Stein 3.8.8 Prove that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

if $a > |b|$ and $a, b \in \mathbb{R}$.

Proof Without loss of generality, assume $b > 0$. Let $z = e^{\mathrm{i}\theta}$ and denote by C the unit circle. Then $\mathrm{d}z = \mathrm{i}z\,\mathrm{d}\theta$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, and

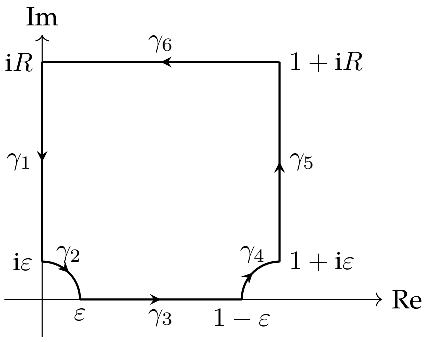
$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} &= \int_C \frac{\mathrm{d}z}{\mathrm{i}z\left[a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right]} \\
&= \frac{2}{b\mathrm{i}} \int_C \frac{\mathrm{d}z}{z^2 + \frac{2a}{b}z + 1} \\
&= \frac{4\pi}{b} \operatorname{Res}\left(\frac{1}{z^2 + \frac{2a}{b}z + 1}, -\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right) \\
&= \frac{4\pi}{b} \lim_{z \to -\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}} \frac{1}{z + \frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}} \\
&= \frac{2\pi}{\sqrt{a^2 - b^2}}.
\end{aligned}$$

□

Stein 3.8.9 Show that

$$\int_0^1 \log(\sin \pi x)\,\mathrm{d}x = -\log 2.$$

Proof Consider the integral of $f(z) = \log(\sin \pi z)$ over the contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5 \cup \gamma_6$ as illustrated below.



By Exercise 3.8.1, the function $f(z)$ is holomorphic in the region bounded by γ, so

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

(1) On γ_1, we have $\sin(\pi \mathrm{i}t) = \dfrac{\mathrm{i}(e^{\pi t} - e^{-\pi t})}{2}$, and then

$$\begin{aligned}
\operatorname{Re}\left\{\int_{\gamma_1} f(z)\,\mathrm{d}z\right\} &= \operatorname{Re}\left\{\int_R^\varepsilon \log\sin(\pi \mathrm{i}t)\mathrm{i}\,\mathrm{d}t\right\} = \int_\varepsilon^R \operatorname{Im}(\log\sin(\pi \mathrm{i}t))\,\mathrm{d}t \\
&= \int_\varepsilon^R \frac{\pi}{2}\,\mathrm{d}t = \frac{\pi}{2}(R - \varepsilon).
\end{aligned}$$

(2) On γ_2, note that $\lim_{z\to 0} z\log\sin\pi z = 0$. If we denote $M(\varepsilon) = \max_{z \in \gamma_2(\varepsilon)} |z \log\sin\pi z|$, then $\lim_{\varepsilon \to 0^+} M(\varepsilon) = 0$.

For $z = \varepsilon e^{\mathrm{i}\theta}$, we have $\mathrm{d}z = \mathrm{i}z\,\mathrm{d}\theta$, and then

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| = \left|\int_{\gamma_2} \frac{z\log\sin\pi z}{z}\,\mathrm{d}z\right| \leqslant \int_0^{\frac{\pi}{2}} M(\varepsilon)\,\mathrm{d}\theta = \frac{\pi}{2}M(\varepsilon) \xrightarrow{\varepsilon\to 0^+} 0.$$

(3) On γ_3, we have

$$\int_{\gamma_3} f(z)\,\mathrm{d}z = \int_\varepsilon^{1-\varepsilon} \log\sin\pi x\,\mathrm{d}x \xrightarrow{\varepsilon\to 0^+} \int_0^1 \log\sin\pi x\,\mathrm{d}x.$$

(4) On γ_4, since $\lim\limits_{z\to 1}(z-1)\log\sin\pi z = 0$, the same argument as in (2) applies. We have

$$\left|\int_{\gamma_4} f(z)\,\mathrm{d}z\right| \xrightarrow{\varepsilon\to 0^+} 0.$$

(5) On γ_5, we have $\sin\pi(1+\mathrm{i}t) = -\dfrac{\mathrm{i}(e^{\pi t}-e^{-\pi t})}{2}$, and then

$$\begin{aligned}\mathrm{Re}\left\{\int_{\gamma_5} f(z)\,\mathrm{d}z\right\} &= \mathrm{Re}\left\{\int_\varepsilon^R \log\sin\pi(1+\mathrm{i}t)\mathrm{i}\,\mathrm{d}t\right\} = -\int_\varepsilon^R \mathrm{Im}(\log\sin\pi(1+\mathrm{i}t))\,\mathrm{d}t\\ &= -\int_\varepsilon^R -\frac{\pi}{2}\,\mathrm{d}t = \frac{\pi}{2}(R-\varepsilon).\end{aligned}$$

(6) On γ_6, we have

$$\sin\pi(t+\mathrm{i}R) = \frac{1}{2}\left(e^{-\pi R}+e^{\pi R}\right)\sin\pi t + \mathrm{i}\cdot\frac{1}{2}\left(e^{\pi R}-e^{-\pi R}\right)\cos\pi t.$$

Thus, we can write

$$\begin{aligned}|\sin\pi(t+\mathrm{i}R)|^2 &= \frac{1}{4}\left(e^{-\pi R}+e^{\pi R}\right)^2\sin^2\pi t + \frac{1}{4}\left(e^{\pi R}-e^{-\pi R}\right)^2\cos^2\pi t\\ &= \frac{1}{4}\left[\left(e^{2\pi R}+e^{-2\pi R}\right)\left(\sin^2\pi t+\cos^2\pi t\right) + 2\left(\sin^2\pi t-\cos^2\pi t\right)\right]\\ &= \frac{1}{4}e^{2\pi R}[1+\mu(R)],\end{aligned}$$

where $\lim\limits_{R\to+\infty}\mu(R) = 0$. Hence,

$$\begin{aligned}2\log|\sin\pi(t+\mathrm{i}R)| = \log|\sin\pi(t+\mathrm{i}R)|^2 &= \log\left(\frac{1}{4}e^{2\pi R}\right) + \log(1+\mu(R))\\ &\xrightarrow{R\to+\infty} 2\pi R - 2\log 2.\end{aligned}$$

Then we obtain

$$\begin{aligned}\mathrm{Re}\left\{\int_{\gamma_6} f(z)\,\mathrm{d}z\right\} &= \mathrm{Re}\left\{\int_1^0 \log\sin\pi(t+\mathrm{i}R)\,\mathrm{d}t\right\} = -\int_0^1 \log|\sin\pi(t+\mathrm{i}R)|\,\mathrm{d}t\\ &\xrightarrow{R\to+\infty} -\frac{1}{2}(2\pi R - 2\log 2) = \log 2 - \pi R.\end{aligned}$$

Finally, combining all the results, we get

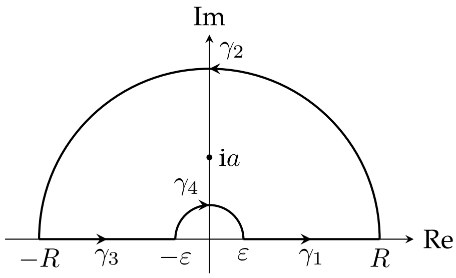
$$0 = \mathrm{Re}\left\{\int_\gamma f(z)\,\mathrm{d}z\right\} \xrightarrow[R\to+\infty]{\varepsilon\to 0^+} \int_0^1 \log(\sin \pi x)\,\mathrm{d}x + \log 2,$$

which implies the desired result. □

Stein 3.8.10 Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{2a}\log a.$$

Proof Consider the integral of $f(z) = \dfrac{\log z}{z^2+a^2}$ over the contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as shown below.



By the residue formula, we have

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,\mathrm{i}a) = 2\pi\mathrm{i}\lim_{z\to\mathrm{i}a}\frac{\log z}{z+\mathrm{i}a} = \frac{\pi\log(\mathrm{i}a)}{a} = \frac{\pi}{a}\log a + \frac{\pi^2}{2a}\mathrm{i}.$$

The integral over the semicircle γ_2 vanishes as $R\to\infty$, since for $R > a$,

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| = \left|\int_0^\pi \frac{\log\big(Re^{\mathrm{i}\theta}\big)\mathrm{i}Re^{\mathrm{i}\theta}}{R^2e^{2\mathrm{i}\theta}+a^2}\,\mathrm{d}\theta\right| \leqslant \int_0^\pi \frac{R\big|\log\big(Re^{\mathrm{i}\theta}\big)\big|}{R^2-a^2}\,\mathrm{d}\theta \leqslant \frac{\pi R|\log R + \mathrm{i}\pi|}{R^2-a^2}.$$

The integral over the semicircle γ_4 vanishes as $\varepsilon\to 0^+$, since for $\varepsilon\in(0,a)$,

$$\left|\int_{\gamma_4} f(z)\,\mathrm{d}z\right| = \left|\int_\pi^0 \frac{\log\big(\varepsilon e^{\mathrm{i}\theta}\big)\mathrm{i}\varepsilon e^{\mathrm{i}\theta}}{\varepsilon^2e^{2\mathrm{i}\theta}+a^2}\,\mathrm{d}\theta\right| \leqslant \int_0^\pi \frac{\varepsilon\big|\log\big(\varepsilon e^{\mathrm{i}\theta}\big)\big|}{a^2-\varepsilon^2}\,\mathrm{d}\theta \leqslant \frac{\pi\varepsilon|\log \varepsilon + \mathrm{i}\pi|}{a^2-\varepsilon^2}.$$

Note that

$$\int_{-\infty}^0 \frac{\log x}{x^2+a^2}\,\mathrm{d}x = \int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x + \mathrm{i}\pi\int_0^\infty \frac{\mathrm{d}x}{x^2+a^2} = \int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x + \frac{\pi^2}{2a}\mathrm{i}.$$

Thus, letting $R\to\infty$ and $\varepsilon\to 0^+$, we obtain

$$2\int_0^\infty \frac{\log x}{x^2+a^2}\,\mathrm{d}x + \frac{\pi^2}{2a}\mathrm{i} = \frac{\pi}{a}\log a + \frac{\pi^2}{2a}\mathrm{i},$$

where the desired result follows. □

Stein 3.8.14 Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$.

Proof Assume that f is an injective entire function. Then f is non-constant, and hence unbounded by Liouville's theorem. So $g(z) := f\left(\frac{1}{z}\right)$ has either a pole or an essential singularity at $z = 0$. We shall show first that the singularity at 0 cannot be an essential singularity. If it were an essential singularity, then the Casorati–Weierstrass theorem would imply that the set $g(\mathbb{D} \setminus \{0\})$ is dense in $\mathbb{C}$. However, $g\big(\mathbb{B}\big(2, \frac{1}{2}\big)\big)$ is an open set by the open mapping theorem. Therefore these two sets intersect, which shows that g and hence f is not injective.

Therefore, the singularity at 0 must be a pole, implying that f is a polynomial and furthermore that f is a monomial by its injectivity. Thus, f takes the form $f(z) = c(z - z_0)^m$. However, for $m \geqslant 2$ such functions are also non-injective:

$$f(z_0 + 1) = c = f\Big(z_0 + e^{\frac{2\pi \mathrm{i}}{m}}\Big).$$

Therefore $m = 1$ and the result follows. □

Stein 3.8.15 Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

(1) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leqslant AR^k + B$$

for all $R > 0$, and for some integer $k \geqslant 0$ and some constants $A, B > 0$, then f is a polynomial of degree $\leqslant k$.

(2) Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg z < \varphi$ as $|z| \to 1$, then $f = 0$.

(3) Let $w_1, \cdots, w_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points w_j, $1 \leqslant j \leqslant n$, is at least 1. Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points w_j, $1 \leqslant j \leqslant n$, is exactly equal to 1.

(4) Show that if the real part of an entire function f is bounded, then f is constant.

Proof (1) By the Cauchy inequalities, we have

$$\left|f^{(n)}(0)\right| \leqslant \frac{n!\big(AR^k + B\big)}{R^n}.$$

Thus, for $n > k$, we can let $R \to \infty$ to obtain $f^{(n)}(0) = 0$. Hence f is a polynomial of degree $\leqslant k$.

(2) Choose $N \in \mathbb{N}$ large enough so that $\frac{N(\varphi - \theta)}{2} > 2\pi$, and consider the function

$$g(z) := f(z) f\Big(e^{\mathrm{i}\frac{\varphi-\theta}{2}} z\Big) f\Big(e^{\mathrm{i}\frac{2(\varphi-\theta)}{2}} z\Big) \cdots f\Big(e^{\mathrm{i}\frac{N(\varphi-\theta)}{2}} z\Big).$$

Note that if we denote by S the sector $\theta < \arg z < \varphi$, then

$$\bigcup_{k=0}^{N} e^{\mathrm{i}\frac{k(\varphi-\theta)}{2}} S \supset \mathbb{D}.$$

Let M be a bound for f on $\mathbb{D}$. Given $\varepsilon > 0$, there exists $0 < r < 1$ such that

$$|f(z)| < \varepsilon, \quad r < |z| < 1,\ \theta < \arg z < \varphi.$$

Then we have

$$|g(z)| < M^N \varepsilon, \quad r < |z| < 1.$$

By the maximum modulus principle, $|g(z)| < M^N\varepsilon$ on $\mathbb{D}$, and since $\varepsilon > 0$ is arbitrary, we conclude that $g(z) = 0$ for all $z \in \mathbb{D}$. If f is not identically zero, then it has countably many zeros in $\mathbb{D}$, and so is g, which is a contradiction. Hence, $f = 0$ in $\mathbb{D}$.

(3) Consider the polynomial

$$p(z) = (z - w_1)\cdots(z - w_n).$$

Since $p(w_k) = 0$ $(1 \leqslant k \leqslant n)$ and $|p(0)| = 1$, it is non-constant. Then by the maximum modulus principle, $|p(z)| > 1$ for some z on the unit circle. Finally, by the intermediate value theorem, there exists a point w on the unit circle such that $|p(w)| = 1$.

(4) Consider the function $g(z) = e^{f(z)}$. Since $|g(z)| = e^{\operatorname{Re} f(z)}$, by our assumption, g is a bounded entire function. By Liouville's theorem, g is constant. Thus, f is locally constant, and hence constant. $\square$

Stein 3.8.16 Suppose f and g are holomorphic in a region containing the disc $|z| \leqslant 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leqslant 1$. Let

$$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

Show that if ε is sufficiently small, then

(1) $f_\varepsilon(z)$ has a unique zero in $|z| \leqslant 1$, and

(2) if z_ε is this zero, the mapping $\varepsilon \mapsto z_\varepsilon$ is continuous.

Proof (1) Since f is non-vanishing on the unit circle, we can pick ε small enough so that $|f(z)| > |\varepsilon g(z)|$ and $f_\varepsilon(z) \neq 0$ for $|z| = 1$. By Rouché's theorem, f and $f + \varepsilon g$ have the same number of zeros in $|z| < 1$. Then $f_\varepsilon(z)$ has a unique zero in $|z| \leqslant 1$.

(2) By (1), there exists $\varepsilon_0 > 0$ small enough, such that $f_\varepsilon(z)$ has a unique zero z_ε in $|z| \leqslant 1$ and $|z_\varepsilon| < 1$. Then we can write $f_\varepsilon(z) = (z - z_\varepsilon)h(z)$ for some holomorphic function h which is non-vanishing in $|z| \leqslant 1$. Note that

$$\int\limits_{|z|=1} \frac{z f_\varepsilon'(z)}{f_\varepsilon(z)}\,\mathrm{d}z = \int\limits_{|z|=1} \left(\frac{z}{z - z_\varepsilon} + \frac{z h'(z)}{h(z)}\right)\mathrm{d}z = 2\pi\mathrm{i}\operatorname{Res}\left(\frac{z}{z - z_\varepsilon}, z_\varepsilon\right) = 2\pi\mathrm{i} z_\varepsilon.$$

Since the integrand $\dfrac{z f_\varepsilon'(z)}{f_\varepsilon(z)}$ is continuous in ε for $|\varepsilon| < \varepsilon_0$, and the unit circle is compact, we see that the integrand is uniformly continuous in ε. Thus, the above integral is continuous in ε. That is, the mapping $\varepsilon \mapsto z_\varepsilon$ is continuous. $\square$

Stein 3.8.17 Let f be non-constant and holomorphic in an open set containing the closed unit disc.

(1) Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.

(2) If $|f(z)| \geqslant 1$ whenever $|z| = 1$ and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

Proof (1) By Exercise 2.6.15, f must vanish at some point in $\mathbb{D}$. For any $w_0 \in \mathbb{D}$, we have

$$|f(z)| = 1 > |w_0|, \quad \forall z \in \partial\mathbb{D},$$

hence by Rouché's theorem, $f(z) - w_0$ has the same number of zeros in $\mathbb{D}$ as $f(z)$, which is at least one. Thus, w_0 is in the image of f.

(2) In the same spirit as (1), we only need to show that f vanishes at some point in $\mathbb{D}$. If it were not the case, then $1/f$ is holomorphic in an open set containing $\overline{\mathbb{D}}$, and by the maximum modulus principle, $|1/f(z)| \leqslant 1$ for $|z| \leqslant 1$. This contradicts the assumption that $|f(z_0)| < 1$. $\square$