Laurent Series Expansion

The Laurent series of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

The Laurent series expansion is discussed in Problem 3.9.3 of the textbook, which we reformulate as the following theorem.

Theorem 1 If $f(z)$ is holomorphic in the annulus $\mathbb{A}_{r,R}(a) := \{z \in \mathbb{C} : r < |z-a| < R\}$, then we can write

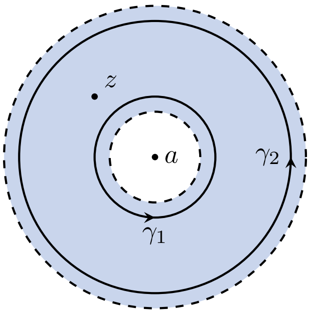
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \tag{1}$$

where

$$c_n = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\zeta - a| = \rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, \mathrm{d}\zeta, \quad r < \rho < R. \tag{2}$$

This series expansion (1) is unique, called the **Laurent series expansion** of $f(z)$ in the annulus $\mathbb{A}_{r,R}(a)$.

Proof We note first that the right-hand side of (2) is independent of the choice of ρ. This is a corollary of Theorem 5.1, Chapter 3.



Given any $z \in \mathbb{A}_{r,R}(a)$, we can take two circles $\gamma_1 : |\zeta - a| = \rho_1$ and $\gamma_2 : |\zeta - a| = \rho_2$ with $\rho_1 < \rho_2$, such that z lies in the annulus between them. By Cauchy's integral formula for annulus,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta. \tag{3}$$

- When $\zeta \in \gamma_2$, we have

  $$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a)\left(1 - \frac{z-a}{\zeta - a}\right)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}}, \tag{4}$$

  and the right-hand side converges uniformly on γ_2 since $\left|\dfrac{z-a}{\zeta - a}\right| = \dfrac{|z-a|}{\rho_2} < 1$.

- When $\zeta \in \gamma_1$, we have

  $$\frac{1}{\zeta - z} = \frac{-1}{(z-a)\left(1 - \frac{\zeta - a}{z-a}\right)} = -\sum_{n=1}^{\infty} \frac{(\zeta - a)^{n-1}}{(z-a)^n}, \tag{5}$$

and the right-hand side converges uniformly on γ_1 since $\left|\frac{\zeta-a}{z-a}\right| = \frac{\rho_1}{|z-a|} < 1$.

Substituting (4) and (5) into (3), and interchanging the order of integration and summation, we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi\mathrm{i}}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,\mathrm{d}\zeta\right)(z-a)^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi\mathrm{i}}\int_{\gamma_1}\frac{f(\zeta)}{(\zeta-a)^{-n+1}}\,\mathrm{d}\zeta\right)(z-a)^{-n} \\ &= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi\mathrm{i}}\int\limits_{|\zeta-a|=\rho}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,\mathrm{d}\zeta\right)(z-a)^n, \quad r<\rho<R. \end{aligned}$$

To show the uniqueness of the Laurent series expansion, we observe that if $f(z)$ can be expressed as (1), then for $r<\rho<R$,

$$\int\limits_{|z-a|=\rho}\frac{f(z)}{(z-a)^{m+1}}\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} c_n \int\limits_{|z-a|=\rho}(z-a)^{n-m-1}\,\mathrm{d}z = 2\pi\mathrm{i}c_m, \tag{6}$$

which implies that c_m is uniquely determined by $f(z)$. Here we have used the fact that

$$\begin{aligned} \int\limits_{|z-a|=\rho}(z-a)^{n-m-1}\,\mathrm{d}z &= \int_0^{2\pi}\left(\rho e^{\mathrm{i}\theta}\right)^{n-m-1}\rho\mathrm{i}e^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}\rho^{n-m}\int_0^{2\pi}e^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta \\ &= \begin{cases} 2\pi\mathrm{i}, & \text{if } n=m, \\ 0, & \text{if } n\neq m. \end{cases} \end{aligned}$$

□

Remark 2 In practice, the integral formula (2) may not offer the most practical method for computing the coefficients c_n for a given function $f(z)$; instead, one often pieces together the Laurent series by combining known Taylor expansions. Because the Laurent expansion of a function is unique whenever it exists, any expression of this form that equals the given function $f(z)$ in some annulus must actually be the Laurent expansion of $f(z)$.

Example 3 Find the Laurent series expansions of the function $f(z) = \frac{z^2-2z+5}{(z-2)(z^2+1)}$ in the annuli $\mathbb{A}_{1,2}(0)$ and $\mathbb{A}_{2,\infty}(0)$.

Solution The partial fraction decomposition of $f(z)$ is $f(z) = \frac{1}{z-2} - \frac{2}{z^2+1}$.

- When $1<|z|<2$, we have

$$f(z) = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{2}{z^2}\frac{1}{1+\frac{1}{z^2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{2}{z^2}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n.$$

- When $|z|>2$, we have

$$f(z) = \frac{1}{z}\frac{1}{1-\frac{2}{z}} - \frac{2}{z^2}\frac{1}{1+\frac{1}{z^2}} = \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{z^{2n}}.$$

□

Residue at Infinity

Given a holomorphic function f on an annulus $\mathbb{A}_{R,\infty}(0)$, the residue at infinity of f can be defined in terms of the usual residue as follows:

$$\operatorname{Res}(f,\infty) := -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Thus, one can transfer the study of $f(z)$ at infinity to the study of $f(1/z)$ at the origin.

Note that the function $f(1/w)$ is holomorphic in the annulus $\mathbb{A}_{0,1/R}(0)$. Hence for $r > R$,

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=r} f(z)\,\mathrm{d}z \xlongequal[\text{reverse orientation}]{w=1/z} \frac{1}{2\pi \mathrm{i}} \int\limits_{|w|=1/r} \frac{1}{w^2} f\left(\frac{1}{w}\right)\mathrm{d}w = -\operatorname{Res}(f,\infty).$$

This shows that for holomorphic functions the sum of the residues at the isolated singularities plus the residue at infinity is zero. That is, if f is holomorphic in $\mathbb{C} \setminus \{z_1, \cdots, z_n\}$, then

$$\sum_{k=1}^{n} \operatorname{Res}(f, z_k) + \operatorname{Res}(f,\infty) = 0. \tag{7}$$

Moreover, from (6) we find that if $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ is the Laurent series expansion of $f(z)$ in the annulus $\mathbb{A}_{R,\infty}(0)$, then

$$\operatorname{Res}(f,\infty) = -c_{-1}.$$

Example 4 Evaluate the integral $I = \int\limits_{|z|=2} \frac{z^5}{1+z^6}\,\mathrm{d}z$.

Solution Tha Laurent series expansion of $f(z) = \frac{z^5}{1+z^6}$ in the annulus $\mathbb{A}_{1,\infty}(0)$ is

$$f(z) = \frac{z^5}{z^6} \frac{1}{1+\frac{1}{z^6}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{1}{z^6}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{6n+1}}.$$

Hence $\operatorname{Res}(f,\infty) = -1$, and by (7) we get

$$I = -2\pi \mathrm{i} \operatorname{Res}(f,\infty) = 2\pi \mathrm{i}.$$

□

Remark 5 One might first guess that the definition of the residue of $f(z)$ at infinity should just be the residue of $f(1/z)$ at $z = 0$. However, the reason that we consider instead $-\frac{1}{z^2} f\left(\frac{1}{z}\right)$ is that one does not take residues of functions, but of *differential forms*, i.e. the residue of $f(z)\,\mathrm{d}z$ at infinity is the residue of $f\left(\frac{1}{z}\right)\mathrm{d}\left(\frac{1}{z}\right) = -\frac{1}{z^2} f\left(\frac{1}{z}\right)\mathrm{d}z$ at $z = 0$.

Definite Integrals Using Residues

Type I: $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$, where $f(x)$ is continuous on $\mathbb{R}$ (Stein 3.8.2, 3.8.6)

Theorem 6 Suppose $f(z)$ is holomorphic in $\mathbb{H}\setminus\{a_1,\cdots,a_n\}$ and continuous on $\mathbb{R}$, where $a_1,\cdots,a_n\in\mathbb{H}$ are isolated singularities of $f(z)$ in the upper half-plane $\mathbb{H}$. If $\lim\limits_{z\to\infty} zf(z)=0$, then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{k=1}^{n}\mathrm{Res}(f,a_k).$$

Corollary 7 If $f(x)=\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are coprime polynomials, $Q(x)$ is non-vanishing in $\mathbb{R}$, and $\deg Q-\deg P\geqslant 2$, then

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{k=1}^{n}\mathrm{Res}\left(\frac{P(z)}{Q(z)},a_k\right),$$

where $a_1,\cdots,a_n$ are all *distinct* roots of $Q(z)$ in $\mathbb{H}$.

Example 8 Evaluate the integral $I=\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+1)^{n+1}}$.

Solution By Corollary 7, we have

$$\begin{aligned} I &= 2\pi\mathrm{i}\,\mathrm{Res}\left(\frac{1}{(z^2+1)^{n+1}},\mathrm{i}\right) = \frac{2\pi\mathrm{i}}{n!}\lim_{z\to\mathrm{i}}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{(z-\mathrm{i})^{n+1}}{(z^2+1)^{n+1}}\right) \\ &= \frac{2\pi\mathrm{i}}{n!}\lim_{z\to\mathrm{i}}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{1}{(z+\mathrm{i})^{n+1}}\right) = \frac{\pi(2n)!}{2^{2n}(n!)^2}. \end{aligned}$$

□

Type II: $\displaystyle\int_{-\infty}^{\infty} f(x)\cos\alpha x\,\mathrm{d}x$ and $\displaystyle\int_{-\infty}^{\infty} f(x)\sin\alpha x\,\mathrm{d}x$ (Stein 3.8.3, 3.8.4)

Theorem 9 Suppose $f(z)$ is holomorphic in $\mathbb{H}\setminus\{a_1,\cdots,a_n\}$ and continuous on $\mathbb{R}$, where $a_1,\cdots,a_n\in\mathbb{H}$ are isolated singularities of $f(z)$ in the upper half-plane $\mathbb{H}$. If $\lim\limits_{z\to\infty} f(z)=0$, then for any $\alpha>0$,

$$\int_{-\infty}^{\infty} f(x)e^{\mathrm{i}\alpha x}\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{k=1}^{n}\mathrm{Res}\big(e^{\mathrm{i}\alpha z}f(z),a_k\big).$$

To prove this theorem, we can use the following lemma.

Lemma 10 (Jordan) If $\lim\limits_{z\to\infty} f(z)=0$, then for any $\alpha>0$,

$$\lim_{R\to\infty}\int_{\gamma_R} e^{\mathrm{i}\alpha z}f(z)=0,$$

where γ_R is the semicircular contour in the upper half-plane centered at the origin with radius R.

Corollary 11 Under the same conditions as Theorem 9, we have

$$\int_{-\infty}^{\infty} f(x)\cos\alpha x\,\mathrm{d}x = -2\pi\,\mathrm{Im}\left\{\sum_{k=1}^{n}\mathrm{Res}\big(e^{\mathrm{i}\alpha z}f(z), a_k\big)\right\},$$
$$\int_{-\infty}^{\infty} f(x)\sin\alpha x\,\mathrm{d}x = 2\pi\,\mathrm{Re}\left\{\sum_{k=1}^{n}\mathrm{Res}\big(e^{\mathrm{i}\alpha z}f(z), a_k\big)\right\}.$$

Example 12 Evaluate the integral $I = \displaystyle\int_{-\infty}^{\infty}\frac{\cos ax}{x^2+b^2}\,\mathrm{d}x$, where $a, b > 0$.

Solution Let $f(z) = \dfrac{1}{z^2+b^2}$. Since $\lim\limits_{z\to\infty} f(z) = 0$, we can apply Corollary 11 to get

$$\begin{aligned} I &= \int_{-\infty}^{\infty} f(x)\cos ax\,\mathrm{d}x = -2\pi\,\mathrm{Im}\big\{\mathrm{Res}\big(e^{\mathrm{i}az}f(z), \mathrm{i}b\big)\big\} \\ &= -2\pi\,\mathrm{Im}\left\{\frac{e^{-ab}}{2\mathrm{i}b}\right\} = \frac{\pi}{b}e^{-ab}. \end{aligned}$$

□

Type III: $\displaystyle\int_0^{2\pi} R(\sin\theta, \cos\theta)\,\mathrm{d}\theta$ (Stein 3.8.7, 3.8.8)

The trick here is to put together some elementary properties of $z = e^{\mathrm{i}\theta}$ on the unit circle.

- ◇ $e^{-\mathrm{i}\theta} = 1/z$.
- ◇ $\cos\theta = \dfrac{e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta}}{2} = \dfrac{z+1/z}{2}$.
- ◇ $\sin\theta = \dfrac{e^{\mathrm{i}\theta - e^{-\mathrm{i}\theta}}}{2\mathrm{i}} = \dfrac{z-1/z}{2\mathrm{i}}$.
- ◇ $\mathrm{d}\theta = \dfrac{\mathrm{d}z}{\mathrm{i}z}$.

Example 13 Evaluate the integral $I = \displaystyle\int_0^{2\pi}\frac{\mathrm{d}\theta}{3+\cos\theta+2\sin\theta}$.

Solution We have

$$\begin{aligned} I &= \int\limits_{|z|=1}\frac{1}{3+\frac{1}{2}(z+1/z)-\mathrm{i}(z-1/z)}\frac{\mathrm{d}z}{\mathrm{i}z} \\ &= -2\mathrm{i}\int\limits_{|z|=1}\frac{\mathrm{d}z}{(1-2\mathrm{i})z^2+6z+(1+2\mathrm{i})} \\ &= -2\mathrm{i}(1+2\mathrm{i})\int\limits_{|z|=1}\frac{\mathrm{d}z}{[z+(1+2\mathrm{i})][5z+(1+2\mathrm{i})]} \\ &= -2\pi\mathrm{i}\cdot\frac{2\mathrm{i}(1+2\mathrm{i})}{5}\frac{1}{-\frac{1+2\mathrm{i}}{5}+(1+2\mathrm{i})} \\ &= \pi. \end{aligned}$$

□

Type IV: $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$, where $f(x)$ has singularities on $\mathbb{R}$ (Stein 2.6.2)

Lemma 14 Suppose that $f(z)$ is continuous in the sector

$$\left\{a+\rho e^{\mathrm{i}\theta} : 0<\rho\leqslant\rho_0,\ \theta_0\leqslant\theta\leqslant\theta_0+\alpha\right\}.$$

If $\lim_{z\to a}(z-a)f(z)=A$, then

$$\lim_{\rho\to 0}\int_{\gamma_\rho} f(z)\,\mathrm{d}z=\mathrm{i}A\alpha,$$

where γ_ρ is the circular arc $\gamma_\rho(\theta)=a+\rho e^{\mathrm{i}\theta}$, with $\theta_0\leqslant\theta\leqslant\theta+\alpha$.

Proof Let $g(z)=(z-a)f(z)-A$. Then $\lim_{z\to a}g(z)=0$. If we denote

$$M_\rho=\sup\left\{|g(z)| : z=a+\rho e^{\mathrm{i}\theta},\ \theta_0\leqslant\theta\leqslant\theta_0+\alpha\right\},$$

then $\lim_{\rho\to 0}M_\rho=0$. Hence, we have

$$\left|\int_{\gamma_\rho}\frac{g(z)}{z-a}\,\mathrm{d}z\right|=\left|\int_{\theta_0}^{\theta_0+\alpha}\frac{g\left(a+\rho e^{\mathrm{i}\theta}\right)}{\rho e^{\mathrm{i}\theta}}\rho\mathrm{i}e^{\mathrm{i}\theta}\,\mathrm{d}\theta\right|\leqslant M_\rho\alpha\xrightarrow{\rho\to 0}0.$$

It follows that

$$\int_{\gamma_\rho} f(z)\,\mathrm{d}z=\mathrm{i}A\alpha+\int_{\gamma_\rho}\frac{g(z)}{z-a}\,\mathrm{d}z\xrightarrow{\rho\to 0}\mathrm{i}A\alpha.$$

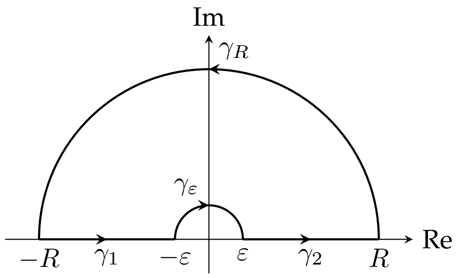
□

Example 15 Evaluate the integral $I=\int_{-\infty}^{\infty}\frac{\sin^3 x}{x^3}\,\mathrm{d}x$.

Solution Rewrite the integral as

$$\begin{aligned}I&=\int_{-\infty}^{\infty}\left(\frac{e^{\mathrm{i}x}-e^{-\mathrm{i}x}}{2\mathrm{i}}\right)^3\frac{\mathrm{d}x}{x^3}\\&=\int_{-\infty}^{\infty}\frac{e^{3\mathrm{i}x}-3e^{\mathrm{i}x}+3e^{-\mathrm{i}x}-e^{-3\mathrm{i}x}}{-8\mathrm{i}x^3}\,\mathrm{d}x\\&=\int_{-\infty}^{\infty}\frac{3\sin x-\sin 3x}{4x^3}\,\mathrm{d}x.\end{aligned}$$

Now, consider the integral of $f(z):=\dfrac{3e^{\mathrm{i}z}-e^{3\mathrm{i}z}}{4z^3}$ over the contour $\gamma:=\gamma_1\cup\gamma_\varphi\cup\gamma_2\cup\gamma_R$ shown below.



By Lemma 10, the integral along γ_R vanishes as $R\to\infty$. Since $f(z)$ is holomorphic in the region enclosed by γ, we have

$$I = \operatorname{Im}\left\{\lim_{\varepsilon\to 0,\, R\to\infty} \int_{\gamma_1\cup\gamma_2} f(z)\,\mathrm{d}z\right\} = -\operatorname{Im}\left\{\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z\right\}.$$

Since

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{3e^{\mathrm{i}z} - e^{3\mathrm{i}z}}{4z^2} = \lim_{z\to 0}\frac{3\frac{(\mathrm{i}z)^2}{2} - \frac{(3\mathrm{i}z)^2}{2}}{4z^2} = \frac{3}{4},$$

applying Lemma 14 (with attention to the contour's orientation), we obtain

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = -\frac{3\pi\mathrm{i}}{4}.$$

Therefore,

$$I = -\operatorname{Im}\left\{-\frac{3\pi\mathrm{i}}{4}\right\} = \frac{3\pi}{4}.$$

□

Koebe Quarter Theorem

Stein 3.9.1 Consider a holomorphic map on the unit disc $f\colon \mathbb{D}\to\mathbb{C}$ which satisfies $f(0)=0$. By the open mapping theorem, the image $f(\mathbb{D})$ contains a small disc centered at the origin. We then ask: does there exist $r>0$ such that for *all* $f\colon\mathbb{D}\to\mathbb{C}$ with $f(0)=0$, we have $D_r(0)\subset f(\mathbb{D})$?

(1) Show that with no further restrictions on f, no such r exists. It suffices to find a sequence of functions $\{f_n\}$ holomorphic in $\mathbb{D}$ such that $\frac{1}{n}\notin f_n(\mathbb{D})$. Compute $f_n'(0)$, and discuss.

(2) Assume in addition that f also satisfies $f'(0)=1$. Show that despite this new assumption, there exists no $r>0$ satisfying the desired condition.

The **Koebe–Bieberbach theorem** states that if in addition to $f(0)=0$ and $f'(0)=1$ we also assume that f is injective, then such an r exists and the best possible value is $r=\frac{1}{4}$.

(3) As a first step, show that if $h(z)=\frac{1}{z}+c_0+c_1z+c_2z^2+\cdots$ is analytic and injective for $0<|z|<1$, then $\sum_{n=1}^{\infty} n|c_n|^2 \leqslant 1$.

(4) If $f(z)=z+a_2z^2+\cdots$ satisfies the hypotheses of the theorem, show that there exists another function g satisfying the hypotheses of the theorem such that $g^2(z)=f(z^2)$.

(5) With the notation of the previous part, show that $|a_2|\leqslant 2$, and that equality holds if and only if

$$f(z)=\frac{z}{(1-e^{\mathrm{i}\theta}z)^2}\quad\text{for some } \theta\in\mathbb{R}.$$

(6) If $h(z)=\frac{1}{z}+c_0+c_1z+c_2z^2+\cdots$ is injective on $\mathbb{D}$ and avoids the values z_1 and z_2, show that $|z_1-z_2|\leqslant 4$.

(7) Complete the proof of the theorem.

Proof (1) Take $f_n(z)=\frac{z}{n}$. Then $\frac{1}{n}\notin f(\mathbb{D})$ and $f_n'(0)=\frac{1}{n}$.

(2) Take $f_\varepsilon(z)=\varepsilon\big(e^{\frac{z}{\varepsilon}}-1\big)$ for $\varepsilon>0$. Then $f_\varepsilon'(0)=1$ but $-\frac{1}{n}\notin f_\varepsilon(\mathbb{D})$.

(3) The choice of c_0 is irrelevant. So assume $c_0 = 0$. Neither the hypothesis nor the conclusion is affected if we replace $h(z)$ by $\lambda h(\lambda z)$ ($|\lambda| = 1$). So we may assume that c_1 is real.

Put $U_r = \{z \in \mathbb{C} : |z| < r\}$, $C_r = \{z \in \mathbb{C} : |z| = r\}$, and $V_r = \{z \in \mathbb{C} : r < |z| < 1\}$, for $0 < r < 1$. Then $h(U_r)$ is a neighborhood of ∞ (by the open mapping theorem, applied to $1/h$); the sets $h(U_r)$, $h(C_r)$, and $h(V_r)$ are disjoint, since h is injective. Write

$$h(z) = \frac{1}{z} + c_1 z + \varphi(z), \quad z \in \mathbb{D}, \tag{8}$$

$h = u + \mathrm{i}v$, and

$$A = \frac{1}{r} + c_1 r, \quad B = \frac{1}{r} - c_1 r. \tag{9}$$

For $z = re^{\mathrm{i}\theta}$, we then obtain

$$u = A\cos\theta + \operatorname{Re}\varphi \quad \text{and} \quad v = -B\sin\theta + \operatorname{Im}\varphi. \tag{10}$$

Divide (10) by A and B, respectively, square, and add:

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 + \frac{2\cos\theta}{A}\operatorname{Re}\varphi + \left(\frac{\operatorname{Re}\varphi}{A}\right)^2 - \frac{2\sin\theta}{B}\operatorname{Im}\varphi + \left(\frac{\operatorname{Im}\varphi}{B}\right)^2.$$

By (8), φ has a zero of order at least 2 at the origin. If we keep account of (9), it follows that there exists $\eta > 0$ such that, for all sufficiently small r,

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} < 1 + \eta r^3, \quad z = re^{\mathrm{i}\theta}.$$

This says that $h(C_r)$ is in the interior of the ellipse E_r, whose semiaxes are $A\sqrt{1+\eta r^3}$ and $B\sqrt{1+\eta r^3}$, and which therefore bounds an area

$$\pi AB(1+\eta r^3) = \pi\left(\frac{1}{r} + c_1 r\right)\left(\frac{1}{r} - c_1 r\right)(1+\eta r^3) \leqslant \frac{\pi}{r^2}(1+\eta r^3). \tag{11}$$

Since $h(C_r)$ is in the interior of E_r, we have $E_r \subset h(U_r)$; hence $h(V_r)$ is in the interior of E_r, so the area of $h(V_r)$ is no larger than (11). Therefore by the area formula we have

$$\begin{aligned}
\frac{\pi}{r^2}(1+\eta r^3) \geqslant & \iint_{V_r} |h'(z)|^2 \,\mathrm{d}x\,\mathrm{d}y \\
= & \int_r^1 \int_0^{2\pi} \left| -\rho^{-2} e^{-2\mathrm{i}\theta} + \sum_{n=1}^{\infty} n c_n \rho^{n-1} e^{\mathrm{i}(n-1)\theta} \right|^2 \rho \,\mathrm{d}\theta\,\mathrm{d}\rho \\
= & \int_r^1 \int_0^{2\pi} \Bigg\{ \rho^{-3} - \sum_{n=1}^{\infty} n c_n \rho^{n-2} e^{\mathrm{i}(n+1)\theta} - \sum_{m=1}^{\infty} m \overline{c_m} \rho^{m-2} e^{-\mathrm{i}(+1)\theta} \\
& + \sum_{n,m=1}^{\infty} nm c_n \overline{c_m} \rho^{n+m-1} e^{\mathrm{i}(n-m)\theta} \Bigg\} \,\mathrm{d}\theta\,\mathrm{d}\rho \\
= & 2\pi \int_r^1 \left(\rho^{-3} + \sum_{n=1}^{\infty} n^2 |c_n|^2 \rho^{2n-1} \right) \mathrm{d}\rho \\
= & \pi \left\{ r^{-2} - 1 + \sum_{n=1}^{\infty} n|c_n|^2 (1 - r^{2n}) \right\}.
\end{aligned} \tag{12}$$

If we divide (12) by π and then subtract r^{-2} from both sides, we obtain

$$\sum_{n=1}^{N} n|c_n|^2\left(1-r^{2n}\right) \leqslant 1+\eta r \tag{13}$$

for all sufficiently small r and for all positive integers N. Let $r \to 0$ in (13), then let $N \to \infty$. This gives the desired result.

(4) Write $f(z) = z\varphi(z)$. Then φ is holomorphic in $\mathbb{D}$, $\varphi(0) = 1$, and φ has no zero in $\mathbb{D}$, since f has no zero in $\mathbb{D} - \{0\}$ by its injectivity. Hence there exists h holomorphic in $\mathbb{D}$ with $h(0) = 1$, $h^2(z) = \varphi(z)$. Put

$$g(z) = zh\left(z^2\right), \quad z \in \mathbb{D}. \tag{14}$$

Then $g^2(z) = z^2h^2\left(z^2\right) = z^2\varphi\left(z^2\right) = f\left(z^2\right)$. It is clear that $g(0) = 0$ and $g'(0) = 1$. We have to show that g is injective.

Suppose z and w are points in $\mathbb{D}$ such that $g(z) = g(w)$. Since f is injective, the identity $g^2(z) = f\left(z^2\right)$ implies that $z^2 = w^2$. So either $z = w$ (which is what we want to prove) or $z = -w$. In the latter case, (14) shows that $g(z) = g(-w) = -g(w)$; it follows that $g(z) = g(w) = 0$, and since g has no zero in $\mathbb{D} - \{0\}$, we have $z = w = 0$.

(5) Let g be the function constructed in part (4), so that g is injective, $g(0) = 0$, and $g'(0) = 1$. Since

$$g^2(z) = f\left(z^2\right) = z^2\left(1 + a_2z^2 + \cdots\right),$$

we get

$$g(z) = z\left(1 + \frac{1}{2}a_2z^2 + \cdots\right).$$

If we take $G := 1/g$, then

$$G(z) = \frac{1}{z}\left(1 - \frac{1}{2}a_2z^2 + \cdots\right) = \frac{1}{z} + c_0 + c_1z + c_2z^2 + \cdots .$$

The result in part (3) applies to G, and this will give

$$\left|-\frac{1}{2}a_2\right|^2 = |c_1|^2 \leqslant \sum_{n=1}^{\infty} n|c_n|^2 \leqslant 1,$$

that is, $|a_2| \leqslant 2$. The equality holds if and only if $c_k = 0$ for all $k \geqslant 2$, that is,

$$G(z) = \frac{1}{z} - \frac{a_2}{2}z,$$

and this is equivalent to

$$f\left(z^2\right) = g^2(z) = \left(\frac{1}{\frac{1}{z} - \frac{a_2}{2}z}\right)^2 = \frac{z^2}{\left(1 - \frac{a_2}{2}z^2\right)^2}.$$

Since $|a_2| = 2$, this is again equivalent to

$$f(z) = \frac{z}{\left(1 - e^{i\theta}z\right)^2} \quad \text{for some } \theta \in \mathbb{R}.$$

(6) Since $z_1 \notin h(\mathbb{D})$, the function $\frac{1}{h(z)-z_1}$ is holomorphic in $\mathbb{D}$ (the origin is a removable singularity), and

$$\frac{1}{h(z)-z_1} = \frac{z}{1+(c_0-z_1)z+c_1z^2+c_2z^3} = z-(c_0-z_1)z^2+\cdots.$$

Hence, $\frac{1}{h(z)-z_1}$ satisfies the hypotheses of the theorem, and by part (5) we have $|c_0-z_1| \leqslant 2$. Similarly, $|c_0-z_2| \leqslant 2$. Therefore,

$$|z_1-z_2| \leqslant |c_0-z_1|+|c_0-z_2| \leqslant 4.$$

(7) Suppose $w \notin f(\mathbb{D})$. Then the function $h(z) := \frac{1}{f(z)}$ satisfies the hypotheses of (3) and (6). Since $h(z)$ avoids the values 0 and $\frac{1}{w}$, by part (6) we have $\left|\frac{1}{w}\right| \leqslant 4$, that is, $|w| \geqslant \frac{1}{4}$. This shows that $D_{1/4}(0) \subset f(\mathbb{D})$. □

Remark 16 The Koebe $1/4$ theorem is named after Paul Koebe, who conjectured the result in 1907. The theorem was proved by Ludwig Bieberbach in 1916. The example of the Koebe function

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$$

shows that the constant $1/4$ in the theorem cannot be improved (increased).