Stein 5.6.4 Let $t > 0$ be given and fixed, and define $F(z)$ by

$$F(z) = \prod_{n=1}^{\infty} \left(1 - e^{-2\pi n t} e^{2\pi \mathrm{i} z}\right).$$

Note that the product defines an entire function of z.

(1) Show that $|F(z)| \leqslant A e^{a|z|^2}$, hence F has an order of growth $\leqslant 2$.

(2) F vanishes exactly when $z = -\mathrm{i}nt + m$ for $n \geqslant 1$ and n, m integers. Thus, if z_n is an enumeration of these zeros we have

$$\sum \frac{1}{|z_n|^2} = \infty \quad \text{but} \quad \sum \frac{1}{|z_n|^{2+\varepsilon}} < \infty.$$

Proof (1) Given z, fix some integer N such that $\frac{|z|}{t} \leqslant N \leqslant \frac{|z|}{t} + 1$, and write $F(z) = F_1(z) F_2(z)$ where

$$F_1(z) = \prod_{n=1}^{N} \left(1 - e^{-2\pi n t} e^{2\pi \mathrm{i} z}\right) \quad \text{and} \quad F_2(z) = \prod_{n=N+1}^{\infty} \left(1 - e^{-2\pi n t} e^{2\pi \mathrm{i} z}\right).$$

Then $e^{-2\pi(N+1)t + 2\pi|z|} < 1$, and

$$\begin{aligned}
|F_2(z)| &\leqslant \prod_{n=N+1}^{\infty} \left(1 + e^{-2\pi n t} e^{2\pi |z|}\right) \\
&\leqslant \prod_{n=N+1}^{\infty} \exp\left(e^{-2\pi n t} e^{2\pi |z|}\right) \\
&= \exp\left\{ e^{-2\pi(N+1)t + 2\pi|z|} \sum_{n=0}^{\infty} e^{-2\pi n t} \right\} \\
&\leqslant \exp\left(\frac{1}{1 - e^{-2\pi t}} \right).
\end{aligned}$$

For the other estimate, using $N \leqslant \frac{|z|}{t} + 1$, we have

$$\begin{aligned}
|F_1(z)| &\leqslant \prod_{n=1}^{N} \left(1 + e^{-2\pi n t} e^{2\pi \mathrm{i} z}\right) \\
&\leqslant \prod_{n=1}^{N} \left(1 + e^{2\pi |z|}\right) \\
&\leqslant \left(2 e^{2\pi|z|}\right)^N \\
&\leqslant 2^{|z|/t+1} e^{2\pi|z|(|z|/t+1)} \\
&= \exp\left\{ \log 2 + \left(2\pi + \frac{\log 2}{t}\right)|z| + \frac{2\pi}{t}|z|^2 \right\}.
\end{aligned}$$

Noticing that

$$\alpha + \beta|z| + \gamma|z|^2 \leqslant (\alpha + \beta) + (\beta + \gamma)|z|^2 \quad \text{whenever } \alpha, \beta, \gamma > 0,$$

we get
$$|F_1(z)| \leqslant \exp\left\{\left(2\pi + \frac{t+1}{t}\log 2\right) + \left(2\pi\frac{t+1}{t} + \frac{\log 2}{t}\right)|z|^2\right\}.$$
Therefore,
$$|F(z)| = |F_1(z)||F_2(z)| \leqslant Ae^{a|z|^2}$$
where
$$A = \exp\left\{\frac{1}{1-e^{-2\pi t}} + 2\pi + \frac{t+1}{t}\log 2\right\} \quad \text{and} \quad a = 2\pi\frac{t+1}{t} + \frac{\log 2}{t}.$$

(2) By Proposition 3.1, F vanishes exactly when $-2\pi nt + 2\pi \mathrm{i}z = 2m\pi\mathrm{i}$ for some $n \geqslant 1$ and n, m integers, i.e., $z = -\mathrm{i}nt + m$. According to Exercise 9.3.3, the first series diverges:
$$\sum \frac{1}{|z_n|^2} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{(m^2+n^2t^2)} = \infty.$$
It then follows from part (1) and Theorem 2.1 (ii) that F is of order 2, and hence the second series converges. □

Stein 5.6.5 Show that if $\alpha > 1$, then
$$F_\alpha(z) = \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi \mathrm{i}zt}\,\mathrm{d}t$$
is an entire function of growth order $\frac{\alpha}{\alpha-1}$.

Proof Interchanging the order of integration, we have
$$\int_T F_\alpha(z)\,\mathrm{d}z = \int_T\int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi \mathrm{i}zt}\,\mathrm{d}t\,\mathrm{d}z = \int_{-\infty}^{\infty}\int_T e^{-|t|^\alpha} e^{2\pi \mathrm{i}zt}\,\mathrm{d}z\,\mathrm{d}t = \int_{-\infty}^{\infty} 0\,\mathrm{d}t = 0$$
for any triangle T. Then Morera's theorem implies that $F_\alpha(z)$ is entire. To find the growth order ρ of $F_\alpha(z)$, we set $A = 4\pi$ and observe that

- If $|t|^{\alpha-1} \leqslant A|z|$, then
$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| \leqslant 2\pi|z|A^{\frac{1}{\alpha-1}}|z|^{\frac{1}{\alpha-1}} = 2\pi A^{\frac{1}{\alpha-1}}|z|^{\frac{\alpha}{\alpha-1}}.$$
- If $|t|^{\alpha-1} > A|z|$, then
$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| = |t|\left(-\frac{|t|^{\alpha-1}}{2} + 2\pi|z|\right) \leqslant |t|\left(-\frac{A|z|}{2} + 2\pi|z|\right) = |t||z|\left(2\pi - \frac{A}{2}\right) = 0.$$

Thus, we have
$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leqslant 2\pi|z||t| \leqslant c|z|^{\frac{\alpha}{\alpha-1}} \tag{5.6.5–1}$$
for some constant $c > 0$.

(1) We begin by showing that $\rho \leqslant \frac{\alpha}{\alpha-1}$. Using (5.6.5–1) we have
$$|F_\alpha(z)| \leqslant \int_{-\infty}^{\infty} e^{-|t|^\alpha + 2\pi|z||t|}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-\frac{|t|^\alpha}{2}} e^{-\frac{|t|^\alpha}{2} + 2\pi|z||t|}\,\mathrm{d}t$$

$$\begin{aligned}
&\leqslant e^{c|z|^{\frac{\alpha}{\alpha-1}}} \int_{-\infty}^{\infty} e^{-\frac{|t|^\alpha}{2}}\,\mathrm{d}t = 2e^{c|z|^{\frac{\alpha}{\alpha-1}}} \int_0^{\infty} e^{-\frac{t^\alpha}{2}}\,\mathrm{d}t \\
&\leqslant 2e^{c|z|^{\frac{\alpha}{\alpha-1}}} \left(1 + \int_1^{\infty} e^{-\frac{t}{2}}\,\mathrm{d}t\right) = 2c\Big(1 + e^{-\frac{1}{2}}\Big) e^{c|z|^{\frac{\alpha}{\alpha-1}}}.
\end{aligned}$$

Hence $\rho \leqslant \frac{\alpha}{\alpha-1}$.

(2) When $z = -\mathrm{i}y$ for $y \in \mathbb{R}$, we have

$$\begin{aligned}
F_\alpha(-\mathrm{i}y) &= \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi yt}\,\mathrm{d}t \\
&\geqslant \int_{\frac{1}{2}y^{1/(\alpha-1)}}^{y^{1/(\alpha-1)}} e^{-|t|^\alpha} e^{2\pi yt}\,\mathrm{d}t \\
&\geqslant e^{-y^{\alpha/(\alpha-1)}} \int_{\frac{1}{2}y^{1/(\alpha-1)}}^{y^{1/(\alpha-1)}} e^{2\pi yt}\,\mathrm{d}t \\
&\geqslant e^{-y^{\alpha/(\alpha-1)}} \left(y^{1/(\alpha-1)} - \frac{1}{2}y^{1/(\alpha-1)}\right) e^{\pi y^{\alpha/(\alpha-1)}} \\
&= \frac{1}{2} y^{1/(\alpha-1)} e^{(\pi-1)y^{\alpha/(\alpha-1)}}.
\end{aligned}$$

Hence $\rho \geqslant \frac{\alpha}{\alpha-1}$. □

Stein 5.6.6 Prove Wallis's product formula

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2m\cdot 2m}{(2m-1)\cdot(2m+1)}\cdots.$$

Proof Evaluate the product formula $\dfrac{\sin \pi z}{\pi} = z\displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$ at $z = \frac{1}{2}$. □

Stein 5.6.7 Establish the following properties of infinite products.

(1) Show that if $\displaystyle\sum_{n=1}^{\infty}|a_n|^2$ converges, then the product $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$ converges to a non-zero limit if and only if $\displaystyle\sum_{n=1}^{\infty}a_n$ converges.

(2) Find an example of a sequence of complex numbers $\{a_n\}$ such that $\displaystyle\sum_{n=1}^{\infty}a_n$ converges but $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$ diverges.

(3) Also find an example such that $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$ converges and $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges.

Solution (1) If $\displaystyle\sum_{n=1}^{\infty}|a_n|^2$ converges, then $\displaystyle\lim_{n\to\infty}a_n = 0$, and

$$\lim_{n\to\infty}\frac{a_n - \log(1+a_n)}{a_n^2} = \frac{1}{2}.$$

By the limit comparison test, the series $\sum_{n=1}^{\infty}[a_n - \log(1+a_n)]$ converges. Hence,

$$\prod_{n=1}^{\infty}(1+a_n) \text{ converges to a non-zero limit } \iff \sum_{n=1}^{\infty}\log(1+a_n) \text{ converges } \iff \sum_{n=1}^{\infty} a_n \text{ converges.}$$

(2) Let $a_n = \dfrac{(-1)^n}{\sqrt{n}}$. Then $\sum_{n=2}^{\infty} a_n$ converges by Leibniz's test for alternating series. Since

$$\prod_{n=2}^{\infty}(1+a_n) = \prod_{k=1}^{\infty}\left(1+\frac{1}{\sqrt{2k}}\right)\left(1-\frac{1}{\sqrt{2k+1}}\right) =: \prod_{k=1}^{\infty} b_k,$$

with $b_k < \left(1+\frac{1}{\sqrt{2k+1}}\right)\left(1-\frac{1}{\sqrt{2k+1}}\right) = 1-\frac{1}{2k+1}$, we have $1-b_k > \frac{1}{2k+1}$ and so $\sum_{k=1}^{\infty}(1-b_k)$ diverges. Note that $b_k \to 1$, therefore

$$\lim_{k\to\infty} -\frac{\log b_k}{1-b_k} = 1.$$

Hence $\sum_{k=1}^{\infty} -\log b_k$ diverges by the limit comparison test, and it follows that

$$\prod_{n=2}^{\infty}(1+a_n) = \prod_{k=1}^{\infty} b_k \text{ diverges.}$$

(3) Let

$$a_n = \begin{cases} -\dfrac{1}{\sqrt{k}}, & n = 2k-1, \\ \dfrac{1}{\sqrt{k}} + \dfrac{1}{k} + \dfrac{1}{k\sqrt{k}}, & n = 2k. \end{cases}$$

Then

$$\sum_{n=1}^{2N} a_n = \sum_{k=1}^{N}(a_{2k-1}+a_{2k}) = \sum_{k=1}^{N}\frac{1}{\sqrt{k}} + \sum_{k=1}^{N}\frac{1}{k\sqrt{k}} \xrightarrow{N\to\infty} +\infty,$$

while

$$\begin{aligned}\prod_{n=2}^{2N}(1+a_n) &= (1+a_2)\prod_{k=2}^{N}(1+a_{2k-1})(1+a_{2k}) = 4\prod_{k=2}^{N}\left(1-\frac{1}{\sqrt{k}}\right)\left(1+\frac{1}{\sqrt{k}}\right)\left(1+\frac{1}{k}\right) \\ &= 4\prod_{k=2}^{N}\frac{k-1}{k}\cdot\frac{k+1}{k} = 4\cdot\frac{N+1}{2N} \xrightarrow{N\to\infty} 2.\end{aligned}$$

□

Stein 5.6.8 Prove that for every z the product below converges, and

$$\cos\left(\frac{z}{2}\right)\cos\left(\frac{z}{4}\right)\cos\left(\frac{z}{8}\right)\cdots = \prod_{k=1}^{\infty}\cos\left(\frac{z}{2^k}\right) = \frac{\sin z}{z}.$$

Proof Since

$$\prod_{n=1}^{n}\cos\left(\frac{z}{2^k}\right) = \frac{\sin z}{2^n \sin\left(\frac{z}{2^n}\right)},$$

taking the limit as $n \to \infty$ gives

$$\prod_{k=1}^{\infty} \cos\left(\frac{z}{2^k}\right) = \lim_{n\to\infty} \frac{\sin z}{2^n \sin\left(\frac{z}{2^n}\right)} = \frac{\sin z}{z}.$$

□

Stein 5.6.11 Show that if f is an entire function of finite order that omits two values, then f is constant. This result remains true for any entire function and is known as Picard's little theorem.

Proof If f omits two values a and b, then by Hadamard's factorization theorem the function $f(z) - a$ is of the form $e^{p(z)}$ where p is a polynomial. It follows that $p(z) \neq \log(b-a)$ for all $z \in \mathbb{C}$, and hence $p(z)$ must be constant by the fundamental theorem of algebra. Therefore, $f(z) = e^{p(z)} + a$ is constant. □

Stein 5.6.12 Suppose f is entire and never vanishes, and that none of the higher derivatives of f ever vanish. Prove that if f is also of finite order, then $f(z) = e^{az+b}$ for some constants a and b.

Proof By Hadamard's factorization theorem, f is of the form $e^{p(z)}$ where p is a polynomial. By assumption, $f'(z) = p'(z)e^{p(z)}$ has no zeros, and hence $p'(z)$ must be constant, i.e., $p(z) = az + b$ for some constants a and b. Therefore, $f(z) = e^{p(z)} = e^{az+b}$. □

Stein 5.6.15 Prove that every meromorphic function in $\mathbb{C}$ is the quotient of two entire functions. Also, if $\{a_n\}$ and $\{b_n\}$ are two disjoint sequences having no finite limit points, then there exists a meromorphic function in the whole complex plane that vanishes exactly at $\{a_n\}$ and has poles exactly at $\{b_n\}$.

Proof

(1) Let f be a meromorphic function in $\mathbb{C}$ with poles at $\{p_n\}$, counted with multiplicities. By Weierstrass's construction, one can find an entire function g such that $g(z) = 0$ exactly at $\{p_n\}$. Then the function $h(z) \coloneqq f(z)g(z)$ is entire, and $f = h/g$.

(2) Let F and G be two entire functions such that $F(z) = 0$ exactly at $\{a_n\}$ and $G(z) = 0$ exactly at $\{b_n\}$. Then the meromorphic function F/G has the required properties. □

Stein 5.6.16 Suppose that

$$Q_n(z) = \sum_{k=1}^{N_n} c_k^n z^k$$

are given polynomials for $n = 1, 2, \cdots$. Suppose also that we are given a sequence of distinct complex numbers $\{a_n\}$ without limit points. Prove that there exists a meromorphic function $f(z)$ whose only poles are at $\{a_n\}$, and so that for each n, the difference

$$f(z) - Q_n\left(\frac{1}{z - a_n}\right)$$

is holomorphic near a_n. In other words, f has prescribed poles and principal parts at each of these poles. This result is due to Mittag-Leffler.

Proof We can add on $Q(1/z)$ by hand, so we may assume each $a_n \neq 0$. Since $Q_n\left(\frac{1}{z-a_n}\right)$ is holomorphic in an open neighborhood of $\overline{\mathbb{B}\left(0, \frac{|a_n|}{2}\right)}$, its Taylor series converges uniformly there, so we can find polynomials $P_n(z)$ such that

$$\sup_{|z| \leqslant \frac{|a_n|}{2}} \left| Q_n\left(\frac{1}{z - a_n}\right) - P_n(z) \right| \leqslant \frac{1}{2^n}.$$

Therefore, the function

$$f(z) := \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly on compact subsets of $\mathbb{C} \setminus \{a_n\}_{n=1}^{\infty}$ and, by construction,

$$f(z) - Q_n\left(\frac{1}{z - a_n}\right)$$

has a removable singularity at a_n. Thus, f has prescribed poles and principal parts at each of these poles. $\square$