

① 在 γ_1 上, 由于 $\lim\limits_{z\to2\pi\mathrm{i}} (z-2\pi\mathrm{i})\dfrac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1} = \lim\limits_{z\to2\pi\mathrm{i}} \dfrac{\mathrm{e}^{\mathrm{i}z}}{\frac{\mathrm{e}^z-1}{z-2\pi\mathrm{i}}} = \dfrac{\mathrm{e}^{\mathrm{i}z}}{(\mathrm{e}^z-1)'}\Big|_{z=2\pi\mathrm{i}} = \mathrm{e}^{-2\pi}$, 若记 $M(\varepsilon) = \max\limits_{\gamma_1(\varepsilon)} \left|\dfrac{\mathrm{e}^{\mathrm{i}z}(z-2\pi\mathrm{i})}{\mathrm{e}^z-1} - \mathrm{e}^{-2\pi}\right|$, 则 $\lim\limits_{\varepsilon\to0^+} M(\varepsilon) = 0$. 当 $z = 2\pi\mathrm{i} + \varepsilon\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z = \mathrm{i}(z-2\pi\mathrm{i})\,\mathrm{d}\theta$, 因此

$$\left|\int\limits_{\gamma_1} \frac{\frac{\mathrm{e}^{\mathrm{i}z}(z-2\pi\mathrm{i})}{\mathrm{e}^z-1} - \mathrm{e}^{-2\pi}}{z-2\pi\mathrm{i}}\,\mathrm{d}z\right| \leqslant \int_{-\frac{\pi}{2}}^{0} M(\varepsilon)\,\mathrm{d}\theta = \frac{\pi}{2}M(\varepsilon) \xrightarrow{\varepsilon\to0^+} 0,$$

即

$$\lim_{\varepsilon\to0^+} \int\limits_{\gamma_1} \frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z = \lim_{\varepsilon\to0^+} \int\limits_{\gamma_1} \frac{\mathrm{e}^{-2\pi}}{z-2\pi\mathrm{i}}\,\mathrm{d}z = -\frac{\pi\mathrm{i}}{2}\mathrm{e}^{-2\pi}.$$

② 在 γ_2 上,

$$\mathrm{Im}\left\{\int\limits_{\gamma_2} \frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^z-1}\,\mathrm{d}z\right\} = \mathrm{Im}\left\{\int_{2\pi-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^{-t}}{\mathrm{e}^{\mathrm{i}t}-1}\,\mathrm{i}\,\mathrm{d}t\right\} \xrightarrow{\varepsilon\to0^+} -\int_0^{2\pi} \mathrm{Re}\left\{\frac{\mathrm{e}^{-t}}{\mathrm{e}^{\mathrm{i}t}-1}\right\}\mathrm{d}t$$

$$= \int_0^{2\pi} \frac{e^{-t}(1-\cos t)}{2-2\cos t}\,dt = \frac{1-e^{-2\pi}}{2}.$$

③ 在 γ_3 上, 同 (1) 可得

$$\int_{\gamma_3} \frac{e^{iz}}{e^z-1}\,dz \xrightarrow{\varepsilon\to 0^+} -\frac{\pi i}{2}.$$

④ 在 γ_4 上,

$$\operatorname{Im}\left\{\int_{\gamma_4} \frac{e^{iz}}{e^z-1}\,dz\right\} \xrightarrow[R\to+\infty]{\varepsilon\to 0^+} \int_0^{+\infty} \operatorname{Im}\left\{\frac{\cos x + i\sin x}{e^x-1}\right\}dx = \int_0^{+\infty} \frac{\sin x}{e^x-1}\,dx.$$

⑤ 在 γ_5 上,

$$\left|\int_{\gamma_5} \frac{e^{iz}}{e^z-1}\,dz\right| = \left|\int_0^{2\pi} \frac{e^{i(R+it)}}{e^{R+it}-1}\,i\,dt\right| \leqslant \int_0^{2\pi} \frac{e^{-t}}{e^R-1}\,dt \leqslant \frac{2\pi}{e^R-1} \xrightarrow{R\to+\infty} 0.$$

⑥ 在 γ_6 上,

$$\begin{aligned}\operatorname{Im}\left\{\int_{\gamma_6} \frac{e^{iz}}{e^z-1}\,dz\right\} &= \operatorname{Im}\left\{\int_R^{\varepsilon} \frac{e^{i(x+2\pi i)}}{e^{x+2\pi i}-1}\,dx\right\} \xrightarrow[R\to+\infty]{\varepsilon\to 0^+} -\int_0^{+\infty} \operatorname{Im}\left\{\frac{e^{ix-2\pi}}{e^x-1}\right\}dx \\ &= -e^{-2\pi}\int_0^{+\infty} \frac{\sin x}{e^x-1}\,dx.\end{aligned}$$

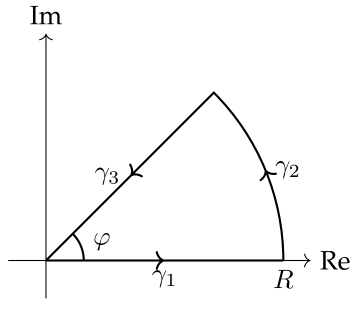
故

$$0 = \operatorname{Im}\left\{\int_{\gamma} \frac{e^{iz}}{e^z-1}\,dz\right\} = -\frac{\pi}{2}e^{-2\pi} + \frac{1-e^{-2\pi}}{2} - \frac{\pi}{2} + \left(1-e^{-2\pi}\right)\int_0^{+\infty} \frac{\sin x}{e^x-1}\,dx,$$

即

$$\int_0^{+\infty} \frac{\sin x}{e^x-1}\,dx = \frac{\pi}{2}\left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right) - \frac{1}{2}.$$

(28) $\int_0^{+\infty} e^{-ax^2}\cos\left(bx^2\right)dx = \operatorname{Re}\left\{\int_0^{+\infty} e^{(-a+bi)x^2}\right\}$. 下证当 $\operatorname{Re}(c) > 0$ 时, $\int_0^{+\infty} e^{-cz^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{c}}$.



选取如图积分路径. 设 $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ 所围区域为 D. 由 $e^{-cz^2} \in \mathcal{H}(D)$ 知 $\int_{\partial D} e^{-cz^2}\,dz = 0$.

① 在 γ_1 上, $\displaystyle\int_{\gamma_1} \mathrm{e}^{-cz^2}\,\mathrm{d}z \xrightarrow{R\to+\infty} \int_0^{+\infty} \mathrm{e}^{-cx^2}\,\mathrm{d}x$.

② 在 γ_2 上, 由于 $\lim\limits_{z\to\infty} z\mathrm{e}^{-cz^2}=0$, 若记 $M(R)=\max\limits_{\gamma_2(R)}\left|z\mathrm{e}^{-cz^2}\right|$, 则 $\lim\limits_{R\to+\infty} M(R)=0$. 当 $z=R\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z=\mathrm{i}z\,\mathrm{d}\theta$, 因此

$$\left|\int_{\gamma_2} \mathrm{e}^{-cz^2}\,\mathrm{d}z\right| = \left|\int_{\gamma_2} \frac{z\mathrm{e}^{-cz^2}}{z}\,\mathrm{d}z\right| \leqslant \int_0^{\varphi} M(R)\,\mathrm{d}\theta = \varphi M(R) \xrightarrow{R\to+\infty} 0.$$

③ 在 $\gamma_3: z=kt$ (待定 $k\in\mathbb{C}$ 于第一象限) 上,

$$\int_{\gamma_3} \mathrm{e}^{-cz^2}\,\mathrm{d}z \xrightarrow{R\to+\infty} \int_{+\infty}^0 \mathrm{e}^{-ck^2t^2}k\,\mathrm{d}t = -k\int_0^{+\infty} \mathrm{e}^{-ck^2t^2}\,\mathrm{d}t.$$

取 $k=\dfrac{1}{\sqrt{c}}$, 则

$$\int_{\gamma_3} \mathrm{e}^{-cz^2}\,\mathrm{d}z \xrightarrow{R\to+\infty} -\frac{1}{\sqrt{c}}\int_0^{+\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = -\frac{1}{\sqrt{c}}\cdot\frac{\sqrt{\pi}}{2}.$$

故

$$\int_0^{+\infty} \mathrm{e}^{-cx^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{c}}.$$

利用此结论即得

$$\int_0^{+\infty} \mathrm{e}^{-(a+b\mathrm{i})x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a-b\mathrm{i}}} = \frac{1}{2}\sqrt{\frac{\pi}{a^2+b^2}}\cdot\sqrt{a+b\mathrm{i}}.$$

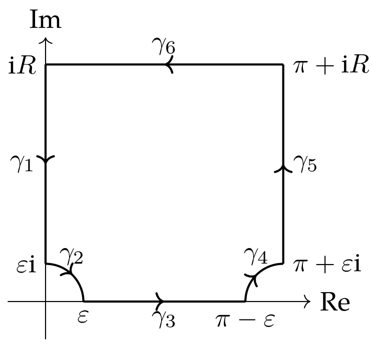
设 $\sqrt{a+b\mathrm{i}}=u+\mathrm{i}v$ $(u,v\in\mathbb{R})$, 则 $a=u^2-v^2$ 且 $b=2uv$, 从而

$$a=u^2-\left(\frac{b}{2u}\right)^2 \implies u^2=\frac{a+\sqrt{a^2+b^2}}{2} \xRightarrow{\text{不妨设 } b\geqslant 0} u=\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}},$$

故

$$\int_0^{+\infty} \mathrm{e}^{-ax^2}\cos\!\left(bx^2\right)\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a^2+b^2}}\cdot u = \frac{\sqrt{2\pi}}{4}\sqrt{\frac{\sqrt{a^2+b^2}+a}{a^2+b^2}}.$$

(29) 选取如图积分路径.



设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4\cup\gamma_5\cup\gamma_6$ 所围区域为 D. 由于 $\sin z$ 在 $\mathbb{C}$ 中零点为 $k\pi$ $(k\in\mathbb{Z})$, 因此 $\log\sin z\in\mathcal{H}(D)$, $\displaystyle\int_{\partial D}\log\sin z\,\mathrm{d}z=0$.

① 在 $\gamma_1: z=\mathrm{i}t$ 上, $\sin(\mathrm{i}t)=\dfrac{\mathrm{i}(\mathrm{e}^t-\mathrm{e}^{-t})}{2}$, 因此

$$\operatorname{Re}\left\{\int\limits_{\gamma_1}\log\sin z\,\mathrm{d}z\right\}=\operatorname{Re}\left\{\int_R^{\varepsilon}\log\sin(\mathrm{i}t)\mathrm{i}\,\mathrm{d}t\right\}=\int_{\varepsilon}^{R}\operatorname{Im}(\log\sin(\mathrm{i}t))\,\mathrm{d}t=\int_{\varepsilon}^{R}\frac{\pi}{2}\,\mathrm{d}t=\frac{\pi}{2}(R-\varepsilon).$$

② 在 γ_2 上, 由于 $\lim\limits_{z\to 0}z\log\sin z=0$, 若记 $M(\varepsilon)=\max\limits_{\gamma_2(\varepsilon)}|z\log\sin z|$, 则 $\lim\limits_{\varepsilon\to 0^+}M(\varepsilon)=0$. 当 $z=\varepsilon\mathrm{e}^{\mathrm{i}\theta}$ 时, $\mathrm{d}z=\mathrm{i}z\,\mathrm{d}\theta$, 因此

$$\left|\int\limits_{\gamma_2}\log\sin z\,\mathrm{d}z\right|=\left|\int\limits_{\gamma_2}\frac{z\log\sin z}{z}\,\mathrm{d}z\right|\leqslant\int_0^{\frac{\pi}{2}}M(\varepsilon)\,\mathrm{d}\theta=\frac{\pi}{2}M(\varepsilon)\xrightarrow{\varepsilon\to 0^+}0.$$

③ 在 γ_3 上,

$$\int\limits_{\gamma_3}\log\sin z\,\mathrm{d}z=\int_{\varepsilon}^{\pi-\varepsilon}\log\sin x\,\mathrm{d}x\xrightarrow{\varepsilon\to 0^+}\int_0^{\pi}\log\sin x\,\mathrm{d}x.$$

④ 在 γ_4 上, 由于 $\lim\limits_{z\to\pi}(z-\pi)\log\sin z=0$, 同 (2) 可得

$$\left|\int\limits_{\gamma_4}\frac{\log\sin z}{z}\right|\xrightarrow{\varepsilon\to 0^+}0.$$

⑤ 在 $\gamma_5: z=\pi+\mathrm{i}t$ 上, $\sin(\pi+\mathrm{i}t)=-\dfrac{\mathrm{i}}{2}(\mathrm{e}^t-\mathrm{e}^{-t})$, 因此

$$\begin{aligned}\operatorname{Re}\left\{\int\limits_{\gamma_5}\log\sin z\,\mathrm{d}z\right\}&=\operatorname{Re}\left\{\int_{\varepsilon}^{R}\log\sin(\pi+\mathrm{i}t)\mathrm{i}\,\mathrm{d}t\right\}=-\int_{\varepsilon}^{R}\operatorname{Im}(\log\sin(\pi+\mathrm{i}t))\,\mathrm{d}t\\&=-\int_{\varepsilon}^{R}-\frac{\pi}{2}\,\mathrm{d}t=\frac{\pi}{2}(R-\varepsilon).\end{aligned}$$

⑥ 在 $\gamma_6: z=t+\mathrm{i}R$ 上, 由

$$\sin(t+\mathrm{i}R)=\frac{1}{2}\left(\mathrm{e}^{-R}+\mathrm{e}^{R}\right)\sin t+\mathrm{i}\cdot\frac{1}{2}\left(\mathrm{e}^{R}-\mathrm{e}^{-R}\right)\cos t$$

可知

$$\begin{aligned}|\sin(t+\mathrm{i}R)|^2&=\frac{1}{4}\left(\mathrm{e}^{-R}+\mathrm{e}^{R}\right)^2\sin^2 t+\frac{1}{4}\left(\mathrm{e}^{R}-\mathrm{e}^{-R}\right)^2\cos^2 t\\&=\frac{1}{4}\left[\left(\mathrm{e}^{2R}+\mathrm{e}^{-2R}\right)\left(\sin^2 t+\cos^2 t\right)+2\left(\sin^2 t-\cos^2 t\right)\right]\\&=\frac{1}{4}\mathrm{e}^{2R}(1+\mu(R)),\end{aligned}$$

其中 $\lim\limits_{R\to+\infty}\mu(R)=0$. 于是

$$\log|\sin(t+\mathrm{i}R)|^2=\log\left(\frac{1}{4}\mathrm{e}^{2R}\right)+\log(1+\mu(R))\xrightarrow{R\to+\infty}2R-2\log 2,$$

从而

$$\mathrm{Re}\left\{\int_{\gamma_6}\log\sin z\,\mathrm{d}z\right\}=\mathrm{Re}\left\{\int_{\pi}^{0}\log\sin(t+\mathrm{i}R)\,\mathrm{d}t\right\}=-\int_0^{\pi}\log|\sin(t+\mathrm{i}R)|\,\mathrm{d}t$$

$$\xrightarrow{R\to+\infty}-\frac{1}{2}(2R-2\log 2)\pi=\pi(\log 2-R).$$

故

$$0=\mathrm{Re}\left\{\int_{\gamma}\log\sin z\,\mathrm{d}z\right\}\xrightarrow[R\to+\infty]{\varepsilon\to0^+}\int_0^{\pi}\log\sin x\,\mathrm{d}x+\pi\log 2,$$

即

$$\int_0^{\frac{\pi}{2}}\log\sin x\,\mathrm{d}x=\frac{1}{2}\int_0^{\pi}\log\sin x\,\mathrm{d}x=-\frac{\pi}{2}\log 2.$$

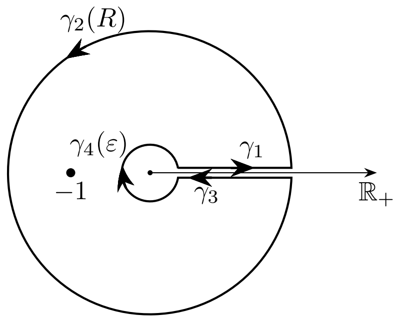
□

习题 5.5.2 设 $f(z)$ 是有理函数, 在 $[0,+\infty)$ 上无极点, 并且 ∞ 是 $f(z)$ 的零点. 证明:

$$\int_0^{+\infty}\frac{f(x)}{(\log x)^2+\pi^2}\,\mathrm{d}x=\sum_{k=1}^{n}\mathrm{Res}\left(\frac{f(z)}{\mathrm{Log}\,z-\pi\mathrm{i}},a_k\right),$$

其中 $a_1=-1,a_2,a_3,\cdots,a_n$ 是 $f(z)$ 在 $\mathbb{C}$ 中的全部彼此不同的极点, $\mathrm{Log}\,z=\log|z|+\mathrm{i}\,\mathrm{Arg}\,z, 0<\mathrm{Arg}\,z<2\pi, z\in\mathbb{C}\setminus[0,+\infty)$.

证明 选取如图“锁钥”路径.



设 $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ 包含 $f(z)$ 的全部极点. 由留数定理,

$$\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\frac{f(z)}{\mathrm{Log}\,z-\pi\mathrm{i}}\,\mathrm{d}z=\sum_{k=1}^{n}\mathrm{Res}\left(\frac{f(z)}{\mathrm{Log}\,z-\pi\mathrm{i}},a_k\right).$$

我们有

$$\int_{\gamma_1}\frac{f(z)}{\mathrm{Log}\,z-\pi\mathrm{i}}\,\mathrm{d}z=\int_{\varepsilon}^{R}\frac{f(z)}{\log x-\pi\mathrm{i}}\,\mathrm{d}x\xrightarrow[\varepsilon\to0^+]{R\to+\infty}\int_0^{+\infty}\frac{f(x)}{\log x-\pi\mathrm{i}}\,\mathrm{d}x,$$

$$\left|\int_{\gamma_2}\frac{f(z)}{\mathrm{Log}\,z-\pi\mathrm{i}}\,\mathrm{d}z\right|\leqslant\int_{\gamma_2}\frac{|f(z)||\,\mathrm{d}z|}{|\mathrm{Log}\,z-\pi\mathrm{i}|}\leqslant\int_{\gamma_2}\frac{|f(z)|}{\log R}|\,\mathrm{d}z|\leqslant\frac{2\pi R\max\limits_{|z|=R}|f(z)|}{\log R}\xrightarrow[f\text{ 有理},\ f(\infty)=0]{R\to+\infty}0,$$

$$\int\limits_{\gamma_3} \frac{f(z)}{\operatorname{Log} z - \pi\mathrm{i}}\,\mathrm{d}z = \int_R^{\varepsilon} \frac{f(x)}{\log x + 2\pi\mathrm{i} - \pi\mathrm{i}}\,\mathrm{d}x \xrightarrow[\varepsilon\to0^+]{R\to+\infty} -\int_0^{+\infty} \frac{f(x)}{\log x + \pi\mathrm{i}}\,\mathrm{d}x,$$

$$\left|\int\limits_{\gamma_4} \frac{f(z)}{\operatorname{Log} z - \pi\mathrm{i}}\,\mathrm{d}z\right| \leqslant \int\limits_{\gamma_4} \frac{|f(z)||\,\mathrm{d}z|}{|\operatorname{Log} z - \pi\mathrm{i}|} \leqslant \frac{2\pi\varepsilon \max\limits_{|z|=\varepsilon} |f(z)|}{|\log\varepsilon|} \xrightarrow{\varepsilon\to0^+} 0.$$

因此在 $\varepsilon \to 0^+, R \to +\infty$ 时就有

$$\frac{1}{2\pi\mathrm{i}}\left\{\int_0^{+\infty} \frac{f(x)}{\log x - \pi\mathrm{i}}\,\mathrm{d}x - \int_0^{+\infty} \frac{f(x)}{\log x + \pi\mathrm{i}}\,\mathrm{d}x\right\} = \sum_{k=1}^{n} \operatorname{Res}\left(\frac{f(z)}{\operatorname{Log} z - \pi\mathrm{i}}, a_k\right),$$

即

$$\int_0^{+\infty} \frac{f(x)}{(\log x)^2 + \pi^2}\,\mathrm{d}x = \sum_{k=1}^{n} \operatorname{Res}\left(\frac{f(z)}{\operatorname{Log} z - \pi\mathrm{i}}, a_k\right).$$

□

习题 6.2.6 证明：$\sum\limits_{n=0}^{\infty} z^{2^n}$ 的收敛圆周上的每个点皆为其和函数的奇异点.

证明 级数收敛半径为 1. 注意到对正整数 k, ℓ, 有

$$\sum_{n=0}^{\infty}\left(\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z\right)^{2^n} = \sum_{n=0}^{k-1}\left(\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z\right)^{2^n} + \sum_{n=k}^{\infty} z^{2^n},$$

因此在收敛圆周上 z 与 $\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z$ 同为奇异点或正则点, 而 1 显然是奇异点, 由 $\left\{\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}}\right\}_{k,\ell\geqslant1}$ 在收敛圆周上稠密即知收敛圆周上的每个点皆为和函数的奇异点. □

习题 6.2.7 证明：$\sum\limits_{n=0}^{\infty} \dfrac{z^{2^n}}{2^n}$ 的收敛圆周上的每个点皆为其和函数的奇异点.

证明 级数收敛半径为 1. 注意到对正整数 k, ℓ, 有

$$\sum_{n=0}^{\infty}\left(\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z\right)^{2^n} = \sum_{n=0}^{k-1}\left(\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z\right)^{2^n} + \sum_{n=k}^{\infty} \frac{z^{2^n}}{2^n},$$

因此在收敛圆周上 z 与 $\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}} z$ 同为奇异点或正则点, 而 2 显然是奇异点, 由 $\left\{\mathrm{e}^{2\pi\mathrm{i}\frac{\ell}{2^k}}\right\}_{k,\ell\geqslant1}$ 在收敛圆周上稠密即知收敛圆周上的每个点皆为和函数的奇异点. □

习题 6.2.8 设 $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ 的收敛半径为 1, $a_n \in \mathbb{R}$ $(n \geqslant 0)$, $S_n = \sum\limits_{k=0}^{n} a_k$. 证明：若 $S_n \to \infty$ $(n \to \infty)$, 则 1 是 $f(z)$ 的奇异点. 举例说明, 仅有 $|S_n| \to \infty$ 不能保证 1 是 $f(z)$ 的奇异点.

证明 (1) 若 1 不是 f 的奇异点, 则 f 在 1 的某个邻域中全纯且 $f(1)$ 存在. 由于 f 限制在实轴上为实值函数, 故 $f(1) \in \mathbb{R}$. 考虑 $g(z) = \dfrac{f(z) - f(1)}{1 - z}$, 则由全纯函数的解析性知 g 在 1 处全纯. 由于 $f(z)$ 在单位圆周上必有奇异点, 因此 $g(z)$ 在单位圆周上必有非 1 的奇异点, 从而 $g(z)$ 的的幂级数的收敛半径仍为 1. 而 $g(z)$ 的幂级数为

$$g(z) = \left(\sum_{n=0}^{\infty} a_n z^n - f(1)\right)\left(\sum_{m=0}^{\infty} z^m\right) = \sum_{n=0}^{\infty}[S_n - f(1)]z^n,$$

由于 $S_n - f(1) \to \infty$, 当 n 充分大时 $S_n - f(1) > 0$, 由定理 6.2.4 知 1 是 $g(z)$ 的奇点, 矛盾.

(2) 考虑 $f(z)=\sum_{n=0}^{\infty}(-1)^n(n+1)z^n$, 其收敛半径为 1, $S_n=(-1)^n\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)$. 但当 $|z|<1$ 时,

$$f(z)=\sum_{n=0}^{\infty}\left[(-1)^n z^{n+1}\right]'=\left(z\sum_{n=0}^{\infty}(-z)^n\right)'=\left(\frac{z}{1+z}\right)'=\frac{1}{(1+z)^2},$$

由于 $\left.\frac{1}{(1+z)^2}\right|_{z=1}=\frac{1}{4}$, 因此 1 是 $f(z)$ 的正则点. □

习题 7.1.1 设 $\{f_n\}$ 是域 D 上的全纯函数列, 并且在 D 上内闭一致有界. 证明：若 $\lim_{n\to\infty}f_n(z)$ 在 D 上处处存在, 则 $\{f_n\}$ 在 D 上内闭一致收敛.

证明 记 $f(z)=\lim_{n\to\infty}f_n(z)$. 由于 $\{f_n\}$ 在 D 上内闭一致有界, 由 Montel 定理, $\{f_n\}$ 是正规族. 若 $\{f_n\}$ 在 D 上非内闭一致收敛, 则存在紧集 $K\subset D$ 与子列 $\{f_{n_k}\}$, 使得

$$\sup_{z\in K}|f_{n_k}(z)-f(z)|\geqslant\varepsilon>0,\quad\forall k.$$

于是该子列 $\{f_{n_k}\}$ 在 K 上无一致收敛子列, 这与 $\{f_n\}$ 是正规族矛盾. □

习题 7.1.2 设 $\{f_n\}$ 是域 D 上的全纯函数列, 并且在 D 上内闭一致有界, $A=\{x+\mathrm{i}y\in D:x,y\in\mathbb{Q}\}$. 证明：若 $\lim_{n\to\infty}f_n(z)$ 在 A 上处处存在, 则 $\{f_n\}$ 在 D 上内闭一致收敛.

证明 用反证法, 假设 $\{f_n\}$ 在 D 上非内闭一致收敛, 则存在紧集 $K\subset D$, 使得 $\{f_n\}$ 在 K 上非一致收敛. 由于 $\{f_n\}$ 在 D 上内闭一致有界, 由 Montel 定理, $\{f_n\}$ 是 D 上的正规族, 因此存在子列 $\{f_{n_k}\}$ 在 K 上一致收敛, 记极限函数为 f. 由于 $\{f_n\}$ 在 K 上不一致收敛, 存在子列 $\{f_{n_j}\}$ 使得

$$\sup_{z\in K}\left|f_{n_j}(z)-f(z)\right|\geqslant\varepsilon>0,\quad\forall j.$$

由于 $\{f_n\}$ 是正规族, 对于子列 $\{f_{n_j}\}$, 存在其子列 $\left\{f_{n_{j_\ell}}\right\}$ 在 K 上一致收敛, 记极限函数为 $\widetilde{f}$. 由于 $f,\widetilde{f}\in\mathcal{H}(K)$, 且 $f|_A=\widetilde{f}|_A$, $A\cap K$ 在 K 中稠密, 由全纯函数零点孤立性即知 $f=\widetilde{f}$. 于是 $f_{n_{j_\ell}}\rightrightarrows f$, 与

$$\sup_{z\in K}\left|f_{n_{j_\ell}}(z)-f(z)\right|\geqslant\varepsilon>0,\quad\forall\ell$$

矛盾. 故 $\{f_n\}$ 在 D 上内闭一致收敛. □

习题 7.1.4 设 $\mathcal{F}$ 是域 D 上的全纯函数族, $z_0\in D$. 证明：若

(1) $\operatorname{Re}f(z)\geqslant 0,\forall z\in D, f\in\mathcal{F}$;

(2) $f(z_0)=g(z_0),\forall f,g\in\mathcal{F}$,

则 $\mathcal{F}$ 是 D 上的正规族. 并举例说明条件 (2) 是不可去掉的.

证明 (1) 由 Montel 定理, 只需证 $\mathcal{F}$ 在 D 上内闭一致有界, 结合有限覆盖定理, 只需证 $\mathcal{F}$ 在 D 中任一圆盘上一致有界, 故不妨设 D 为单位圆盘, $z_0=0$, $f(0)=w$, 其中 $f\in\mathcal{F}$. 进一步地, 可不妨设 $|w|\leqslant 1$. 考虑 $g(z)=\frac{f(z)-1}{f(z)+1}$, 令 $h(z)=\frac{g(0)-g(z)}{1-\overline{g(0)}g(z)}$, 则 $h(0)=0$ 且 $|h(z)|\leqslant 1$, 由 Schwarz 引理知 $|h(z)|\leqslant|z|$. 由此可得 $\mathcal{F}$ 在 D 上内闭一致有界, 结论得证.

(2) 考虑 $f_n(z)=n$, 则 $\{f_n(z)\}_{n=0}^{\infty}$ 是 $D=\{z\in\mathbb{C}:\operatorname{Re} z>0\}$ 上的全纯函数列, 条件 (2) 显然不满足, 此时 $\{f_n(z)\}$ 在 D 上不一致有界, 因此不是正规族. □

习题 7.1.5 设 $\mathcal{F}$ 是域 D 上的正规全纯函数族, g 是整函数. 证明: $\{g\circ f: f\in\mathcal{F}\}$ 也是 D 上的正规族.

证明 $\{g\circ f: f\in\mathcal{F}\}$ 在 D 上显然内闭一致有界, 因此是 D 上的正规族. □

习题 7.1.6 设 D 是有界域, $0<M<+\infty$. 证明:

$$\mathcal{F}=\left\{f\in\mathcal{H}(D):\iint\limits_D |f(z)|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant M\right\}$$

是 D 上的正规族.

证明 对任意紧集 $K\subset D$, 由有限覆盖定理可知, 存在 $R>0$, 使得对任意 $z\in D$ 均有 $\mathbb{B}(z,R)\subset D$. 由定理 8.4.5 知, 对任意 $f\in\mathcal{F}$, $|f|^2$ 都是 D 上的次调和函数, 因此

$$|f(z)|^2\leqslant\frac{1}{\pi R^2}\int\limits_{\mathbb{B}(z,R)}|f(\zeta)|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{1}{\pi R^2}\int\limits_D|f(\zeta)|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{M}{\pi R^2},$$

因此 $f(z)$ 在 D 上内闭一致有界, 由 Montel 定理, $\mathcal{F}$ 是 D 上的正规族. □

习题 7.2.1 (推广的 Liouville 定理) 设 D 是异于 $\mathbb{C}$ 的单连通域. 证明: 若 f 是整函数, 并且 $f(\mathbb{C})\subset D$, 则 f 是常值函数.

证明 由 Riemann 映射定理, 可取双全纯变换 $g:D\to\mathbb{B}(0,1)$, 则 $g\circ f$ 为有界整函数, 由 Liouville 定理, $g\circ f$ 为常值函数, 从而 f 为常值函数. □

习题 7.2.2 设 D 是异于 $\mathbb{C}$ 的单连通域, $a\in D$. 证明: 若 f 将 D 双全纯地映为 $\mathbb{B}(0,1)$, 并且 $f(a)=0$, $f'(a)>0$, 则

$$\min_{z\in\partial D}|z-a|\leqslant\frac{1}{f'(a)}\leqslant\max_{z\in\partial D}|z-a|.$$

称 $\dfrac{1}{f'(a)}$ 为 D 在 a 处的映射半径.

证明 令 $F(w)=\begin{cases}\dfrac{f^{-1}(w)-a}{w}, & w\in\mathbb{B}(0,1)\setminus\{0\},\\ \dfrac{1}{f'(a)}, & w=0.\end{cases}$ 由 Morera 定理易知 $F\in\mathcal{H}(\mathbb{B}(0,1))$. 由最大模原理,

$$\min_{|w|=1}\left|f^{-1}(w)-a\right|=\min_{|w|=1}|F(w)|\leqslant|F(0)|\leqslant\max_{|w|=1}|F(w)|=\max_{|w|=1}\left|f^{-1}(w)-a\right|.$$

由边界对应定理, f^{-1} 将 $\partial\mathbb{B}(0,1)$ 一一地映为 ∂D, 因此上式可改写为

$$\min_{z\in\partial D}|z-a|\leqslant\frac{1}{f'(a)}\leqslant\max_{z\in\partial D}|z-a|.$$

□

习题 7.2.3 设 D 是异于 $\mathbb{C}$ 的单连通域, $a\in D$, f 将 D 双全纯地映为 $\mathbb{B}(0,1)$, 并且 $f(a)=0, f'(a)>0$. 证明: 若 g 将 D 双全纯地映为 $\mathbb{B}(0,1)$, $p=g^{-1}(0)$, 则

$$g(z)=\frac{g'(a)}{|g'(a)|}\cdot\frac{f(z)-f(p)}{1-\overline{f(p)}f(z)}.$$

证明 由于 $g\circ f^{-1}\in \mathrm{Aut}(\mathbb{D})$, 故它具有形式 $g\circ f^{-1}(z)=\mathrm{e}^{\mathrm{i}\theta}\dfrac{z-z_0}{1-\overline{z_0}z}$, 其中 $|z_0|<1,\theta\in\mathbb{R}$ 待定. 由于 $g\circ f^{-1}(f(p))=g(p)=0$, 因此 $z_0=f(p)$. 由

$$\frac{g'(a)}{f'(a)}=\left(g\circ f^{-1}\right)'(0)=\mathrm{e}^{\mathrm{i}\theta}\left(\frac{z-f(p)}{1-\overline{f(p)}z}\right)'\Bigg|_{z=0}=\mathrm{e}^{\mathrm{i}\theta}\left(1-|f(p)|^2\right)$$

及 $f'(a)>0,|f(p)|<1$ 可知 $\mathrm{e}^{\mathrm{i}\theta}=\dfrac{g'(a)}{|g'(a)|}$. 故

$$g\circ f^{-1}(z)=\frac{g'(a)}{|g'(a)|}\cdot\frac{z-f(p)}{1-\overline{f(p)}z}\xrightarrow{z\to f(z)}g(z)=\frac{g'(a)}{|g'(a)|}\cdot\frac{f(z)-f(p)}{1-\overline{f(p)}f(z)}.$$

□

习题 7.2.4 设 D 为异于 $\mathbb{C}$ 的凸域, $a\in D$, $\mathcal{F}=\{f\in\mathcal{H}(D):f(a)=0,f'(a)>0\}$. 证明: $\mathcal{F}$ 中满足 $f(D)=\mathbb{B}(0,1)$ 和 $\mathrm{Re}\,f'(z)\geqslant 0\ (\forall z\in D)$ 的 f 最多只有一个.

证明 对任意 $f\in\mathcal{F}$, 若 $\mathrm{Re}\,f'(z)\geqslant 0,\forall z\in D$, 我们证明 f 必为单叶函数. 用反证法, 假设存在不同的两点 $z_1,z_2\in D$, 使得 $f(z_1)=f(z_2)$, 由于 D 为凸域 (从而为单连通域), 我们有

$$0=f(z_2)-f(z_1)=\int\limits_{[z_1,z_2]}f'(\zeta)\,\mathrm{d}\zeta=\int_0^1 f'(z_1+t(z_2-z_1))(z_2-z_1)\,\mathrm{d}t,$$

因此

$$\int_0^1 f'(z_1+t(z_2-z_1))\,\mathrm{d}t=0\implies\int_0^1\mathrm{Re}\,f'(z_1+t(z_2-z_1))\,\mathrm{d}t=0$$
$$\xRightarrow{\mathrm{Re}\,f'(z)\geqslant 0}\mathrm{Re}\,f'(z)=0,\quad\forall z\in[z_1,z_2].$$

同习题 2.2.2 (1) 可知 $f'(z)$ 在 $[z_1,z_2]$ 上为常数, 再由全纯函数零点孤立性可知 $f'(z)$ 在 D 上为常数, 从而在 D 上 $\mathrm{Re}\,f'(z)\equiv 0$, 这与 $\mathrm{Re}\,f'(a)=f'(a)>0$ 矛盾. 故 f 为单叶函数, 结合 $f(D)=\mathbb{B}(0,1)$, 由 Riemann 映射定理, f 唯一. □

习题 7.2.7 设 D 是异于 $\mathbb{C}$ 的单连通域, $a\in D$, R 为 D 在 a 处的映射半径 (定义见习题 7.2.2). 证明: 若 $F\in\mathcal{H}(D),F(a)=0,F'(a)=1$, 则

$$\iint\limits_D\left|F'(z)\right|^2\mathrm{d}x\,\mathrm{d}y\geqslant\pi R^2.$$

等号成立当且仅当 F 是将 D 映为 $\mathbb{B}(0,R)$ 的双全纯映射.

证明 令 f 为从 $\mathbb{B}(0,1)$ 到 D 的双全纯映射, 满足 $f(0)=a,f'(0)>0$. 由于 f 作为 $\mathbb{R}^2$ 上映射的 Jacobi 行列式为 $|f'|^2$, 且由定理 8.4.5, $|(F\circ f)'|^2$ 为次调和函数, 我们有

$$\iint\limits_D\left|F'(z)\right|^2\mathrm{d}x\,\mathrm{d}y=\iint\limits_{\mathbb{B}(0,1)}\left|F'\circ f(w)\right|^2\left|f'(w)\right|^2\mathrm{d}x\,\mathrm{d}y=\iint\limits_{\mathbb{B}(0,1)}\left|(F\circ f)'(w)\right|^2\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant\pi\left|(F\circ f)'(0)\right|^2=\pi\left|F'(a)f'(0)\right|^2=\frac{\pi}{\left|(f^{-1})'(a)\right|^2}=\pi R^2.$$

等号成立当且仅当 $|(F\circ f)'|$ 为常值函数, 由习题 2.2.2, $(F\circ f)'$ 为常值函数, 结合 $F\circ f(0)=F(a)=0$ 即

知 $F\circ f(z)=cz$, 其中 $c\in\mathbb{C}$, 故 $F(z)=cf^{-1}(z)$ 是将 D 映为 $\mathbb{B}(0,R)$ 的双全纯映射. □

习题 7.3.1 利用 Schwarz 对称原理和边界对应定理证明：将 $\mathbb{B}(0,1)$ 映为自身的双全纯映射一定是分式线性变换.

证明 任取将 $\mathbb{B}(0,1)$ 映为自身的双全纯映射 f, 由边界对应定理, f 可延拓为 $\overline{\mathbb{B}(0,1)}$ 上的连续函数, 且将 $\partial\mathbb{B}(0,1)$ 一一地映为 $\partial\mathbb{B}(0,1)$. 于是 $f(z)$ 可延拓为 $\widetilde{f}(z)=\begin{cases} f(z), & |z|\leqslant 1,\\ \dfrac{1}{\overline{f\left(\frac{1}{\overline{z}}\right)}}, & |z|>1.\end{cases}$ 由于 f 在 $\mathbb{B}(0,1)$ 上有且仅有 1 个零点, $\widetilde{f}$ 在 $\partial\mathbb{B}(0,1)$ 上连续, 由 Painlevé 原理可知 $\widetilde{f}$ 为 $\overline{\mathbb{C}}$ 上的亚纯函数, 进而 $\widetilde{f}\in\mathrm{Aut}(\overline{\mathbb{C}})$. 由定理 5.3.5, $\widetilde{f}$ 为分式线性变换, 从而 f 为分式线性变换. □

习题 7.3.3 设 D 是由简单闭曲线所围成的单连通域, $z_1,z_2,z_3\in\partial D$ 是彼此不同的三点, 按 ∂D 的正向排列. 证明：若 $w_1,w_2,w_3\in\partial\mathbb{B}(0,1)$ 是彼此不同的三点, 按 $\partial\mathbb{B}(0,1)$ 的正向排列, 则存在唯一的 φ, 将 D 双全纯地映为 $\mathbb{B}(0,1)$, 将 $\overline{D}$ 同胚地映为 $\overline{\mathbb{B}(0,1)}$, 并且 $f(z_k)=w_k, k=1,2,3$.

证明 **(存在性)** 由 Riemann 映射定理与边界对应定理, 存在函数 f, 将 D 双全纯地映为 $\mathbb{B}(0,1)$, 并将 $\overline{D}$ 同胚地映为 $\overline{\mathbb{B}(0,1)}$, 再取分式线性变换 g 使得 $g(f(z_i))=w_i$ $(i=1,2,3)$, 由分式线性变换的保圆性即知 $\varphi\coloneqq g\circ f$ 为所求.

(唯一性) 设函数 φ_1,φ_2 均满足题意, 则 $\varphi_1\circ\varphi_2^{-1}$ 是 $\mathbb{B}(0,1)$ 的全纯自同构 (从而为分式线性变换), 且 $\varphi_1\circ\varphi_2^{-1}(w_i)=w_i$ $(i=1,2,3)$, 由于三点可确定一个分式线性变换, $\varphi_1\circ\varphi_2^{-1}=\mathrm{Id}$, 即 $\varphi_1=\varphi_2$. □

习题 7.3.5 设 $f\in\mathcal{H}(\mathbb{B}(0,1)), f(0)=0, f'(0)=a>0$. 证明：若 $f(\mathbb{B}(0,1))\subset\mathbb{B}(0,1)$, 则 f 在 $\mathbb{B}\left(0,\dfrac{a}{1+\sqrt{1-a^2}}\right)$ 上双全纯.

证明 由 Schwarz 引理知 $a=f'(0)\in(0,1)$. 设 $f(z)$ 在 $\mathbb{B}(0,\rho)$ 上非单叶函数, 则存在不同的两点 $z_1,z_2\in\mathbb{B}(0,\rho)$ 使得 $f(z_1)=f(z_2)$. 由于 z_1,z_2 均为 $f(z)-f(z_1)$ 的零点, 由定理 4.4.1,

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=\rho}\frac{f'(z)}{f(z)-f(z_1)}\,\mathrm{d}z\geqslant 2.$$

记 $\gamma_\rho=f(\partial\mathbb{B}(0,\rho))$, 则 γ_ρ 不是简单闭曲线, 否则

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=\rho}\frac{f'(z)}{f(z)-f(z_1)}\,\mathrm{d}z=\frac{1}{2\pi\mathrm{i}}\int\limits_{|z|=\rho}\mathrm{d}\,\mathrm{Log}(f(z)-f(z_1))\overset{w=f(z)}{=\!=\!=}\frac{1}{2\pi}\Delta_{\gamma_\rho}\mathrm{Arg}(w-f(z_1))=1,$$

与前一式矛盾. 因此 γ_ρ 自交, 即存在不同的两点 $\zeta_1,\zeta_2\in\partial\mathbb{B}(0,\rho)$, 使得 $f(\zeta_1)=f(\zeta_2)$. 由习题 4.5.20 即得 $|f(\zeta_1)|\leqslant\rho^2$. 而由习题 4.5.21,

$$|\zeta_1|\frac{a-|\zeta_1|}{1-a|\zeta_1|}\leqslant|f(\zeta_1)|\implies\rho\cdot\frac{a-\rho}{1-a\rho}\leqslant|f(\zeta_1)|\leqslant\rho^2\implies\rho\geqslant\frac{1-\sqrt{1-a^2}}{a}=\frac{a}{1+\sqrt{1-a^2}}.$$

故 f 在 $\mathbb{B}\left(0,\dfrac{a}{1+\sqrt{1-a^2}}\right)$ 上双全纯. □

补充题 1 求分式线性变换 $T\in\mathrm{Aut}(\mathbb{D})$, 使得 $T(1)=\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}$ 且 $T(a)=\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$, 其中 $|a|=1$.

解答 注意到 $\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}$ 与 $\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$ 为对径点, 故先求分式线性变换 $w\in\mathrm{Aut}(\mathbb{D})$ 使得 $w(1)=1, w(-1)=a$. 设

$w(z)=\mathrm{e}^{\mathrm{i}\theta}\dfrac{z-z_0}{1-\overline{z_0}z}$, 其中 $|z_0|<1,\theta\in\mathbb{R}$ 待定. 我们有

$$\begin{cases}\mathrm{e}^{\mathrm{i}\theta}\dfrac{1-z_0}{1-\overline{z_0}}=1,\\ \mathrm{e}^{\mathrm{i}\theta}\dfrac{-1-z_0}{1+\overline{z_0}}=a\end{cases}\implies \overline{z_0}=1-\mathrm{e}^{\mathrm{i}\theta}(1-z_0)\xRightarrow{\text{回代}} z_0=\frac{a-1}{a+1}-\frac{2a}{a+1}\mathrm{e}^{-\mathrm{i}\theta}$$
$$\xRightarrow{\text{回代}}\mathrm{e}^{\mathrm{i}\theta}=\frac{a+1}{\bar{a}+1}\xRightarrow{\text{回代}} z_0=\frac{a-3}{a+1}.$$

由 $w^{-1}:1\mapsto 1,a\mapsto -1$ 知

$$T(z)=\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}w^{-1}(z)=\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}\cdot\frac{z+\mathrm{e}^{\mathrm{i}\theta}z_0}{\mathrm{e}^{\mathrm{i}\theta}+\overline{z_0}z}=\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}\cdot\frac{(\bar{a}+1)z+(a-3)}{(\bar{a}-3)z+(a+1)}.$$

□

补充题 2 对 $t>0$ 定义 $\vartheta(t)=\displaystyle\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\pi n^2 t}$, 证明：$\vartheta(t)=t^{-\frac{1}{2}}\vartheta\left(\frac{1}{t}\right)$.

证明 令 $f(z)=\mathrm{e}^{-\pi z^2 t}$, 则

$$\begin{aligned}\hat{f}(\xi)&=\int_{\mathbb{R}}\mathrm{e}^{-\pi x^2 t}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x=\mathrm{e}^{-\frac{\pi\xi^2}{t}}\int_{\mathbb{R}}\mathrm{e}^{-\pi t\left(x+\frac{\mathrm{i}\xi}{t}\right)^2}\,\mathrm{d}x\\&=\mathrm{e}^{-\frac{\pi\xi^2}{t}}\int_{\mathbb{R}}\mathrm{e}^{-\pi t x^2}\,\mathrm{d}x\xlongequal{t>0}2\mathrm{e}^{-\frac{\pi\xi^2}{t}}\cdot\frac{1}{2\sqrt{t}}=\frac{1}{\sqrt{t}}\mathrm{e}^{-\frac{\pi\xi^2}{t}}.\end{aligned}$$

由于 $f\in\mathfrak{F}$, 由 Poisson 求和公式得

$$\vartheta(t)=\sum_{n=-\infty}^{\infty}f(n)=\sum_{n=-\infty}^{\infty}\hat{f}(n)=\frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi n^2}{t}}=t^{-\frac{1}{2}}\vartheta\left(\frac{1}{t}\right).$$

□

补充题 3 设 $t>0,a\in\mathbb{R}$. 证明：$\displaystyle\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}an}}{\cosh\left(\frac{\pi n}{t}\right)}=\sum_{n=-\infty}^{\infty}\frac{t}{\cosh(\pi(n+a)t)}$.

证明 由于 $\dfrac{1}{\cosh\pi x}$ 是 Fourier 变换的不动点,

$$\int_{\mathbb{R}}\frac{\mathrm{e}^{-2\pi\mathrm{i}x\xi}}{\cosh\pi x}\,\mathrm{d}x=\frac{1}{\cosh\pi\xi},$$

由此可知 $f(z)=\dfrac{\mathrm{e}^{-2\pi\mathrm{i}az}}{\cosh\left(\frac{\pi z}{t}\right)}$ 的 Fourier 变换为

$$\hat{f}(\xi)=\int_{\mathbb{R}}\frac{\mathrm{e}^{-2\pi\mathrm{i}x(a+\xi)}}{\cosh\left(\frac{\pi x}{t}\right)}\,\mathrm{d}x\xlongequal{x=ty}t\int_{\mathbb{R}}\frac{\mathrm{e}^{-2\pi\mathrm{i}y[t(a+\xi)]}}{\cosh(\pi y)}\,\mathrm{d}y=\frac{t}{\cosh(\pi(\xi+a)t)}.$$

由

$$|f(x)|=\left|\frac{\mathrm{e}^{-2\pi\mathrm{i}ax}}{\cosh\left(\frac{\pi x}{t}\right)}\right|=\frac{2}{\mathrm{e}^{\frac{\pi x}{t}}+\mathrm{e}^{-\frac{\pi x}{t}}}\leqslant 2\mathrm{e}^{-\frac{\pi|x|}{t}}$$

可见 $f\in\mathfrak{F}$, 故由 Poisson 求和公式得

$$\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}an}}{\cosh\left(\frac{\pi n}{t}\right)}=\sum_{n=-\infty}^{\infty}\frac{t}{\cosh(\pi(n+a)t)}.$$

□

补充题 4 补充用 Phragmén-Lindelöf 定理实现 Paley-Wiener 定理证明中 Step 3 的细节：

$$\begin{cases} |f(x)| \leqslant 1, \\ |f(z)| \leqslant \mathrm{e}^{2\pi M|z|} \end{cases} \implies |f(z)| \leqslant \mathrm{e}^{2\pi M|y|}.$$

证明 通过乘恰当的旋转因子可知, Phragmén-Lindelöf 定理中的角状区域可换为第一象限. 令 $F(z) = f(z)\mathrm{e}^{2\pi \mathrm{i} Mz}$, 注意到 $F(z)$ 在第一象限的边界上有上界 1：

$$|F(x)| = |f(x)| \leqslant 1, \quad \forall x \in \mathbb{R}_+,$$
$$|F(\mathrm{i}y)| = |f(\mathrm{i}y)|\mathrm{e}^{-2\pi My} \leqslant \mathrm{e}^{2\pi M|y|}\mathrm{e}^{-2\pi My} = 1, \quad \forall y \in \mathbb{R}_+,$$

又 $|F(z)| = |f(z)|\left|\mathrm{e}^{2\pi\mathrm{i}Mz}\right| \leqslant \mathrm{e}^{4\pi M|z|}$, 由 Phragmén-Lindelöf 定理知, 在第一象限中有 $|F(z)| \leqslant 1$, 即 $|f(z)| \leqslant \left|\mathrm{e}^{-2\pi\mathrm{i}Mz}\right| = \left|\mathrm{e}^{-2\pi\mathrm{i}M(x+\mathrm{i}y)}\right| = \mathrm{e}^{2\pi My}$. 对余下三个象限类似讨论可得结论成立. □

Stein 4.4.1 Suppose f is continuous and of moderate decrease, and $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Show that $f = 0$ by completing the following outline:

(1) For each fixed real number t consider the two functions

$$A(z) = \int_{-\infty}^{t} f(x)\mathrm{e}^{-2\pi\mathrm{i}z(x-t)}\,\mathrm{d}x \quad \text{and} \quad B(z) = -\int_{t}^{\infty} f(x)\mathrm{e}^{-2\pi\mathrm{i}z(x-t)}\,\mathrm{d}x.$$

Show that $A(\xi) = B(\xi)$ for all $\xi \in \mathbb{R}$.

(2) Prove that the function F equal to A in the closed upper half-plane, and B in the lower half-plane, is entire and bounded, thus constant. In fact, show that $F = 0$.

(3) Deduce that

$$\int_{-\infty}^{t} f(x)\,\mathrm{d}x = 0,$$

for all t, and conclude that $f = 0$.

Proof (1) We have

$$A(\xi) - B(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi\mathrm{i}\xi(x-t)}\,\mathrm{d}x = \hat{f}(\xi) = 0.$$

(2) By Symmetry principle, the function $F(z) \coloneqq \begin{cases} A(z), & \operatorname{Im} z \geqslant 0, \\ B(z), & \operatorname{Im} z < 0 \end{cases}$ is an entire function. Since f is of moderate decrease, we see that

$$|A(z)| \leqslant \int_{-\infty}^{t} |f(x)|\mathrm{e}^{2\pi\,\mathrm{Im}(z)(x-t)}\,\mathrm{d}x \leqslant \int_{-\infty}^{t} \frac{A}{1+x^2}\,\mathrm{d}x \leqslant \pi A$$

is bounded in the closed upper-half plane. Similarly,

$$|B(z)| \leqslant \int_{t}^{\infty} |f(x)|\mathrm{e}^{2\pi\,\mathrm{Im}(z)(x-t)}\,\mathrm{d}x \leqslant \int_{t}^{\infty} \frac{A}{1+x^2}\,\mathrm{d}x \leqslant \pi A$$

is bounded in the lower-half plane. So F is both entire and bounded, thus constant by Liouville's theorem. Let $z = \mathrm{i}s$ for $s \geqslant 0$, we have

$$A(\mathrm{i}s) = \int_{-\infty}^{t} f(x)\mathrm{e}^{2\pi s(x-t)}\,\mathrm{d}x \xrightarrow{s\to\infty} 0$$

by DCT. So $F = 0$.

(3) Take $z = 0$ we find $\displaystyle\int_{-\infty}^{t} f(x)\,\mathrm{d}x = F(0) = 0$ for all t, hence $f = 0$. □

Stein 4.4.3 Show, by contour integration, that if $a > 0$ and $\xi \in \mathbb{R}$ then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = \mathrm{e}^{-2\pi a|\xi|},$$

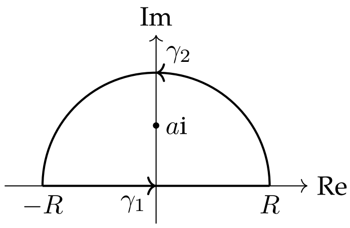
and check that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi a|\xi|}\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi = \frac{1}{\pi}\frac{a}{a^2+x^2}.$$

Proof Let $f(z) = \dfrac{a}{a^2+z^2}\mathrm{e}^{-2\pi\mathrm{i}z\xi}$.

(1) If $\xi = 0$ then LHS $= \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} \frac{a}{a^2+x^2}\,\mathrm{d}x = 1 =$ RHS.

(2) For $\xi < 0$, choose upper semicircle contour, from the residue formula we get



$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f, a\mathrm{i}) = 2\pi\mathrm{i}\lim_{z\to a\mathrm{i}} \frac{a}{z+a\mathrm{i}}\mathrm{e}^{-2\pi\mathrm{i}z\xi} = \pi\mathrm{e}^{-2\pi a|\xi|}.$$

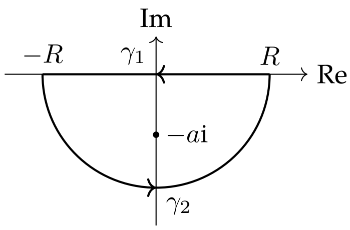
Since when $R \to +\infty$,

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leqslant \int_0^{\pi} \left|\frac{a}{a^2+R^2\mathrm{e}^{2\mathrm{i}\theta}}\mathrm{e}^{2\pi R\xi\sin\theta}\right|\mathrm{d}\theta \leqslant \frac{\pi a}{R^2-a^2} \to 0,$$

it follows that

$$\int_{\mathbb{R}} \frac{a}{a^2+x^2}\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = \pi\mathrm{e}^{-2\pi a|\xi|}.$$

(3) For $\xi > 0$, choose lower semicircle contour, like in (2) we get



$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f, -a\mathrm{i}) = 2\pi\mathrm{i}\lim_{z\to -a\mathrm{i}} \frac{a}{z-a\mathrm{i}}\mathrm{e}^{-2\pi\mathrm{i}z\xi} = -\pi\mathrm{e}^{-2\pi a|\xi|}.$$

When $R \to +\infty$,

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leqslant \int_{-\pi}^{0}\left|\frac{a}{a^2+R^2\mathrm{e}^{2\mathrm{i}\theta}}\mathrm{e}^{2\pi R\xi\sin\theta}\right|\mathrm{d}\theta \leqslant \frac{\pi a}{R^2-a^2} \to 0,$$

hence

$$\int_{\mathbb{R}} \frac{a}{a^2+x^2}\,\mathrm{d}x = -\left(-\pi\mathrm{e}^{-2\pi a|\xi|}\right) = \pi\mathrm{e}^{-2\pi a|\xi|}.$$

For the second part of the exercise, notice $f \in \mathfrak{F}$, so Fourier inversion implies the result. □

Stein 4.4.7 The Poisson summation formula applied to specific examples often provides interesting identities.

(1) Let τ be fixed with $\mathrm{Im}(\tau) > 0$. Apply the Poisson summation formula to

$$f(z) = (\tau+z)^{-k},$$

where k is an integer $\geqslant 2$, to obtain

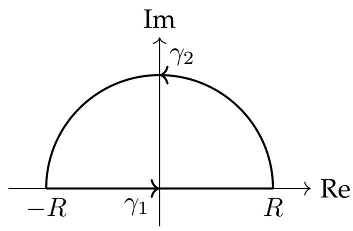
$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}\mathrm{e}^{2\pi\mathrm{i}m\tau}.$$

(2) Set $k = 2$ in the above formula to show that if $\mathrm{Im}(\tau) > 0$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(3) Can one conclude that the above formula holds true whenever τ is any complex number that is not an integer?

Proof (1) ① For $\xi \leqslant 0$, choose upper semicircle contour.



Since $(\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}$ is holomorphic in the upper half-plane, we have

$$\int_{\gamma_1} (\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z + \int_{\gamma_2} (\tau+z)^{-k}\mathrm{e}^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z = 0.$$

When $R \to +\infty$,

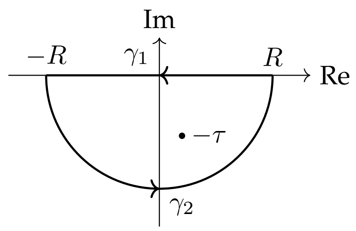
$$\left|\int_{\gamma_2} (\tau+z)^{-k} \mathrm{e}^{-2\pi \mathrm{i} z \xi} \,\mathrm{d}z\right| = \left|\int_0^{\pi} \frac{\mathrm{e}^{-2\pi \mathrm{i}\xi R\mathrm{e}^{\mathrm{i}\theta}} R\mathrm{i}}{(\tau+R\mathrm{e}^{\mathrm{i}\theta})^k} \,\mathrm{d}\theta\right| \leqslant \int_0^{\pi} \frac{R\left|\mathrm{e}^{-2\pi \mathrm{i}\xi R\mathrm{e}^{\mathrm{i}\theta}}\right|}{(R-|\tau|)^k} \,\mathrm{d}\theta$$

$$\leqslant \frac{\pi R^2 \mathrm{e}^{2\pi\xi R\sin\theta}}{(R-|\tau|)^k} \overset{\xi\leqslant 0}{\leqslant} \frac{\pi R^2}{(R-|\tau|)^k} \xrightarrow{k\geqslant 2} 0,$$

hence when $\xi \leqslant 0$ we get

$$\hat{f}(\xi) = \int_{\mathbb{R}} (\tau+x)^{-k} \mathrm{e}^{-2\pi \mathrm{i} x\xi} \,\mathrm{d}x = 0.$$

② For $\xi > 0$, choose lower semicircle contour.



The residue at $-\tau$ is

$$\operatorname{Res}\big((\tau+z)^{-k}\mathrm{e}^{-2\pi \mathrm{i} z\xi}, -\tau\big) = \frac{1}{(k-1)!}\big(\mathrm{e}^{-2\pi \mathrm{i} z\xi}\big)^{(k-1)}\Big|_{z=-\tau} = \frac{(-2\pi \mathrm{i}\xi)^{k-1}}{(k-1)!}\mathrm{e}^{2\pi \mathrm{i}\tau\xi},$$

thus

$$\int_{\gamma_1} (\tau+z)^{-k} \mathrm{e}^{-2\pi \mathrm{i} z \xi} \,\mathrm{d}z + \int_{\gamma_2} (\tau+z)^{-k} \mathrm{e}^{-2\pi \mathrm{i} z \xi} \,\mathrm{d}z = -\frac{(-2\pi \mathrm{i})^k \xi^{k-1}}{(k-1)!}\mathrm{e}^{2\pi \mathrm{i}\tau\xi}.$$

When $R \to +\infty$,

$$\left|\int_{\gamma_2} (\tau+z)^{-k} \mathrm{e}^{-2\pi \mathrm{i} z \xi} \,\mathrm{d}z\right| = \left|\int_{-\pi}^{0} \frac{\mathrm{e}^{-2\pi \mathrm{i}\xi R\mathrm{e}^{\mathrm{i}\theta}} R\mathrm{i}}{(\tau+R\mathrm{e}^{\mathrm{i}\theta})^k} \,\mathrm{d}\theta\right| \leqslant \int_{-\pi}^{0} \frac{R\left|\mathrm{e}^{-2\pi \mathrm{i}\xi R\mathrm{e}^{\mathrm{i}\theta}}\right|}{(R-|\tau|)^k} \,\mathrm{d}\theta$$

$$\leqslant \frac{\pi R^2 \mathrm{e}^{2\pi\xi R\sin\theta}}{(R-|\tau|)^k} \overset{\xi> 0}{\leqslant} \frac{\pi R^2}{(R-|\tau|)^k} \xrightarrow{k\geqslant 2} 0,$$

hence when $\xi > 0$ we get

$$\hat{f}(\xi) = \int_{\mathbb{R}} (\tau+x)^{-k} \mathrm{e}^{-2\pi \mathrm{i} x\xi} \,\mathrm{d}x = \frac{(-2\pi \mathrm{i})^k \xi^{k-1}}{(k-1)!}\mathrm{e}^{2\pi \mathrm{i}\tau\xi}.$$

Since $f \in \mathfrak{F}$, by Poisson summation formula we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi \mathrm{i})^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} \mathrm{e}^{2\pi \mathrm{i} m\tau}.$$

(2) Set $k = 2$ in the above formula, we get

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = -4\pi^2 \sum_{m=1}^{\infty} m\mathrm{e}^{2\pi \mathrm{i} m\tau}.$$

To finish the proof, notice that when $\mathrm{Im}(\tau) > 0$ we have $\left|\mathrm{e}^{2\pi\mathrm{i}\tau}\right| = \mathrm{e}^{-2\pi\,\mathrm{Im}(\tau)} < 1$, hence

$$\begin{aligned}\sum_{m=1}^{\infty} m\mathrm{e}^{2\pi\mathrm{i}m\tau} &= \frac{1}{2\pi\mathrm{i}} \sum_{m=1}^{\infty} \frac{\partial}{\partial\tau}\left(\mathrm{e}^{2\pi\mathrm{i}m\tau}\right) = \frac{1}{2\pi\mathrm{i}} \frac{\partial}{\partial\tau}\left(\sum_{m=1}^{\infty} \mathrm{e}^{2\pi\mathrm{i}m\tau}\right) = \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial\tau}\left(\frac{\mathrm{e}^{2\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2\pi\mathrm{i}\tau}}\right)\\ &= \frac{\mathrm{e}^{2\pi\mathrm{i}\tau}}{(1-\mathrm{e}^{2\pi\mathrm{i}\tau})^2} = \frac{1}{(\mathrm{e}^{\pi\mathrm{i}\tau}-\mathrm{e}^{-\pi\mathrm{i}\tau})^2} = \frac{1}{-4\sin^2(\pi\tau)}.\end{aligned}$$

(3) For the case that $\mathrm{Im}(\tau) < 0$, by replacing τ with $-\tau$, we see the formula in (2) still holds. When τ is a real number that is not an integer, the same formula holds by the isolating property of the zeros of a holomorphic function. □

Stein 4.4.9 Here are further results similar to the Phragmén-Lindelöf theorem.

(1) Let F be a holomorphic function in the right half-plane that extends continuously to the boundary, that is, the imaginary axis. Suppose that $|F(\mathrm{i}y)| \leqslant 1$ for all $y \in \mathbb{R}$, and

$$|F(z)| \leqslant C\mathrm{e}^{c|z|^{\gamma}}$$

for some $c, C > 0$ and $\gamma < 1$. Prove that $|F(z)| \leqslant 1$ for all z in the right half-plane.

(2) More generally, let S be a sector whose vertex is the origin, and forming an angle of $\frac{\pi}{\beta}$. Let F be a holomorphic function in S that is continuous on the closure of S, so that $|F(z)| \leqslant 1$ on the boundary of S and

$$|F(z)| \leqslant C\mathrm{e}^{c|z|^{\alpha}} \quad \text{for all} \quad z \in S$$

for some $c, C > 0$ and $0 < \alpha < \beta$. Prove that $|F(z)| \leqslant 1$ for all $z \in S$.

Proof We prove (2) directly. Let $F_\varepsilon(z) = F(z)\mathrm{e}^{-\varepsilon z^r}$, where $r \in (\alpha, \beta) \cap \mathbb{Q}$ and $\varepsilon > 0$. Then

$$|F_\varepsilon(z)| = |F(z)|\mathrm{e}^{-\varepsilon|z|^r\cos(r\arg z)} \leqslant C\mathrm{e}^{c|z|^{\alpha}-\varepsilon|z|^r\cos(r\arg z)}.$$

Without loss of generality, we consider the sector

$$S = \left\{z \in \mathbb{C} : -\frac{\pi}{2\beta} < \arg z < \frac{\pi}{2\beta}\right\},$$

then $r\arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\cos(r\arg z) > 0$. Hence $|F_\varepsilon(z)| \to 0$ as $|z| \to \infty$, and we can conclude that $|F_\varepsilon(z)|$ achieves its maximum on $\overline{S}$ at some point $z_0 \neq \infty$. Using the maximum modulus principle on some region with compact closure that contains z_0, we see that z_0 must lie on the boundary of S. Thus $|F_\varepsilon(z)| \leqslant |F_\varepsilon(z_0)| \leqslant 1$, and by letting $\varepsilon \to 0$ we get $|F(z)| \leqslant 1$ for all $z \in S$. □

Stein 4.4.11 One can give a neater formulation of the result in Exercise 10 by proving the following fact.

Suppose $f(z)$ is an entire function of strict order 2, that is,

$$f(z) = O\left(e^{c_1|z|^2}\right)$$

for some $c_1 > 0$. Suppose also that for x real,

$$f(x) = O\left(e^{-c_2|x|^2}\right)$$

for some $c_2 > 0$. Then

$$|f(x+iy)| = O\left(e^{-ax^2+by^2}\right)$$

for some $a, b > 0$. The converse is obviously true.

Proof For $z = x + iy$, if $x^2 \leqslant y^2$, then

$$c_1|z|^2 = c_1\left(x^2+y^2\right) \leqslant 2c_1y^2 \leqslant 3c_1y^2 - c_1x^2,$$

and so we already have

$$|f(z)| = O\left(e^{c_1|z|^2}\right) = O\left(e^{-c_1x^2+3c_1y^2}\right),$$

which is the desired result. So we may assume $x^2 > y^2$. By symmetry, we can only focus on the sector $S = \left\{z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{4}\right\}$. Let

$$g_\varepsilon(z) = f(z)e^{[c_2-\varepsilon+i(c_1+\varepsilon)]z^2} \quad \text{for } \varepsilon > 0,$$

then $|g_\varepsilon(z)| \leqslant e^{c|z|^2}$ in S for some $c > 0$. And on the boundary of S, we have

$$\begin{aligned} |g_\varepsilon(x)| &= |f(x)|e^{(c_2-\varepsilon)x^2} \leqslant C_2e^{-c_2x^2}e^{(c_2-\varepsilon)x^2} = C_2e^{-\varepsilon x^2}, \quad x \geqslant 0, \\ \left|g_\varepsilon\left(Re^{i\frac{\pi}{4}}\right)\right| &= |f(x)|e^{-(c_1+\varepsilon)R^2} \leqslant C_1e^{c_1R^2}e^{-(c_1+\varepsilon)R^2} = C_1e^{-\varepsilon R^2}, \quad R \geqslant 0. \end{aligned}$$

So $|g_\varepsilon(z)| \leqslant Ce^{-\varepsilon|z|^2}$ on the boundary of S for some $C > 0$. Now we can apply Exercise 4.4.9 (2), where we take $\alpha = 2$ and $\beta = 4$, to the function $g_\varepsilon(z)e^{\varepsilon|z|^2}$ (recall $\left|g_\varepsilon(z)e^{\varepsilon|z|^2}\right| \leqslant e^{(c+\varepsilon)|z|^2}$), and conclude that

$$\left|g_\varepsilon(z)e^{\varepsilon|z|^2}\right| \leqslant C \quad \text{for all } z \in S.$$

Then let $\varepsilon \to 0$ we get

$$|f(z)| \cdot \left|e^{(c_2+ic_1)(x+iy)^2}\right| = |f(z)|e^{c_2\left(x^2-y^2\right)-2c_1xy} \leqslant C \quad \text{for } x+iy \in S.$$

Hence

$$|f(x)| \leqslant Ce^{-c_2\left(x^2-y^2\right)+2c_1xy} \overset{\lambda>0}{\leqslant} Ce^{-c_2\left(x^2-y^2\right)+c_1\lambda x^2+\frac{c_1}{\lambda}y^2} = Ce^{-(c_2-c_1\lambda)x^2+\left(c_2+\frac{c_1}{\lambda}\right)y^2},$$

choosing $\lambda < \dfrac{c_2}{c_1}$ we complete the proof with $a = c_2 - c_1\lambda$ and $b = c_2 + \dfrac{c_1}{\lambda}$. □

Stein 4.4.12 The principle that a function and its Fourier transform cannot both be too small at infinity is illustrated by the following theorem of Hardy.

If f is a function on $\mathbb{R}$ that satisfies

$$f(x) = O\left(\mathrm{e}^{-\pi x^2}\right) \quad \text{and} \quad \hat{f}(\xi) = O\left(\mathrm{e}^{-\pi \xi^2}\right),$$

then f is a constant multiple of $\mathrm{e}^{-\pi x^2}$. As a result, if $f(x) = O\left(\mathrm{e}^{-\pi A x^2}\right)$, and $\hat{f}(\xi) = O\left(\mathrm{e}^{-\pi B \xi^2}\right)$, with $AB > 1$ and $A, B > 0$, then f is identically zero.

(1) If f is even, show that $\hat{f}$ extends to an even entire function. Moreover, if $g(z) = \hat{f}\left(z^{\frac{1}{2}}\right)$, then g satisfies

$$|g(x)| \leqslant c\mathrm{e}^{-\pi x} \quad \text{and} \quad |g(z)| \leqslant c\mathrm{e}^{\pi R \sin^2 \frac{\theta}{2}} \leqslant c\mathrm{e}^{\pi |z|}$$

when $x \in \mathbb{R}$ and $z = R\mathrm{e}^{\mathrm{i}\theta}$ with $R \geqslant 0$ and $\theta \in \mathbb{R}$.

(2) Apply the Phragmén-Lindelöf principle to the function

$$F(z) = g(z)\mathrm{e}^{\gamma z} \quad \text{where } \gamma = \mathrm{i}\pi \frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{2\beta}}}{\sin \frac{\pi}{2\beta}}$$

and the sector $0 \leqslant \theta \leqslant \frac{\pi}{\beta} < \pi$, and let $\beta \to 1$ to deduce that $\mathrm{e}^{\pi z} g(z)$ is bounded in the closed upper half-plane. The same result holds in the lower half-plane, so by Liouville's theorem $\mathrm{e}^{\pi z} g(z)$ is constant, as desired.

(3) If f is odd, then $\hat{f}(0) = 0$, and apply the above argument to $\dfrac{\hat{f}(z)}{z}$ to deduce that $f = \hat{f} = 0$. Finally, write an arbitrary f as an appropriate sum of an even function and an odd function.

Proof (1) Since $\hat{f}(\xi) = O\left(\mathrm{e}^{-\pi \xi^2}\right)$, $\hat{f}$ can be extended to an entire function by Theorem 3.1. Moreover, when f is even,

$$\hat{f}(-\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{2\pi \mathrm{i} x \xi}\,\mathrm{d}x = \int_{\mathbb{R}} f(-x)\mathrm{e}^{-2\pi \mathrm{i} x \xi}\,\mathrm{d}x = \hat{f}(\xi)$$

for all $\xi \in \mathbb{R}$, which implies that $\hat{f}(z) - \hat{f}(-z)$ is identically zero in the whole complex plane. So $\hat{f}$ extends to an even entire function. For $g(z) = \hat{f}\left(z^{\frac{1}{2}}\right)$, we have

$$|g(x)| = \left|\hat{f}\left(x^{\frac{1}{2}}\right)\right| \leqslant c\mathrm{e}^{-\pi x}$$

and

$$\begin{aligned}\left|\hat{f}\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| &= \left|\int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi \mathrm{i} x R(\cos\theta + \mathrm{i}\sin\theta)}\,\mathrm{d}x\right| \leqslant \int_{\mathbb{R}} |f(x)|\mathrm{e}^{2\pi x R \sin\theta}\,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}} c\mathrm{e}^{-\pi x^2 + 2\pi x R \sin\theta}\,\mathrm{d}x = c\mathrm{e}^{\pi R^2 \sin^2\theta} \int_{\mathbb{R}} \mathrm{e}^{-\pi(x - R\sin\theta)^2}\,\mathrm{d}x \\ &= c\mathrm{e}^{\pi R^2 \sin^2\theta},\end{aligned}$$

and so

$$\left|g\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| = \left|f\left(R^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\left(\frac{\theta}{2} + k\pi\right)}\right)\right| \leqslant c\mathrm{e}^{\pi R \sin^2 \frac{\theta}{2}} \leqslant c\mathrm{e}^{\pi R}.$$

(2) First we show that

$$\left|F\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| = \left|g\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| \cdot \left|\mathrm{e}^{\mathrm{i}\frac{\pi R}{\sin\frac{\pi}{2\beta}}\mathrm{e}^{\mathrm{i}\left(\theta - \frac{\pi}{2\beta}\right)}}\right| = \left|g\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{e}^{-\frac{\pi R}{\sin\frac{\pi}{2\beta}}\sin\left(\theta - \frac{\pi}{2\beta}\right)}$$

$$\overset{(1)}{\leqslant} c\mathrm{e}^{\pi R(1-\varepsilon_\theta)}, \quad \text{where } \varepsilon_\theta = \frac{\sin\left(\theta - \frac{\pi}{2\beta}\right)}{\sin\frac{\pi}{2\beta}}.$$

For $\beta > 1$, consider the sector $S = \left\{z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{\beta}\right\}$, on its boundary we have

$$|F(x)| = |g(x)| \cdot \left|\mathrm{e}^{\mathrm{i}\frac{\pi x}{\sin\frac{\pi}{2\beta}}\left(\cos\frac{\pi}{2\beta} - \mathrm{i}\sin\frac{\pi}{2\beta}\right)}\right| = |g(x)|\mathrm{e}^{\pi x} \overset{(1)}{\leqslant} c\mathrm{e}^{-\pi x} \cdot \mathrm{e}^{\pi x} = c, \quad \forall x \geqslant 0,$$

$$\left|F\left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{\beta}}\right)\right| \leqslant c\mathrm{e}^{\pi R(1-1)} = c, \quad \forall R \geqslant 0.$$

Hence $|F(z)| \leqslant 1$ on the boundary of S. Note that $|\varepsilon_\theta| \leqslant 1$ for $0 \leqslant \theta \leqslant \frac{\pi}{\beta}$, so

$$\left|F\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant c\mathrm{e}^{2\pi R}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{\beta}.$$

Since $\beta > 1$, we can apply result in Exercise 4.4.9 (2) to $\dfrac{F(z)}{c}$ to get $|F(z)| \leqslant c$ for all z in S. Let $\beta \to 1$, then $\gamma \to \pi$ and we conclude that $|g(z)\mathrm{e}^{\pi z}| \leqslant c$ for all z in the upper half-plane. The same result holds in the lower half-plane, so by Liouville's theorem $\mathrm{e}^{\pi z}g(z)$ is constant.

(3) If f is odd, then $\hat{f}(0) = 0$, $\hat{f}$ extends to an odd entire function by the same argument in (1), and $\dfrac{\hat{f}(z)}{z}$ is even. Let $h(z) = \hat{f}\left(z^{\frac{1}{2}}\right)z^{-\frac{1}{2}}$ and we get the same bound as in (1), then follow the same argument in (2) to conclude that $h(z)$ is constant for all $z \in \mathbb{C}$. Hence from $\hat{f}(0) = 0$ we see $\hat{f} \equiv 0$ and then $f \equiv 0$ by Fourier inversion.

Finally, for an arbitrary f, by decomposing f into even and odd parts, we see that f is a constant multiple of $\mathrm{e}^{-\pi x^2}$. □

Stein 4.5.3 In this problem, we investigate the behavior of certain bounded holomorphic functions in an infinite strip. The particular result described here is sometimes called the three-lines lemma.

(1) Suppose $F(z)$ is holomorphic and bounded in the strip $0 < \mathrm{Im}(z) < 1$ and continuous on its closure. If $|F(z)| \leqslant 1$ on the boundary lines, then $|F(z)| \leqslant 1$ throughout the strip.

(2) For the more general F, let $\sup_{x\in\mathbb{R}} |F(x)| = M_0$ and $\sup_{x\in\mathbb{R}} |F(x+\mathrm{i})| = M_1$. Then,

$$\sup_{x\in\mathbb{R}} |F(x+\mathrm{i}y)| \leqslant M_0^{1-y}M_1^y, \quad \text{if } 0 \leqslant y \leqslant 1.$$

(3) As a consequence, prove that $\log \sup_{x\in\mathbb{R}} |F(x+\mathrm{i}y)|$ is a convex function of y when $0 \leqslant y \leqslant 1$.

Proof (1) Let $F_\varepsilon(z) = F(z)\mathrm{e}^{-\varepsilon z^2}$ for some $\varepsilon > 0$, then

$$|F_\varepsilon(z)| = |F(z)|\mathrm{e}^{-\varepsilon\left(x^2 - y^2\right)} \to 0 \quad \text{as } x \to \infty.$$

Hence $|F_\varepsilon(z)|$ achieves its maximum in the strip at some point $z_0 \neq \infty$. Using the maximum modulus principle on some region with compact closure that contains z_0, we see that z_0 must lie on the boundary of the strip. Thus $|F_\varepsilon(z)| \leqslant |F_\varepsilon(z_0)| \leqslant 1$, and by letting $\varepsilon \to 0$ we get $|F(z)| \leqslant 1$ throughout strip.

(2) Let $G(z)=M_0^{-\mathrm{i}z-1}M_1^{\mathrm{i}z}F(z)$, then $G(z)$ satisfies the conditions of (1), i.e.

$$|G(x)|=\left|M_0^{-\mathrm{i}x-1}\right|\cdot\left|M_1^{\mathrm{i}x}\right|\cdot|F(x)|=M_0^{-1}|F(x)|\leqslant 1,\quad \forall x\in\mathbb{R},$$
$$|G(x+\mathrm{i})|=\left|M_0^{-\mathrm{i}(x+\mathrm{i})-1}\right|\cdot\left|M_1^{\mathrm{i}(x+\mathrm{i})}\right|\cdot|F(x+\mathrm{i})|=M_1^{-1}|F(x+\mathrm{i})|\leqslant 1,\quad \forall x\in\mathbb{R}.$$

By (1), we have $|G(z)|\leqslant 1$ throughout the strip, i.e.

$$|G(z)|=\left|M_0^{-\mathrm{i}(x+\mathrm{i}y)-1}\right|\cdot\left|M_1^{\mathrm{i}(x+\mathrm{i}y)}\right|\cdot|F(z)|=M_0^{y-1}M_1^{-y}|F(x+\mathrm{i}y)|\leqslant 1,$$

which implies the desired result.

(3) Set $M(y)=\sup\limits_{x\in\mathbb{R}}|F(x+\mathrm{i}y)|$ for $y\in[0,1]$. For $0\leqslant y_1<y_2\leqslant 1$, by scaling we see the result in (2) applies to the strip $y_1<\operatorname{Im}z<y_2$, i.e., for all $y\in[y_1,y_2]$,

$$\log M(y)\leqslant\log\Big(M(y_1)^{\frac{y_2-y}{y_2-y_1}}M(y_2)^{\frac{y-y_1}{y_2-y_1}}\Big)=\frac{y_2-y}{y_2-y_1}\log M(y_1)+\frac{y-y_1}{y_2-y_1}\log M(y_2),$$

which implies the convexity of $\log M(y)$. □

补充题 5 设 $|w|\leqslant 1$, 估计使 $|1-\mathrm{e}^w|\leqslant c|w|$ 成立的常数 c.

解答 记 $f(w)=\dfrac{1-\mathrm{e}^w}{w}$, 由于 0 是可去奇点, 因此 $f\in\mathcal{H}(\mathbb{B}(0,1))$, 作幂级数展开可得

$$\mathrm{e}^w-1=\sum_{n=1}^{\infty}\frac{w^n}{n!}\implies|f(w)|=\left|\sum_{n=0}^{\infty}\frac{w^n}{(n+1)!}\right|\leqslant\sum_{n=0}^{\infty}\frac{1}{(n+1)!}=\mathrm{e}-1.$$

因此可取 $c=\mathrm{e}-1$ (代入 $w=1$ 可知这是最佳常数). □

Stein 5.6.1 Give another proof of Jensen's formula in the unit disc using the functions (called Blaschke factors)

$$\psi_\alpha(z)=\frac{\alpha-z}{1-\bar{\alpha}z}.$$

Proof Let Ω be an open set that contains the closure of a disc D_R and suppose that f is holomorphic in Ω, $f(0)\neq 0$, and f vanishes nowhere on the circle C_R. Let $z_1,\cdots,z_N$ denote the zeros of f inside the disc (counted with multiplicities), we want to show that

$$\log|f(0)|=\sum_{k=1}^{N}\log\left(\frac{|z_k|}{R}\right)+\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(R\mathrm{e}^{\mathrm{i}\theta})\right|\mathrm{d}\theta.$$

(1) First, we observe that if f_1 and f_2 are two functions satisfying the hypotheses and the conclusion of the theorem, then so does their product f_1f_2.

(2) By setting $\widetilde{f}(z)=f(Rz)$, what we want to prove can be reduced to the specific case when $R=1$. Note that the function

$$g(z)=\frac{f(z)}{\psi_{z_1}(z)\cdots\psi_{z_N}(z)}$$

initially defined on $\Omega\setminus\{z_1,\cdots,z_N\}$, is bounded near each z_j. Therefore each z_j is a removable singularity, and hence we can write

$$f(z)=\psi_{z_1}(z)\cdots\psi_{z_N}(z)g(z)$$

where g is holomorphic in Ω and nowhere vanishing in $\overline{\mathbb{B}(0,1)}$. By (1) above, it suffices to prove Jensen's formula for functions like g that vanish nowhere, and for Blaschke factors.

(3) The case of functions that vanish nowhere follows from the mean value theorem for holomorphic functions. So it remains to show the result for Blaschke factors. We have

$$\log|\psi_\alpha(0)| = \log|\alpha| = \log|\alpha| + \frac{1}{2\pi}\int_0^{2\pi} \log\left|\psi_\alpha(\mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\theta$$

since $|\psi_\alpha(z)| = 1$ for $z \in \partial\mathbb{B}(0,1)$. □

Stein 5.6.3 Show that if τ is fixed with $\operatorname{Im}(\tau) > 0$, then the Jacobi theta function

$$\Theta(z \mid \tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi \mathrm{i} n^2 \tau} \mathrm{e}^{2\pi \mathrm{i} n z}$$

is of order 2 as a function of z.

Proof We have

$$\begin{aligned}
|\Theta(z \mid \tau)| &\leqslant \sum_{n=-\infty}^{\infty} \left|\mathrm{e}^{\pi i n^2 \tau} \mathrm{e}^{2\pi i n z}\right| \leqslant \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 \operatorname{Im}(\tau) + 2\pi n|z|} \\
&= \underbrace{\sum_{n < \frac{4|z|}{\operatorname{Im}(\tau)}} \mathrm{e}^{-\pi n^2 \operatorname{Im}(\tau) + 2\pi n|z|}}_{2\pi n|z| \leqslant \frac{8\pi|z|^2}{\operatorname{Im}(\tau)} \text{ when } n < \frac{4|z|}{\operatorname{Im}(\tau)}} + \underbrace{\sum_{n \geqslant \frac{4|z|}{\operatorname{Im}(\tau)}} \mathrm{e}^{-\pi n^2 \operatorname{Im}(\tau) + 2\pi n|z|}}_{-n^2 \operatorname{Im}(\tau) + 2n|z| \leqslant -\frac{n^2 \operatorname{Im}(\tau)}{2} \text{ when } n \geqslant \frac{4|z|}{\operatorname{Im}(\tau)}} \\
&\leqslant \mathrm{e}^{\frac{8\pi|z|^2}{\operatorname{Im}(\tau)}} \sum_{n < \frac{4|z|}{\operatorname{Im}(\tau)}} \mathrm{e}^{-\pi n^2 \operatorname{Im}(\tau)} + \sum_{n \geqslant \frac{4|z|}{\operatorname{Im}(\tau)}} \mathrm{e}^{-\frac{\pi n^2 \operatorname{Im}(\tau)}{2}} \\
&\overset{\mathrm{e}^{-x} \leqslant \frac{1}{x+1}}{\leqslant} \mathrm{e}^{\frac{8\pi|z|^2}{\operatorname{Im}(\tau)}} \sum_{n < \frac{4|z|}{\operatorname{Im}(\tau)}} \frac{1}{\pi n^2 \operatorname{Im}(\tau) + 1} + \sum_{n \geqslant \frac{4|z|}{\operatorname{Im}(\tau)}} \frac{1}{\frac{\pi n^2 \operatorname{Im}(\tau)}{2} + 1} \\
&\leqslant C_1 \mathrm{e}^{\frac{8\pi|z|^2}{\operatorname{Im}(\tau)}} + C_2.
\end{aligned}$$

It remains to show that the order is at least 2. We use repeatedly Proposition 1.1 (iii) in Chapter 10, which is about the quasi-periodicity of $\Theta(z \mid \tau)$, to see that

$$\Theta(x + m\tau \mid \tau) = \mathrm{e}^{-2\pi \mathrm{i} m x - \pi \mathrm{i} m^2 \tau} \Theta(x \mid \tau).$$

Then take $x = 0$ to get

$$|\Theta(m\tau \mid \tau)| = \mathrm{e}^{\pi m^2 \operatorname{Im}(\tau)} \Theta(0 \mid \tau) = A \mathrm{e}^{B|m\tau|^2},$$

which shows that the order of $\Theta(z \mid \tau)$ is at least 2. □

Stein 5.6.5 Show that if $\alpha > 1$, then

$$F_\alpha(z) = \int_{-\infty}^{\infty} \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{2\pi \mathrm{i} z t} \,\mathrm{d}t$$

is an entire function of growth order $\dfrac{\alpha}{\alpha - 1}$.

Proof By Fubini's theorem we have

$$\int_\gamma F_\alpha(z)\,\mathrm{d}z = \int_\gamma \int_{-\infty}^{\infty} \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{2\pi \mathrm{i} zt}\,\mathrm{d}t\,\mathrm{d}z = \int_{-\infty}^{\infty} \int_\gamma \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{2\pi \mathrm{i} zt}\,\mathrm{d}z\,\mathrm{d}t = \int_{-\infty}^{\infty} 0\,\mathrm{d}t = 0$$

for all closed curves γ, hence from Morera's theorem we see that $F_\alpha(z)$ is an entire function. To approximate the order of $F_\alpha(z)$, we first set $A = 4\pi$ and observe that

- ◇ If $|t|^{\alpha-1} \leqslant A|z|$, then

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| \leqslant 2\pi|z|A^{\frac{1}{\alpha-1}}|z|^{\frac{1}{\alpha-1}} = 2\pi A^{\frac{1}{\alpha-1}}|z|^{\frac{\alpha}{\alpha-1}}.$$

- ◇ If $|t|^{\alpha-1} > A|z|$, then

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| = |t|\left(-\frac{|t|^{\alpha-1}}{2} + 2\pi|z|\right) \leqslant |t|\left(-\frac{A|z|}{2} + 2\pi|z|\right) = |t||z|\left(2\pi - \frac{A}{2}\right) = 0.$$

So we can conclude that

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| \leqslant c|z|^{\frac{\alpha}{\alpha-1}} \tag{5.6.5–1}$$

for some constant $c > 0$. Denote ρ the order of growth of $F_\alpha(z)$.

(1) We first show that $\rho \leqslant \dfrac{\alpha}{\alpha-1}$. Using (5.6.5–1) we have

$$\begin{aligned} |F_\alpha(z)| &\leqslant \int_{\mathbb{R}} \mathrm{e}^{-|t|^\alpha + 2\pi|z||t|}\,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\frac{|t|^\alpha}{2}} \mathrm{e}^{-\frac{|t|^\alpha}{2} + 2\pi|z||t|}\,\mathrm{d}t \\ &\leqslant \mathrm{e}^{c|z|^{\frac{\alpha}{\alpha-1}}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{|t|^\alpha}{2}}\,\mathrm{d}t = 2\mathrm{e}^{c|z|^{\frac{\alpha}{\alpha-1}}} \int_0^{+\infty} \mathrm{e}^{-\frac{t^\alpha}{2}}\,\mathrm{d}t \\ &\leqslant 2\mathrm{e}^{c|z|^{\frac{\alpha}{\alpha-1}}} \left(1 + \int_1^{+\infty} \mathrm{e}^{-\frac{t}{2}}\,\mathrm{d}t\right) = 2c\left(1 + \mathrm{e}^{-\frac{1}{2}}\right)\mathrm{e}^{c|z|^{\frac{\alpha}{\alpha-1}}}, \end{aligned}$$

hence $\rho \leqslant \dfrac{\alpha}{\alpha-1}$.

(2) Next we show that $\rho \geqslant \dfrac{\alpha}{\alpha-1}$. For simplicity we consider $G_\alpha(z) = F_\alpha\left(\frac{z}{2\pi\mathrm{i}}\right) = \int_{\mathbb{R}} \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{zt}\,\mathrm{d}t$ and it has the same order of growth as $F_\alpha(z)$. Suppose to the contrary that $\rho < \dfrac{\alpha}{\alpha-1}$, and that

$$|G_\alpha(z)| \leqslant A\mathrm{e}^{B|z|^\rho}, \quad \forall z \in \mathbb{C}$$

for some positive constants A and B. For $R \in \mathbb{R}_{>0}$, we have

$$G_\alpha(R) = \int_{\mathbb{R}} \mathrm{e}^{-|t|^\alpha} \mathrm{e}^{Rt}\,\mathrm{d}t > \int_0^{+\infty} \mathrm{e}^{-t^\alpha} \mathrm{e}^{Rt}\,\mathrm{d}t > \int_0^{\frac{R^{\frac{1}{\alpha-1}}}{2}} \mathrm{e}^{-t^\alpha} \mathrm{e}^{Rt}\,\mathrm{d}t > \mathrm{e}^{-\frac{R^\rho}{2^\alpha}} \int_0^{\frac{R^{\frac{1}{\alpha-1}}}{2}} \mathrm{e}^{Rt}\,\mathrm{d}t.$$

Therefore we have

$$G_\alpha(R) > \mathrm{e}^{-\frac{R^{\frac{\alpha}{\alpha-1}}}{2^\alpha}} \frac{1}{R}\left(\mathrm{e}^{\frac{R^{\frac{\alpha}{\alpha-1}}}{2}} - 1\right) = \frac{1}{R}\left(\mathrm{e}^{\left(\frac{1}{2} - \frac{1}{2^\alpha}\right)R^{\frac{\alpha}{\alpha-1}}} - 1\right).$$

But we know that

$$G_\alpha(R) \leqslant A\mathrm{e}^{BR^\rho} \implies \frac{1}{R}\left(\mathrm{e}^{\left(\frac{1}{2}-\frac{1}{2^\alpha}\right)R^{\frac{\alpha}{\alpha-1}}} - 1\right) < A\mathrm{e}^{BR^\rho},$$

which does not hold for large R by our assumption that $\rho < \dfrac{\alpha}{\alpha-1}$.

Now we conclude that $F_\alpha(z)$ is an entire function of growth order $\dfrac{\alpha}{\alpha-1}$. □

Stein 5.6.7 Establish the following properties of infinite products.

(1) Show that if $\sum_{n=1}^{\infty} |a_n|^2$ converges, then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges to a non-zero limit if and only if $\sum_{n=1}^{\infty} a_n$ converges.

(2) Find an example of a sequence of complex numbers $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n$ converges but $\prod_{n=1}^{\infty}(1+a_n)$ diverges.

(3) Also find an example such that $\prod_{n=1}^{\infty}(1+a_n)$ converges and $\sum_{n=1}^{\infty} a_n$ diverges.

Solution (1) If $\sum_{n=1}^{\infty} |a_n|^2$ converges, then $\lim_{n\to\infty} a_n = 0$, hence

$$\lim_{n\to\infty} \frac{a_n - \log(1+a_n)}{a_n^2} = \frac{1}{2}.$$

By the limit comparison test, we see that $\sum_{n=1}^{\infty}[a_n - \log(1+a_n)]$ converges, then

$$\prod_{n=1}^{\infty}(1+a_n) \text{ converges to a non-zero limit} \iff \sum_{n=1}^{\infty}\log(1+a_n) \text{ converges} \iff \sum_{n=1}^{\infty} a_n \text{ converges.}$$

(2) Let $a_n = \dfrac{(-1)^n}{\sqrt{n}}$, then $\sum_{n=2}^{\infty} a_n$ converges by the Leibniz's test for alternating series, but

$$\prod_{n=2}^{\infty} a_n = \prod_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2k}}\right)\left(1-\frac{1}{\sqrt{2k+1}}\right) =: \prod_{k=1}^{\infty} b_k,$$

since $b_k < \left(1+\frac{1}{\sqrt{2k+1}}\right)\left(1-\frac{1}{\sqrt{2k+1}}\right) = 1-\frac{1}{2k+1}$, we have $1-b_k > \frac{1}{2k+1}$ and hence $\sum_{k=1}^{\infty}(1-b_k)$ diverges. Note that $b_k \to 1$, therefore

$$\lim_{k\to\infty} -\frac{\log b_k}{1-b_k} = 1.$$

Hence $\sum_{k=1}^{\infty} -\log b_k$ diverges by the limit comparison test, and it follows that

$$\sum_{k=1}^{\infty} \log b_k \text{ diverges} \implies \prod_{k=1}^{\infty} b_k = \prod_{n=2}^{\infty} a_n \text{ diverges.}$$

(3) Let

$$a_n = \begin{cases} -\dfrac{1}{\sqrt{k}}, & n = 2k-1, \\ \dfrac{1}{\sqrt{k}} + \dfrac{1}{k} + \dfrac{1}{k\sqrt{k}}, & n = 2k. \end{cases}$$

Then

$$\sum_{n=1}^{2N} a_n = \sum_{k=1}^{N} (a_{2k-1} + a_{2k}) = \sum_{k=1}^{N} \frac{1}{\sqrt{k}} + \sum_{k=1}^{N} \frac{1}{k\sqrt{k}} \xrightarrow{N\to\infty} +\infty,$$

but

$$\begin{aligned} \prod_{n=2}^{2N} (1+a_n) &= (1+a_2) \prod_{k=2}^{N} (1+a_{2k-1})(1+a_{2k}) = 4 \prod_{k=2}^{N} \left(1 - \frac{1}{\sqrt{k}}\right)\left(1 + \frac{1}{\sqrt{k}}\right)\left(1 + \frac{1}{k}\right) \\ &= 4 \prod_{k=2}^{N} \frac{k-1}{k} \cdot \frac{k+1}{k} = 4 \cdot \frac{N+1}{2N} \xrightarrow{N\to\infty} 2. \end{aligned}$$

□

Stein 5.6.9 Prove that if $|z| < 1$, then

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \prod_{k=0}^{\infty} \left(1 + z^{2^k}\right) = \frac{1}{1-z}.$$

Proof If we denote $P_n = \prod_{k=0}^{n-1} \left(1 + z^{2^n}\right)$, then

$$(1-z)P_n = (1-z)(1+z)(1+z^2)\cdots\left(1 + z^{2^{n-1}}\right) = 1 - z^{2n}.$$

Hence $P_n = \dfrac{1-z^{2n}}{1-z}$ and by taking the limit as $n \to \infty$ we get the desired result when $|z| < 1$. □

Stein 5.6.10 Find the Hadamard products for:

(1) $\mathrm{e}^z - 1$;

(2) $\cos \pi z$.

Solution (1) Since $\mathrm{e}^z - 1$ has growth order 1 and $\mathrm{e}^z - 1 = 0 \iff z = 2\pi \mathrm{i} n$ for $n \in \mathbb{Z}$, by Hadamard's factorization theorem we see it has the form

$$\mathrm{e}^z - 1 = \mathrm{e}^{Az+B} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{2\pi \mathrm{i} n}\right)\left(1 + \frac{z}{2\pi \mathrm{i} n}\right) \mathrm{e}^{\frac{z}{2\pi \mathrm{i} n} - \frac{z}{2\pi \mathrm{i} n}} = \mathrm{e}^{Az+B} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Then

$$\mathrm{e}^{\frac{z}{2}} - \mathrm{e}^{-\frac{z}{2}} = \mathrm{e}^{\left(A-\frac{1}{2}\right)z+B} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Since LHS is odd we get $A = \frac{1}{2}$, and from

$$1 = \lim_{z\to 0} \frac{e^z - 1}{z} = \lim_{z\to 0} e^B \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

we see that $B = 0$. So we have

$$e^z - 1 = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

(2) Since $\cos \pi z$ has growth order 1 and $\cos \pi z = 0 \iff z = n + \frac{1}{2}$ for $n \in \mathbb{Z}$, by Hadamard's factorization theorem we see it has the form

$$\cos \pi z = e^{Az+B} \prod_{n\in\mathbb{Z}} \left(1 - \frac{z}{n+\frac{1}{2}}\right) e^{\frac{z}{n+\frac{1}{2}}} = e^{Az+B} \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

Since LHS is even we get $A = 0$, and by letting $z = 0$ we see that $B = 0$. So we have

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

□

Stein 5.6.13 Show that the equation $e^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.

Proof Suppose to the contrary that $e^z - z = 0$ has only finitely many solutions, then since $e^z - z$ is entire and has growth order 1, by Hadamard's factorization theorem we have $e^z - z = e^{Az+B} P(z)$ for some polynomial $P(z)$. Then $P(z) = \dfrac{e^z - z}{e^{Az+B}} = O\left(e^{(1-A)z}\right)$, which is possible only when $P(z)$ is constant and $A = 1$, and hence $z = e^z\left(1 - e^B C\right)$ for some constant C, which is impossible. □

Stein 5.6.14 Deduce from Hadamard's theorem that if F is entire and of growth order ρ that is non-integral, then F has infinitely many zeros.

Proof Let $k = \lfloor \rho \rfloor$, then $k < \rho < k+1$. Suppose to the contrary that F has only finitely many zeros, then by Hadamard's factorization theorem we have $F(z) = e^{P(z)} Q(z)$ for some polynomials with $\deg P \leqslant k$. However, this implies that F has growth order at most k, which is a contradiction. □

Stein 5.7.1 Prove that if f is holomorphic in the unit disc, bounded and not identically zero, and $z_1, z_2, \cdots, z_n, \cdots$ are its zeros ($|z_k| < 1$), then

$$\sum_n (1 - |z_n|) < \infty.$$

Proof Without loss of generality, we may assume that $f(0) \neq 0$ (otherwise just factor out z^m) and the number of zeros is infinite. Fix $k \in \mathbb{N}$ and consider $r \in (0,1)$ such that $\mathfrak{n}(r) > k$ and f vanishes nowhere on the circle $|z| = r$, where $\mathfrak{n}(r)$ denotes the number of zeros of f (counted with their multiplicities) inside the disc $|z| < r$. Recall Jensen's formula:

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta - \log|f(0)| = \sum_{n=1}^{\mathfrak{n}(r)} \log\left(\frac{r}{|z_n|}\right).$$

The boundedness of f implies that there exists $M > 0$ such that

$$|f(0)| \prod_{n=1}^{k} \frac{r}{|z_n|} \leqslant |f(0)| \prod_{n=1}^{\mathfrak{n}(r)} \frac{r}{|z_n|} = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta \right\} \leqslant M.$$

Let $r \to 1^-$ to see that

$$\prod_{n=1}^{k} |z_n| \geqslant \frac{|f(0)|}{M} \quad \text{for all } k \in \mathbb{N}.$$

Then by taking $k \to \infty$ we find

$$\prod_{n=1}^{\infty} |z_n| \geqslant \frac{|f(0)|}{M} > 0.$$

Therefore, by taking the logarithm we have

$$\sum_{n=1}^{\infty} (-\log|z_n|) < \infty$$

and $\lim_{n\to\infty} |z_n| = 1$. Hence $\lim_{n\to\infty} \frac{-\log|z_n|}{1-|z_n|} = 1$ and by the comparison test we get

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

□