Stein 3.8.11 Show that if $|a|<1$, then

$$\int_0^{2\pi} \log\left|1-ae^{\mathrm{i}\theta}\right| \mathrm{d}\theta = 0.$$

Then, prove that the above result remains true if we assume only that $|a| \leqslant 1$.

Proof (1) If $|a|<1$, let us consider the function $f(z)=1-az$, which vanishes nowhere in the closed unit disc. By Theorem 6.2, there exists a holomorphic function g in a disc of radius greater than 1 such that $f(z)=e^{g(z)}$. Then $|f|=e^{\mathrm{Re}(g)}$, and therefore $\log|f|=\mathrm{Re}(g)$. By Corollary 7.3, we have

$$\int_0^{2\pi} \log\left|1-ae^{\mathrm{i}\theta}\right| \mathrm{d}\theta = 2\pi\log|f(0)| = 0.$$

(2) For $a=e^{\mathrm{i}\varphi}$, we have

$$\begin{aligned}\int_0^{2\pi} \log\left|1-ae^{\mathrm{i}\theta}\right| \mathrm{d}\theta &= \int_0^{2\pi} \log\left|1-e^{\mathrm{i}(\theta+\varphi)}\right| \mathrm{d}\theta = \int_0^{2\pi} \log\left|1-e^{\mathrm{i}\theta}\right| \mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi} \log(2-2\cos\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \log\left(2\sin\frac{\theta}{2}\right)\mathrm{d}\theta \\ &\overset{x=\frac{\theta}{2\pi}}{=\!=} 2\pi\left(\log 2 + \int_0^1 \log(\sin\pi x)\,\mathrm{d}x\right) \overset{\text{Exercise 3.8.9}}{=\!=\!=\!=} 0.\end{aligned}$$

$\square$

Stein 3.8.12 Suppose u is not an integer. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin\pi u)^2}$$

by integrating

$$f(z) = \frac{\pi\cot\pi z}{(u+z)^2}$$

over the circle $|z|=R_N=N+\frac{1}{2}$ (N integral, $N\geqslant|u|$), adding the residues of f inside the circle, and letting N tend to infinity.

Proof The function $f(z)=\dfrac{\pi\cot\pi z}{(u+z)^2}$ has simple poles at $z=n$ for $n\in\mathbb{Z}$, and a pole of order 2 at $z=-u$. The corresponding residues are given by

$$\begin{aligned}\mathrm{Res}(f,n) &= \lim_{z\to n}(z-n)f(z) = \lim_{z\to n}\frac{\pi\cos\pi z}{(u+z)^2(\sin\pi z)'} = \frac{1}{(u+n)^2},\\ \mathrm{Res}(f,-u) &= \lim_{z\to -u}\frac{\mathrm{d}}{\mathrm{d}z}(z+u)^2 f(z) = \pi\lim_{z\to -u}\frac{\mathrm{d}}{\mathrm{d}z}\cot\pi z = -\frac{\pi^2}{(\sin\pi u)^2}.\end{aligned}$$

By the residue formula, we have

$$I_N := \int\limits_{|z|=R_N} f(z)\,\mathrm{d}z = \sum_{n=-N}^{N}\frac{1}{(u+n)^2} - \frac{\pi^2}{(\sin\pi u)^2}.$$

It remains to show that $I_N\to 0$ as $N\to\infty$. It suffices to show that $\cot\pi z$ is bounded when $|z|=R_N$, independent of N. Suppose $z=x+\mathrm{i}y$.

(1) Consider first the case $y > 0$. We use the expression

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} = \mathrm{i}\frac{e^{2\pi \mathrm{i} z} + 1}{e^{2\pi \mathrm{i} z} - 1}. \tag{3.8.12–1}$$

◇ If $0 < y \leqslant 1$, then

$$R_N - |x| \leqslant R_N - \sqrt{R_N^2 - 1} = \frac{1}{R_N + \sqrt{R_N^2 - 1}} \xrightarrow{N \to \infty} 0.$$

Hence, for N sufficiently large, we have

$$|\cos 2\pi(x - R_N)| < \frac{1}{2},$$

and therefore

$$\begin{aligned}\left|e^{2\pi \mathrm{i}(x+\mathrm{i}y)} - 1\right| &\geqslant \mathrm{Re}\Big(1 - e^{2\pi \mathrm{i}(x+\mathrm{i}y)}\Big) \\ &= 1 - e^{-2\pi y}\cos(2\pi x) \\ &= 1 + e^{-2\pi y}\cos 2\pi(x - R_N) \\ &> 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

◇ If $y > 1$, then

$$\left|e^{2\pi \mathrm{i}(x+\mathrm{i}y)} - 1\right| \geqslant 1 - e^{-2\pi y} \geqslant 1 - e^{-2\pi}.$$

These estimates, along with the fact that

$$\left|e^{2\pi \mathrm{i}(x+\mathrm{i}y)} + 1\right| \leqslant e^{-2\pi y} + 1 < 2,$$

imply that (3.8.12–1) is bounded for $y > 0$, independent of N.

(2) When $y = 0$, that is, $|x| = N + \frac{1}{2}$, we have

$$\cot \pi z = \cot \pi\big(N + \tfrac{1}{2}\big) = 0.$$

(3) When $y < 0$, we rewrite the expression for $\cot \pi z$ as

$$\cot \pi z = \mathrm{i}\frac{1 + e^{-2\pi \mathrm{i} z}}{1 - e^{-2\pi \mathrm{i} z}},$$

which reduces to case (1) above.

Thus, we conclude that $\cot \pi z$ is bounded when $|z| = R_N$, independent of N, and the result follows. □

Stein 3.8.19 Prove the maximum principle for harmonic functions, that is:

(1) If u is a non-constant real-valued harmonic function in a region Ω, then u cannot attain a maximum (or a minimum) in Ω.

(2) Suppose that Ω is a region with compact closure $\overline{\Omega}$. If u is harmonic in Ω and continuous in $\overline{\Omega}$, then

$$\sup_{z\in\Omega}|u(z)| \leqslant \sup_{z\in\overline{\Omega}-\Omega}|u(z)|.$$

Proof (1) Assume that u attains a maximum at $z_0 \in \Omega$. Let f be holomorphic near z_0 with $u = \mathrm{Re}(f)$. Since f is non-constant, it is an open map. However, we can find an open neighborhood U of z_0 in Ω such that $f(U)$ is contained in the left half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \leqslant u(z_0)\}$. This contradicts the fact that f is an open map.

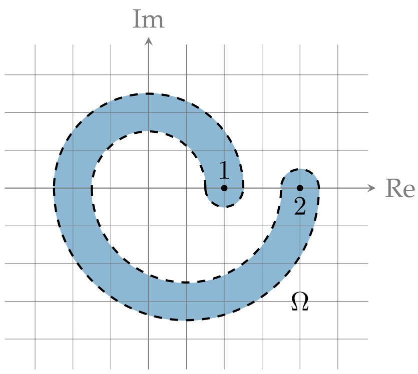
(2) Since u is continuous in the compact set $\overline{\Omega}$, it attains its maximum at some point $z_0 \in \overline{\Omega}$. If u is constant, then the result is trivial. Otherwise, by (1) we have $z_0 \in \overline{\Omega} - \Omega$. That is,

$$\sup_{z\in\Omega}|u(z)| \leqslant \sup_{z\in\overline{\Omega}}|u(z)| = \sup_{z\in\overline{\Omega}-\Omega}|u(z)|.$$

□

Stein Page 98 In Theorem 6.1 (iii) of Chapter 3, do we have $F(r) = \ln r$ for all $r \in \Omega \cap \mathbb{R}_{>0}$?

Solution This doesn't hold in general. For example, if we let Ω be the region illustrated in the figure below, then $F(2) - F(1) = \ln 2 + 2\pi\mathrm{i}$.



□