Selected Exercises from the Textbook

Stein 1.4.13 Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

(1) $\mathrm{Re}(f)$ is constant;

(2) $\mathrm{Im}(f)$ is constant;

(3) $|f|$ is constant;

one can conclude that f is constant.

Proof Suppose that $f = u + \mathrm{i}v$ where u and v are real-valued functions.

(1) If $\mathrm{Re}(f)$ is constant, then $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} \equiv 0$ and by the Cauchy–Riemann equation we get $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} \equiv 0$. Hence f is constant.

(2) Apply (1) to the holomorphic function $\mathrm{i}f$.

(3) Suppose $|f(z)| \equiv C > 0$. Since $\dfrac{\partial \bar{f}}{\partial z} = 0$ by the Cauchy–Riemann equation, we have

$$0 = \frac{\partial}{\partial z}\left(f(z)\overline{f(z)}\right) = \frac{\partial f}{\partial z}\overline{f(z)} + f(z)\frac{\partial \bar{f}}{\partial z} = \frac{\partial f}{\partial z}\overline{f(z)}.$$

By our assumption, $\overline{f(z)}$ is always non-zero, where $\dfrac{\partial f}{\partial z} \equiv 0$ follows. □

Stein 2.6.7 Suppose $f\colon \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup\limits_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2|f'(0)| \leqslant d. \tag{2.6.7–1}$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Proof The Cauchy integral formula gives

$$f'(0) = \frac{1}{2\pi \mathrm{i}} \int_{C_r} \frac{f(\zeta)}{\zeta^2}\,\mathrm{d}\zeta, \tag{2.6.7–2}$$

where C_r is the circle centered at the origin with radius r. Replace ζ by $-\zeta$ in (2.6.7–2) to get

$$f'(0) = -\frac{1}{2\pi \mathrm{i}} \int_{C_r} \frac{f(-\zeta)}{\zeta^2}\,\mathrm{d}\zeta. \tag{2.6.7–3}$$

Adding (2.6.7–2) and (2.6.7–3) gives

$$\begin{aligned} |2f'(0)| &= \left|\frac{1}{2\pi\mathrm{i}} \int_{C_r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2}\,\mathrm{d}\zeta\right| \leqslant \frac{1}{2\pi}\int_{C_r} \left|\frac{f(\zeta) - f(-\zeta)}{\zeta^2}\right| \mathrm{d}\zeta \\ &\leqslant \frac{d}{2\pi}\int_{C_r} \frac{\mathrm{d}\zeta}{r^2} = \frac{d}{r}. \end{aligned}$$

Letting $r \to 1$ gives (2.6.7–1). It is clear that the equality holds when f is linear.

To show that the equality holds only when f is linear, we let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and denote

$$N(r) \coloneqq \frac{1}{\pi r^2} \int\limits_{\mathbb{B}(0,r)} |f'(z)|^2 \,\mathrm{d}x\,\mathrm{d}y$$

for $r \in [0,1]$. If $f'(0) = 0$, then $d = 0$ and f is constant. Otherwise, we have

$$\lim_{r \to 0^+} N(r) = |f'(0)|^2 > 0.$$

This shows that f is locally injective near the origin, and by the area formula we have

$$\begin{aligned}
\frac{\mathrm{Area}(f(\mathbb{B}(0,r)))}{\pi r^2} = N(r) &= \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \left| \sum_{n=1}^{\infty} n a_n \rho^{n-1} e^{\mathrm{i}(n-1)\theta} \right|^2 \rho \,\mathrm{d}\theta\,\mathrm{d}\rho \\
&= \frac{1}{\pi r^2} \sum_{n,m=1}^{\infty} nm a_n \overline{a_m} \int_0^r \int_0^{2\pi} \rho^{n+m-2} e^{\mathrm{i}(n-m)\theta} \rho \,\mathrm{d}\theta\,\mathrm{d}\rho \\
&= \frac{1}{\pi r^2} \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^r \int_0^{2\pi} \rho^{2n-1} \,\mathrm{d}\theta\,\mathrm{d}\rho \\
&= \sum_{n=1}^{\infty} n |a_n|^2 r^{2n-2}
\end{aligned}$$

and

$$N'(r) = \sum_{n=2}^{\infty} n(2n-2)|a_n|^2 r^{2n-3}$$

for all r small enough. If f is not linear, i.e., there exists $n \geqslant 2$ such that $a_n \neq 0$, then $N'(r) > 0$ and $N(r)$ is strictly increasing in r for r small enough. Hence

$$|f'(0)|^2 = N(0) < N(r) = \frac{\mathrm{Area}(f(\mathbb{B}(0,r)))}{\pi r^2} \leqslant \frac{\pi[d(r)/2]^2}{\pi r^2} = \left(\frac{d(r)}{2r}\right)^2, \tag{2.6.7–4}$$

where the "$\leqslant$" sign is due to the isodiametric inequality, and

$$d(r) \coloneqq \sup_{z,w \in \mathbb{B}(0,r)} |f(z) - f(w)|.$$

Meanwhile, by the maximum modulus principle, we have

$$\frac{d(r)}{r} = \sup_{\theta \in [0,2\pi]} \sup_{|z|=r} \left| \frac{f(e^{\mathrm{i}\theta} z) - f(z)}{z} \right|.$$

For any fixed θ, the function $\dfrac{f(e^{\mathrm{i}\theta} z) - f(z)}{z}$ is holomorphic in $\mathbb{D}$. By the maximum modulus principle, the supremum of its modulus over $|z| = r$ is a nondecreasing function of r. Taking the supremum over θ, we conclude that $\dfrac{d(r)}{r}$ is a nondecreasing function of r. So if the equality holds in (2.6.7–1), then for small r we have

$$\frac{d(r)}{r} \leqslant \frac{d(1)}{1} = d = 2|f'(0)|,$$

which contradicts (2.6.7–4). Therefore f must be linear. □

Stein 2.6.14 Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Proof We may assume $z_0 = 1$ for otherwise we can take $w = \frac{z}{z_0}$ and consider the function

$$g(w) \coloneqq f(z) = \sum_{n=0}^{\infty} a_n (z_0 w)^n = \sum_{n=0}^{\infty} (a_n z_0^n) w^n.$$

Since $z_0 = 1$ is the only pole of f in this open set, we can write

$$f(z) = g(z) + \frac{b_{-m}}{(z-1)^m} + \cdots + \frac{b_{-1}}{z-1} + g(z),$$

where g is holomorphic in this open set and $b_{-m} \neq 0$. Suppose that in this open set

$$g(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then $\lim\limits_{n\to\infty} c_n = 0$. Whenever $|z| < 1$, one has

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{m} \frac{b_{-n}}{(z-1)^n}. \tag{2.6.14–1}$$

Note that

$$\begin{aligned}
\frac{1}{(z-1)^n} &= \frac{(-1)^{n-1}}{(n-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} \frac{1}{z-1} = \frac{(-1)^n}{(n-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} \sum_{k=0}^{\infty} z^k \\
&= \frac{(-1)^n}{(n-1)!} \sum_{k=n-1}^{\infty} k(k-1)\cdots(k-n+2) z^{k-n+1} \\
&= \frac{(-1)^n}{(n-1)!} \sum_{s=0}^{\infty} \frac{(s+n-1)!}{s!} z^s,
\end{aligned}$$

which converges absolutely for every compact subset of $\mathbb{D}$, so we can rearrange the series and use (2.6.14–1) to get

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{m} \frac{(-1)^n b_{-n}}{(n-1)!} \sum_{s=0}^{\infty} \frac{(s+n-1)!}{s!} z^s \\
&= \sum_{n=0}^{\infty} c_n z^n + \sum_{s=0}^{\infty} \underbrace{\sum_{n=1}^{m} \frac{(-1)^n b_{-n}}{(n-1)!} \frac{(s+n-1)!}{s!}}_{\text{polynomial in } s} z^s \\
&= \sum_{s=0}^{\infty} [c_s + P(s)] z^s,
\end{aligned}$$

where P is a polynomial of degree $m-1$ since $b_{-m} \neq 0$. Hence we have $a_n = c_n + P(n)$ for each $n \geqslant 0$ and it follows from $\lim\limits_{n\to\infty} c_n = 0$ that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{c_n + P(n)}{c_{n+1} + P(n+1)} = \lim_{n\to\infty} \frac{P(n)}{P(n+1)} = 1. \qquad \square$$

Stein 2.7.1 Here are some examples of analytic functions on the unit disc that cannot be extended analytically past the unit circle. The following definition is needed. Let f be a function defined in the unit disc $\mathbb{D}$, with boundary circle C. A point w on C is said to be *regular* for f if there is an open neighborhood U of w and an analytic function g on U, so that $f = g$ on $\mathbb{D} \cap U$. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f.

(1) Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \quad \text{for } |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc.

(2) Fix $0 < \alpha < \infty$. Show that the analytic function f defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \quad \text{for } |z| < 1$$

extends continuously to the unit circle, but cannot be analytically continued past the unit circle.

Proof (1) Note that if $z_0 \in C$ is a regular point of f, then there exists an open neighborhood U of z_0 in C in which all the points are regular for f. Hence, by the denseness of the set

$$\bigcup_{n=1}^{\infty} \left\{ z \in \mathbb{C} : z^{2^n} = 1 \right\}$$

in C, it suffices to show that all points in this set are irregular for f.

- The point 1 is irregular for f, since $\lim\limits_{x\to 1^-} f(x) = +\infty$.
- Note that

$$f(z) = z + f(z^2) = z + z^2 + f(z^4) = z + z^2 + z^4 + f(z^8) = \cdots,$$

so the roots of

$$z^2 = 1, \quad z^4 = 1, \quad z^8 = 1, \quad \cdots$$

are all irregular for f by the last point.

(2) Since for $z \in C$ we have

$$|f(z)| \leqslant \sum_{n=0}^{\infty} z^{-n\alpha} = \frac{1}{1-2^{-\alpha}} < \infty,$$

the function f extends to the unit circle. Fix any $z_0 \in C$. For any $\varepsilon > 0$, we choose $N \in \mathbb{N}$ such that

$\sum_{n=N+1}^{\infty} 2^{-n\alpha} < \varepsilon$. Then

$$\begin{aligned}|f(z) - f(z_0)| &\leqslant \sum_{n=0}^{N} 2^{-n\alpha}\left|z^{2^n} - z_0^{2^n}\right| + \sum_{n=N+1}^{\infty} 2^{-n\alpha}\left|z^{2^n} - z_0^{2^n}\right| \\ &< \sum_{n=0}^{N} 2^{-n\alpha}\left|z^{2^n} - z_0^{2^n}\right| + 2\varepsilon \to 2\varepsilon \quad \text{as } z \to z_0.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, f extends continuously to C.

We refer to Theorem 3.1 in Chapter 4, Book I of this series for the following result:

If $0 < \alpha < 1$, *then the function*

$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{\mathrm{i}2^n x}$$

is nowhere differentiable.

And Problem 8 of Chapter 5 in Book I gives a refinement of the above result, which states that $f_\alpha(x)$ is nowhere differentiable even in the case $\alpha = 1$. Now, for $\alpha \in (1, 2]$, consider the function

$$zf'(z) = \sum_{n=0}^{\infty} 2^{-n(\alpha-1)} z^{2^n}.$$

Since $0 < \alpha - 1 \leqslant 1$, the function $zf'(z)$ is not differentiable at any point on the unit circle. Therefore, f cannot be analytically continued past the unit circle. □

The Cauchy–Pompeiu Formula

A corrected version of Cauchy's integral formula is the Cauchy–Pompeiu formula, and holds for smooth functions as well, as it is based on Stokes' theorem. Let D be a disk in $\mathbb{C}$ and suppose that f is a complex-valued $\mathcal{C}^\infty$ function in an open neighborhood of $\overline{D}$, then

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi} \int_D \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{\mathrm{d}x \wedge \mathrm{d}y}{\zeta - z}$$

for $z \in D$. This reduces to the Cauchy integral formula when $\overline{\partial} f = 0$.

To prove this, you may need the following analogue of Goursat's theorem:

$$\int_{\partial D} f(z)\,\mathrm{d}z = 2\mathrm{i} \int_D \frac{\partial f}{\partial \bar{z}}(z)\,\mathrm{d}x \wedge \mathrm{d}y.$$

Universal Entire Functions

Our interest here will be in what has come to be called *hypercylic operators* on the space $\mathcal{H}(\mathbb{C})$ of entire functions of one complex variable. This subject has its origins in 1929 with the paper *Démonstration d'un théorème elémentaire sur les fonctions entiéres* by G. D. Birkhoff, in which he proved that there is $f \in \mathcal{H}(\mathbb{C})$ such that the set of all translates $\{f(z), f(1+z), \cdots, f(n+z), \cdots\}$ is dense in $\mathcal{H}(\mathbb{C})$. About 25 years later, G. MacLane proved in *Sequences of derivatives and normal families* an analogous result for derivatives: There is an entire function f such that the set of all derivatives $\left\{f, f', \cdots, f^{(n)}, \cdots\right\}$ is dense in $\mathcal{H}(\mathbb{C})$.

Recall that an operator $T\colon X \to X$ is said to be *hypercylic* if there is some vector $x \in X$ such that $\{x, Tx, \cdots, T^n x, \cdots\}$ is dense in X. These two results can be restated in terms of hypercyclic operators on $\mathcal{H}(\mathbb{C})$, by simply noting that Birkhoff's result means that the translation operator

$$T\colon \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}), \quad T(h)(z) := h(1+z)$$

is hypercyclic. Likewise, MacLane's result just says that the differentiation operator is hypercyclic.

Theorem 1 There is a function $f \in \mathcal{H}(\mathbb{C})$ with the following property: For every $g \in \mathcal{H}(\mathbb{C})$ and every $R, \varepsilon > 0$, there is $n \in \mathbb{N}$ such that $|f(z+n) - g(z)| < \varepsilon$ for every $z \in \mathbb{C}$ with $|z| \leqslant R$.

Proof Let $(P_j)_j$ be a dense sequence of polynomials in $\mathcal{H}(\mathbb{C})$. To simplify the argument, we assume that each P_j occurs infinitely often in this sequence. Let $(D_j)_j$ be a sequence of disjoint closed discs, each of radius j, such that the centers $(c_j)_j$ form an increasing sequence on the positive real axis. Let E_j be a sequence of closed discs, each centered at the origin, such that $D_j \subset E_j$ and $D_{j+1} \cap E_j = \varnothing$.

Define $Q_1 = P_1$. By Runge's theorem, there is a polynomial Q_2 such that $\|Q_2\|_{E_1} < \frac{1}{2}$ and

$$|Q_2(z) - [P_2(z - c_2) - Q_1(z)]| < \frac{1}{2}, \quad \forall z \in D_2.$$

Next, choose a polynomial Q_3 such that $\|Q_3\|_{E_2} < \frac{1}{2^2}$ and

$$|Q_3(z) - [P_3(z - c_3) - Q_1(z) - Q_2(z)]| < \frac{1}{2^2}, \quad \forall z \in D_3.$$

In general, let Q_n be a polynomial such that $\|Q_n\|_{E_{n-1}} < \frac{1}{2^{n-1}}$ and

$$\left|Q_n(z) - \left[P_n(z - c_n) - \sum_{i=1}^{n-1} Q_i(z)\right]\right| < \frac{1}{2^{n-1}}, \quad \forall z \in D_n.$$

We claim that the function $f(z) = \sum_{n=1}^{\infty} Q_n(z)$ has the desired property. It is easy to see that f is entire, and so it remains to show that if $g \in \mathcal{H}(\mathbb{C})$, $R > 0$, and $\varepsilon > 0$ are arbitrary, then for some j,

$$|f(z + c_j) - g(z)| < \varepsilon, \quad |z| \leqslant R.$$

In fact, it is enough to demonstrate this for $g = P = P_k$ for some k. Since there are infinitely many k for which $P = P_k$, we can choose k large enough so that

$$\left\| f - \sum_{i=1}^{k} Q_i \right\|_{E_k} = \left\| \sum_{i=k+1}^{\infty} Q_i \right\|_{E_k} \leqslant \sum_{i=k+1}^{\infty} \|Q_i\|_{E_{i-1}} \leqslant \sum_{i=k+1}^{\infty} \frac{1}{2^{i-1}} < \frac{\varepsilon}{2},$$

and also

$$\left| \sum_{i=1}^{k} Q_i(z) - P(z - c_k) \right| = \left| Q_k(z) - \left[P(z - c_k) - \sum_{i=1}^{k-1} Q_i(z) \right] \right| < \frac{1}{2^{k-1}} < \frac{\varepsilon}{2}, \quad \forall z \in D_k.$$

Then, by the triangle inequality, we obtain

$$|f(z) - P(z - c_k)| < \frac{\varepsilon}{2}, \quad \forall z \in D_k.$$

The result follows by a change of variables. □

In the same spirit, one can show that there is an entire function whose collection of derivatives is dense in $\mathcal{H}(\mathbb{C})$.

Theorem 2 There is an entire function f such that the set $\left\{ f^{(n)} : n \in \mathbb{N} \right\}$ is dense in $\mathcal{H}(\mathbb{C})$.

Remark 3 Godefroy and Shapiro generalized the above results in *Operators with dense, invariant, cyclic vector manifolds* to show that every continuous linear operator $L\colon \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ which commutes with translations and which is not a multiple of the identity is hypercyclic.