Stein 4.4.1 Suppose f is continuous and of moderate decrease, and $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Show that $f = 0$ by completing the following outline:

(1) For each fixed real number t consider the two functions

$$A(z) = \int_{-\infty}^{t} f(x) e^{-2\pi \mathrm{i} z(x-t)} \,\mathrm{d}x \quad \text{and} \quad B(z) = -\int_{t}^{\infty} f(x) e^{-2\pi \mathrm{i} z(x-t)} \,\mathrm{d}x.$$

Show that $A(\xi) = B(\xi)$ for all $\xi \in \mathbb{R}$.

(2) Prove that the function F equal to A in the closed upper half-plane, and B in the lower half-plane, is entire and bounded, thus constant. In fact, show that $F = 0$.

(3) Deduce that

$$\int_{-\infty}^{t} f(x) \,\mathrm{d}x = 0,$$

for all t, and conclude that $f = 0$.

Proof (1) For all $\xi \in \mathbb{R}$, we have

$$A(\xi) - B(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi \mathrm{i} \xi (x-t)} \,\mathrm{d}x = e^{2\pi \mathrm{i} \xi t} \hat{f}(\xi) = 0.$$

(2) The function F is entire by the symmetry principle. Since f is of moderate decrease, we have

$$|A(z)| \leqslant \int_{-\infty}^{t} |f(x)| e^{2\pi \operatorname{Im}(z)(x-t)} \,\mathrm{d}x \leqslant \int_{-\infty}^{t} \frac{C}{1+x^2} \,\mathrm{d}x \leqslant \pi C \quad \text{whenever } \operatorname{Im}(z) \geqslant 0,$$

and similarly

$$|B(z)| \leqslant \int_{t}^{\infty} |f(x)| e^{2\pi \operatorname{Im}(z)(x-t)} \,\mathrm{d}x \leqslant \int_{t}^{\infty} \frac{C}{1+x^2} \,\mathrm{d}x \leqslant \pi C \quad \text{whenever } \operatorname{Im}(z) < 0.$$

Thus F is a bounded entire function, which must be constant by Liouville's theorem. Now, take $z = \mathrm{i}s$ for $s \geqslant 0$ and note that

$$A(\mathrm{i}s) = \int_{-\infty}^{t} f(x) e^{2\pi s(x-t)} \,\mathrm{d}x \xrightarrow{s \to \infty} 0$$

by Lebesgue's dominated convergence theorem. Therefore $F = 0$.

(3) By (2), $F(0) = \displaystyle\int_{-\infty}^{t} f(x) \,\mathrm{d}x = 0$ for all $t \in \mathbb{R}$, which implies that $f = 0$. □

Stein 4.4.2 If $f \in \mathfrak{F}_a$ with $a > 0$, then for any positive integer n one has $f^{(n)} \in \mathfrak{F}_b$ whenever $0 < b < a$.

Proof Only the second condition in the definition of $\mathfrak{F}_b$ needs to be checked. For any fixed b, if we take $r = a - b > 0$, then

$$\overline{\mathbb{B}(x + \mathrm{i}y, r)} \subset \{z \in \mathbb{C} : |\operatorname{Im}(z)| < a\} \quad \text{for all } x \in \mathbb{R} \text{ and } |y| < b.$$

By the Cauchy inequalities,

$$\left|f^{(n)}(x+\mathrm{i}y)\right| \leqslant \frac{n!}{r^n} \sup_{|\zeta-(x+\mathrm{i}y)|=r} |f(\zeta)|.$$

Since $f \in \mathfrak{F}_a$, we have

$$|f(x+\mathrm{i}y)| \leqslant \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R} \text{ and } |y|<a.$$

Then, for $|y|<b$,

$$\left|f^{(n)}(x+\mathrm{i}y)\right| \leqslant \frac{n!A}{r^n} \sup_{\theta\in[0,2\pi]} \frac{1}{1+(x+r\cos\theta)^2} = \frac{n!A}{r^n}\frac{1}{[1+(|x|-r)^2]}.$$

Finally, note that since

$$\lim_{x\to\infty} \frac{1+x^2}{1+(|x|-r)^2} = 1,$$

there exists a constant $C>0$ such that

$$1+(|x|-r)^2 \geqslant C\big(1+x^2\big), \quad \forall x \in \mathbb{R}.$$

Combining all the above estimates, we obtain

$$\left|f^{(n)}(z)\right| \leqslant \frac{n!A}{Cr^n}\frac{1}{1+x^2}.$$

Therefore, we conclude that $f^{(n)} \in \mathfrak{F}_b$ for all $0<b<a$. □

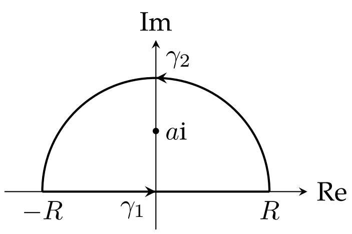
Stein 4.4.3 Show, by contour integration, that if $a>0$ and $\xi \in \mathbb{R}$ then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = e^{-2\pi a|\xi|},$$

and check that

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi = \frac{1}{\pi}\frac{a}{a^2+x^2}.$$

Proof We may assume that $\xi<0$. Consider the integral of $f(z) = \dfrac{a}{a^2+z^2}e^{-2\pi\mathrm{i}z\xi}$ along the upper semicircle contour as shown below:



By the residue formula,

$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,a\mathrm{i}) = 2\pi\mathrm{i}\lim_{z\to a\mathrm{i}} \frac{a}{z+a\mathrm{i}}e^{-2\pi\mathrm{i}z\xi} = \pi e^{-2\pi a|\xi|}.$$

Since

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leqslant \int_0^\pi \left|\frac{a}{a^2+R^2e^{2\mathrm{i}\theta}}e^{2\pi R\xi\sin\theta}\right|\mathrm{d}\theta \leqslant \frac{\pi a}{R^2-a^2} \xrightarrow{R\to+\infty} 0,$$

we get

$$\int_{\mathbb{R}} \frac{a}{a^2+x^2}e^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = \pi e^{-2\pi a|\xi|}.$$

To check the second identity, we can use Lemma 2.3 to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-2\pi a|\xi|}e^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi &= \int_{-\infty}^{0} e^{2\pi\xi(a+\mathrm{i}x)}\,\mathrm{d}\xi + \int_0^\infty e^{2\pi\xi(-a+\mathrm{i}x)}\,\mathrm{d}\xi \\
&= \int_0^\infty e^{-(2\pi a+2\pi x\mathrm{i})\xi}\,\mathrm{d}\xi + \int_0^\infty e^{-(2\pi a-2\pi x\mathrm{i})\xi}\,\mathrm{d}\xi \\
&= \frac{1}{2\pi a+2\pi\mathrm{i}x} + \frac{1}{2\pi a-2\pi\mathrm{i}x} \\
&= \frac{1}{\pi}\frac{a}{a^2+x^2}.
\end{aligned}$$

□

Stein 4.4.4 Suppose Q is a polynomial of degree $\geqslant 2$ with distinct roots, none lying on the real axis. Calculate

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi\mathrm{i}x\xi}}{Q(x)}\,\mathrm{d}x, \quad \xi\in\mathbb{R}$$

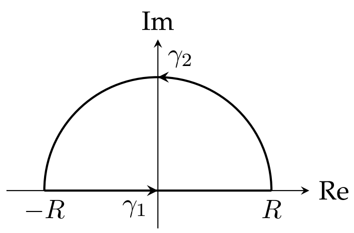
in terms of the roots of Q. What happens when several roots coincide?

Solution Let U be the set of roots of Q in the upper half-plane, and L be the set of roots of Q in the lower half-plane.

Assume first $\xi \leqslant 0$. Choose R large enough so that

- ◇ $|Q(z)| \geqslant C|z|^2$ for some constant R and all $|z| \geqslant R$,
- ◇ all the roots of Q are contained in $\mathbb{B}(0,R)$.

Now, consider the integral of $f(z) = \dfrac{e^{-2\pi\mathrm{i}z\xi}}{Q(z)}$ along the contour $\gamma = \gamma_1 \cup \gamma_2$ as shown below:



Since

$$\begin{aligned}
\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| &= \left|\int_0^\pi \frac{\exp\bigl(-2\pi\mathrm{i}Re^{\mathrm{i}\theta}\xi\bigr)}{Q(Re^{\mathrm{i}\theta})}\mathrm{i}Re^{\mathrm{i}\theta}\,\mathrm{d}\theta\right| \\
&\leqslant \int_0^\pi \frac{\exp(2\pi R\xi\sin\theta)}{CR^2}R\,\mathrm{d}\theta \\
&\leqslant \frac{\pi}{CR} \xrightarrow{R\to+\infty} 0,
\end{aligned}$$

we obtain by the residue formula that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{Q(x)}\,dx = \lim_{R\to+\infty}\int_\gamma f(z)\,dz = 2\pi i\sum_{z\in U}\operatorname{Res}(f,z) = 2\pi i\sum_{z\in U}\frac{e^{-2\pi i z\xi}}{Q'(z)}.$$

By considering the polynomial $Q(-x)$ and substituting $-x$ for x, we have for $\xi \geqslant 0$ that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{Q(x)}\,dx = -2\pi i\sum_{z\in L}\frac{e^{-2\pi i z\xi}}{Q'(z)}.$$

For multiple roots, the idea is the same, but the formula for the residues would be more complicated.□

Stein 4.4.6 Prove that

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}$$

whenever $a>0$. Hence show that the sum equals $\coth \pi a$.

Proof Let $f(z) = \dfrac{1}{\pi}\dfrac{a}{a^2+z^2}$. Note that

$$|f(x+iy)| = \frac{a}{\pi}\frac{1}{|a^2+(x+iy)^2|} \leqslant \frac{a}{\pi}\frac{1}{|a^2+x^2-y^2|} \leqslant \frac{a}{\pi}\frac{1}{\frac{3a^2}{4}+x^2}$$

whenever $|y|<\frac{a}{2}$. Since

$$\lim_{x\to\infty}\frac{1+x^2}{\frac{3a^2}{4}+x^2} = 1,$$

there exists a constant $C>0$ such that

$$|f(x+iy)| \leqslant \frac{a}{\pi}\frac{C}{1+x^2} \quad \text{whenever } |y|<\frac{a}{2}.$$

This shows that $f\in\mathfrak{F}$. By the Poisson summation formula, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}}\hat{f}(n) \xlongequal{\text{Exercise 4.4.3}} \sum_{n\in\mathbb{Z}} e^{-2\pi a|n|}.$$

Hence the sum equals

$$-1+2\sum_{n=0}^{\infty}e^{-2\pi a n} = -1+\frac{2}{1-e^{-2\pi a}} = \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} = \coth \pi a.$$

□

Stein 4.4.7 The Poisson summation formula applied to specific examples often provides interesting identities.

(1) Let τ be fixed with $\operatorname{Im}(\tau)>0$. Apply the Poisson summation formula to

$$f(z) = (\tau+z)^{-k},$$

where k is an integer $\geqslant 2$, to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi\mathrm{i})^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi\mathrm{i}m\tau}.$$

(2) Set $k=2$ in the above formula to show that if $\mathrm{Im}(\tau) > 0$, then

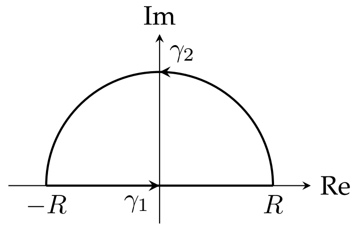
$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}. \tag{4.4.7–1}$$

(3) Can one conclude that the above formula holds true whenever τ is any complex number that is not an integer?

Proof (1) Let us find the Fourier transform of f:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} (\tau+x)^{-k} e^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x.$$

① For $\xi \leqslant 0$, we choose the upper semicircle contour.



Since $(\tau+z)^{-k}e^{-2\pi\mathrm{i}z\xi}$ is holomorphic in the upper half-plane, we have

$$\int_{\gamma_1} (\tau+z)^{-k} e^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z + \int_{\gamma_2} (\tau+z)^{-k} e^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z = 0.$$

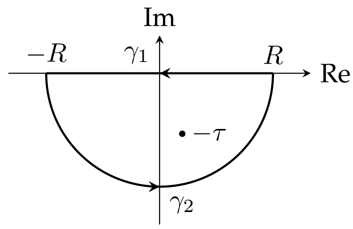
When $R \to +\infty$,

$$\begin{aligned}\left|\int_{\gamma_2} (\tau+z)^{-k} e^{-2\pi\mathrm{i}z\xi}\,\mathrm{d}z\right| &= \left|\int_0^{\pi} \frac{e^{-2\pi\mathrm{i}\xi Re^{\mathrm{i}\theta}} R\mathrm{i}}{(\tau+Re^{\mathrm{i}\theta})^k}\,\mathrm{d}\theta\right| \leqslant \int_0^{\pi} \frac{R\left|e^{-2\pi\mathrm{i}\xi Re^{\mathrm{i}\theta}}\right|}{(R-|\tau|)^k}\,\mathrm{d}\theta \\ &\leqslant \frac{\pi R^2 e^{2\pi\xi R\sin\theta}}{(R-|\tau|)^k} \overset{\xi\leqslant 0}{\leqslant} \frac{\pi R^2}{(R-|\tau|)^k} \xrightarrow{k\geqslant 2} 0.\end{aligned}$$

Hence, when $\xi \leqslant 0$ we get

$$\hat{f}(\xi) = \int_{\mathbb{R}} (\tau+x)^{-k} e^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = 0.$$

② For $\xi > 0$, we choose the lower semicircle contour.



The residue at $-\tau$ is given by

$$\operatorname{Res}\big((\tau+z)^{-k}e^{-2\pi i z\xi}, -\tau\big) = \frac{1}{(k-1)!}\big(e^{-2\pi i z\xi}\big)^{(k-1)}\Big|_{z=-\tau} = \frac{(-2\pi i\xi)^{k-1}}{(k-1)!}e^{2\pi i\tau\xi}.$$

Thus, we have

$$\int_{\gamma_1}(\tau+z)^{-k}e^{-2\pi i z\xi}\,dz + \int_{\gamma_2}(\tau+z)^{-k}e^{-2\pi i z\xi}\,dz = -\frac{(-2\pi i)^k\xi^{k-1}}{(k-1)!}e^{2\pi i\tau\xi}.$$

When $R \to +\infty$,

$$\left|\int_{\gamma_2}(\tau+z)^{-k}e^{-2\pi i z\xi}\,dz\right| = \left|\int_{-\pi}^{0}\frac{e^{-2\pi i\xi Re^{i\theta}}Ri}{(\tau+Re^{i\theta})^k}\,d\theta\right| \leqslant \int_{-\pi}^{0}\frac{R\left|e^{-2\pi i\xi Re^{i\theta}}\right|}{(R-|\tau|)^k}\,d\theta$$
$$\leqslant \frac{\pi R^2 e^{2\pi\xi R\sin\theta}}{(R-|\tau|)^k} \overset{\xi>0}{\leqslant} \frac{\pi R^2}{(R-|\tau|)^k} \xrightarrow{k\geqslant 2} 0.$$

Hence, when $\xi > 0$ we get

$$\hat{f}(\xi) = \int_{\mathbb{R}}(\tau+x)^{-k}e^{-2\pi i x\xi}\,dx = \frac{(-2\pi i)^k\xi^{k-1}}{(k-1)!}e^{2\pi i\tau\xi}.$$

Given that $f \in \mathfrak{F}$, the Poisson summation formula gives

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^k} = \sum_{m\in\mathbb{Z}}\hat{f}(m) = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}e^{2\pi i m\tau}.$$

(2) Setting $k = 2$ in the formula above, we get

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^2} = -4\pi^2\sum_{m=1}^{\infty}me^{2\pi i m\tau}.$$

To complete the proof, notice that when $\operatorname{Im}(\tau) > 0$ we have $\left|e^{2\pi i\tau}\right| = e^{-2\pi\operatorname{Im}(\tau)} < 1$, hence

$$\sum_{m=1}^{\infty}me^{2\pi i m\tau} = \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{\partial}{\partial\tau}\big(e^{2\pi i m\tau}\big) = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\left(\sum_{m=1}^{\infty}e^{2\pi i m\tau}\right) = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\left(\frac{e^{2\pi i\tau}}{1-e^{2\pi i\tau}}\right)$$
$$= \frac{e^{2\pi i\tau}}{(1-e^{2\pi i\tau})^2} = \frac{1}{(e^{\pi i\tau}-e^{-\pi i\tau})^2} = \frac{1}{-4\sin^2(\pi\tau)}.$$

(3) Recall from Exercise 3.8.1 that the complex zeros of $\sin(\pi\tau)$ are exactly at the integers. Hence, both

sides of (4.4.7–1) are holomorphic functions of τ in $\mathbb{C} \setminus \mathbb{Z}$. Since they agree for $\mathrm{Im}(\tau) > 0$, by the identity theorem, they must agree for all $\tau \in \mathbb{C} \setminus \mathbb{Z}$. □

Remark Part (3) has already been proved in Exercise 3.8.12.