Stein 3.8.1 Using Euler's formula

$$\sin \pi z = \frac{e^{\mathrm{i}\pi z} - e^{-\mathrm{i}\pi z}}{2\mathrm{i}},$$

show that the complex zeros of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.

Calculate the residue of $\frac{1}{\sin \pi z}$ at $z = n \in \mathbb{Z}$.

Proof By Euler's formula,

$$\sin \pi z = 0 \iff e^{\mathrm{i}\pi z} = e^{-\mathrm{i}\pi z} \iff e^{2\mathrm{i}\pi z} = 1 \iff \begin{cases} \mathrm{Im}(z) = 0, \\ e^{2\mathrm{i}\pi \, \mathrm{Re}(z)} = 1. \end{cases} \iff z \in \mathbb{Z}.$$

To check the order of the zero at $z = n \in \mathbb{Z}$ is 1, it suffices to note that

$$(\sin \pi z)'|_{z=n} = \pi \cos \pi z|_{z=n} = (-1)^n \pi \neq 0.$$

Since n is a simple pole of $\frac{1}{\sin \pi z}$, we have

$$\mathrm{Res}\left(\frac{1}{\sin \pi z}, n\right) = \lim_{z \to n} \frac{z - n}{\sin \pi z} = \frac{1}{\pi \cos \pi n} = \frac{(-1)^n}{\pi}.$$

□

Stein 3.8.2 Evaluate the integral

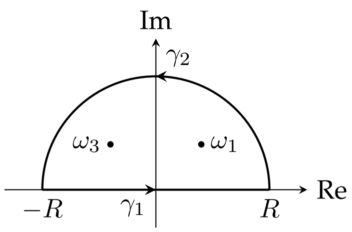
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1 + x^4}.$$

Where are the poles of $\frac{1}{1+z^4}$?

Solution The (simple) poles of $f(z) = \frac{1}{1+z^4}$ are at the fourth roots of -1, which are the complex numbers $\omega_k = e^{\mathrm{i}\frac{k\pi}{4}}$ $(k = 1, 3, 5, 7)$. The corresponding residues are

$$\mathrm{Res}(f, \omega_k) = \lim_{z \to \omega_k} \frac{z - \omega_k}{1 + z^4} = \frac{1}{4\omega_k^3} = -\frac{\omega_k}{4}, \quad k = 1, 3, 5, 7.$$

Let us consider the contour $\gamma = \gamma_1 \cup \gamma_2$ as shown below.



By the residue formula, we have

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}[\mathrm{Res}(f, \omega_1) + \mathrm{Res}(f, \omega_3)] = -\frac{\pi\mathrm{i}}{2}(\omega_1 + \omega_3) = \frac{\pi}{\sqrt{2}}.$$

The integral over the semicircle γ_2 vanishes as $R \to \infty$, since for $R > 1$,

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| = \left|\int_0^\pi \frac{\mathrm{i}Re^{\mathrm{i}\theta}\,\mathrm{d}\theta}{1 + R^4 e^{4\mathrm{i}\theta}}\right| \leqslant \int_0^\pi \frac{R\,\mathrm{d}\theta}{R^4 - 1} = \frac{\pi R}{R^4 - 1}.$$

Therefore, we obtain

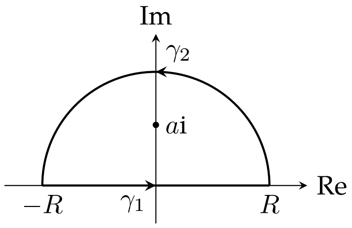
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4} = \lim_{R\to\infty} \int_{\gamma_1} f(z)\,\mathrm{d}z = \frac{\pi}{\sqrt{2}}.$$

□

Stein 3.8.3 Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x = \pi\frac{e^{-a}}{a}, \quad \text{for all } a>0.$$

Proof Consider the integral of $f(z) = \dfrac{e^{\mathrm{i}z}}{z^2+a^2}$ over the contour $\gamma = \gamma_1 \cup \gamma_2$ as shown below.



By the residue theorem, we have

$$\int_{\gamma} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,a\mathrm{i}) = 2\pi\mathrm{i}\lim_{z\to a\mathrm{i}}(z-a\mathrm{i})f(z) = 2\pi\mathrm{i}\lim_{z\to a\mathrm{i}}\frac{e^{\mathrm{i}z}}{z+a\mathrm{i}} = \pi\frac{e^{-a}}{a}.$$

The integral over the semicircle γ_2 vanishes as $R\to\infty$, since for $R>a$,

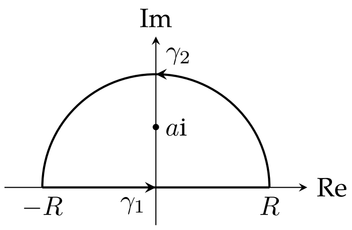
$$\begin{aligned}\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| &= \left|\int_0^{\pi} \frac{\exp\left(\mathrm{i}Re^{\mathrm{i}\theta}\right)\mathrm{i}Re^{\mathrm{i}\theta}}{R^2e^{2\mathrm{i}\theta}+a^2}\,\mathrm{d}\theta\right| \leqslant \int_0^{\pi}\frac{Re^{-R\sin\theta}}{R^2-a^2}\,\mathrm{d}\theta \\ &\leqslant \frac{2R}{R^2-a^2}\int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}\,\mathrm{d}\theta = \frac{\pi\left(1-e^{-R}\right)}{R^2-a^2}.\end{aligned}$$

The proof is complete by letting $R\to\infty$ and taking the real part of the integral. □

Stein 3.8.4 Show that

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x = \pi e^{-a}, \quad \text{for all } a>0.$$

Proof Consider the integral of $f(z) = \dfrac{ze^{\mathrm{i}z}}{z^2+a^2}$ over the contour $\gamma = \gamma_1 \cup \gamma_2$ as shown below.



By the residue theorem, we have

$$\int_{\gamma} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,a\mathrm{i}) = 2\pi\mathrm{i}\lim_{z\to a\mathrm{i}}(z-a\mathrm{i})f(z) = 2\pi\mathrm{i}\lim_{z\to a\mathrm{i}}\frac{ze^{\mathrm{i}z}}{z+a\mathrm{i}} = \mathrm{i}\pi e^{-a}.$$

The integral over the semicircle γ_2 vanishes as $R \to \infty$, since for $R > a$,

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| = \left|\int_0^{\pi} \frac{\mathrm{i}R^2 e^{2\mathrm{i}\theta}\exp\big(\mathrm{i}Re^{\mathrm{i}\theta}\big)}{R^2e^{2\mathrm{i}\theta}+a^2}\,\mathrm{d}\theta\right| \leqslant \int_0^{\frac{\pi}{2}} \frac{R^2 e^{-R\sin\theta}}{R^2-a^2}\,\mathrm{d}\theta$$
$$\leqslant \frac{2R^2}{R^2-a^2}\int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}\,\mathrm{d}\theta = \frac{\pi R\big(1-e^{-R}\big)}{R^2-a^2}.$$

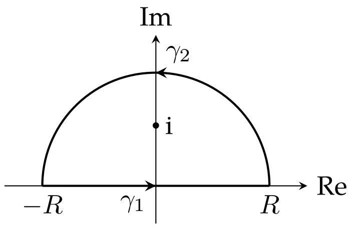
The proof is complete by letting $R \to \infty$ and taking the imaginary part of the integral. □

Stein 3.8.5 Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi\mathrm{i}x\xi}}{(1+x^2)^2}\,\mathrm{d}x = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}$$

for all ξ real.

Proof We may assume $\xi \leqslant 0$, and consider the integral of $f(z) = \dfrac{e^{-2\pi\mathrm{i}z\xi}}{(1+z^2)^2}$ over the contour $\gamma = \gamma_1 \cup \gamma_2$ as shown below.



By the residue theorem, we have

$$\int_{\gamma} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,\mathrm{i}) = 2\pi\mathrm{i}\lim_{z\to\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}\big[(z-\mathrm{i})^2 f(z)\big] = 2\pi\mathrm{i}\lim_{z\to\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}\frac{e^{-2\pi\mathrm{i}z\xi}}{(z+\mathrm{i})^2} = \frac{\pi}{2}(1-2\pi\xi)e^{2\pi\xi}.$$

The integral over the semicircle γ_2 vanishes as $R \to \infty$, since for $R > 1$,

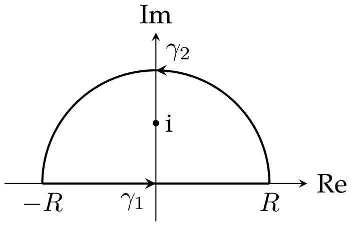
$$\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| = \left|\int_0^{\pi} \frac{\mathrm{i}R\exp\big(\mathrm{i}\theta - 2\pi\mathrm{i}Re^{\mathrm{i}\theta}\xi\big)}{\big(1+R^2e^{2\mathrm{i}\theta}\big)^2}\,\mathrm{d}\theta\right| \leqslant \int_0^{\pi} \frac{R\exp(2\pi R\xi\sin\theta)}{\big(R^2-1\big)^2}\,\mathrm{d}\theta$$
$$\leqslant \frac{R}{\big(R^2-1\big)^2}\int_0^{\pi} e^{4R\xi\theta}\,\mathrm{d}\theta = \begin{cases} \dfrac{\pi R}{\big(R^2-1\big)^2}, & \text{if } \xi = 0, \\ \dfrac{e^{4R\xi\pi}-1}{4\xi\big(R^2-1\big)^2}, & \text{if } \xi < 0. \end{cases}$$

The proof is complete by letting $R \to \infty$. □

Stein 3.8.6 Show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\pi.$$

Proof Consider the integral of $f(z) = \dfrac{1}{(1+z^2)^{n+1}}$ over the contour $\gamma = \gamma_1 \cup \gamma_2$ as shown below.



The point i is a pole of order $n+1$, and the residue at this pole is given by

$$
\begin{aligned}
\operatorname{Res}(f, \mathrm{i}) &= \frac{1}{n!} \lim_{z \to \mathrm{i}} \frac{\mathrm{d}^n}{\mathrm{d}z^n} (z-\mathrm{i})^{n+1} f(z) = \frac{1}{n!} \lim_{z \to \mathrm{i}} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \frac{1}{(z+\mathrm{i})^{n+1}} \\
&= \frac{1}{n!} \lim_{z \to \mathrm{i}} (-1)^n \frac{(2n)!}{n!} \frac{1}{(z+\mathrm{i})^{2n+1}} = \frac{(2n)!}{2\mathrm{i}(2^n n!)^2} \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{1}{2\mathrm{i}}.
\end{aligned}
$$

The integral over the semicircle γ_2 vanishes as $R \to \infty$, since for $R > 1$,

$$
\left| \int_{\gamma_2} f(z)\, \mathrm{d}z \right| = \left| \frac{\mathrm{i}Re^{\mathrm{i}\theta}\, \mathrm{d}\theta}{(1+R^2 e^{2\mathrm{i}\theta})^{n+1}} \right| \leqslant \int_0^{\pi} \frac{R\, \mathrm{d}\theta}{(R^2-1)^{n+1}} = \frac{\pi R}{(R^2-1)^{n+1}}.
$$

Therefore, by the residue formula, we get

$$
\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{n+1}} = 2\pi \mathrm{i} \operatorname{Res}(f, \mathrm{i}) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.
$$

□

Stein 3.8.7 Prove that

$$
\int_0^{2\pi} \frac{\mathrm{d}\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{\frac{3}{2}}}, \quad \text{whenever } a > 1.
$$

Proof Let $z = e^{\mathrm{i}\theta}$ and denote by C the unit circle. Then $\mathrm{d}z = \mathrm{i}z\, \mathrm{d}\theta$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, and

$$
\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{(a+\cos\theta)^2} &= \int_C \frac{\mathrm{d}z}{\mathrm{i}z\left(a + \frac{z+1/z}{2}\right)^2} \\
&= \frac{4}{\mathrm{i}} \int_C \frac{z\, \mathrm{d}z}{(z^2+2az+1)^2} \\
&= 8\pi \operatorname{Res}\left(\frac{z}{(z^2+2az+1)^2}, -a+\sqrt{a^2-1} \right) \\
&= 8\pi \lim_{z \to -a+\sqrt{a^2-1}} \frac{\mathrm{d}}{\mathrm{d}z} \frac{z}{\left(z+a+\sqrt{a^2-1}\right)^2} \\
&= 8\pi \lim_{z \to -a+\sqrt{a^2-1}} \frac{a+\sqrt{a^2-1}-z}{\left(a+\sqrt{a^2-1}+z\right)^3}
\end{aligned}
$$

$$= \frac{2\pi a}{(a^2-1)^{\frac{3}{2}}}. \qquad \square$$

Stein 3.8.13 Suppose $f(z)$ is holomorphic in a punctured disc $D_r(z_0) - \{z_0\}$. Suppose also that

$$|f(z)| \leqslant A|z-z_0|^{-1+\varepsilon}$$

for some $\varepsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

Proof Consider the function $g(z) = (z-z_0)f(z)$. We have

$$|g(z)| \leqslant A|z-z_0|^{\varepsilon}$$

for all z near z_0. By Riemann's theorem on removable singularities, the singularity of g at z_0 is removable. Now, g is a holomorphic function with z_0 its zero. Thus, we can write $g(z) = (z-z_0)h(z)$ for some holomorphic function h in the disc $D_r(z_0)$. This implies that $f(z) = h(z)$ is holomorphic in the disc $D_r(z_0)$, and hence the singularity of f at z_0 is removable. $\square$

Stein Page 83 Show that

$$\int_{-\infty}^{\infty} e^{-2\pi i x\xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} \, \mathrm{d}x = \frac{2\sinh 2\pi a\xi}{\sinh 2\pi\xi}$$

whenever $0 < a < 1$ and $\xi \in \mathbb{R}$.

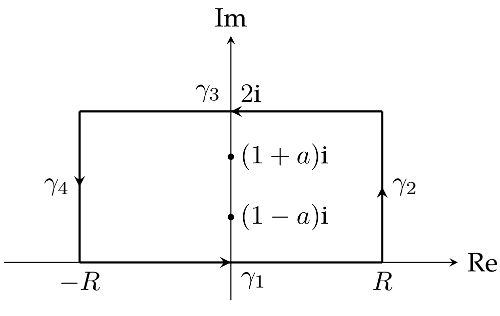
Proof Let $f(z) = e^{-2\pi i z\xi} \dfrac{\sin \pi a}{\cosh \pi z + \cos \pi a}$. To identify the poles of $f(z)$, we note that

$$\begin{aligned} \cosh \pi z + \cos \pi a = 0 \iff \frac{e^{\pi z} + e^{-\pi z}}{2} + \cos \pi a = 0 \iff e^{2\pi z} + (2\cos \pi a)e^{\pi z} + 1 = 0 \\ \iff e^{\pi z} = -\cos \pi a \pm i \sin \pi a = -e^{\mp i\pi a} \iff e^{\pi(z \pm ia)} = -1 \\ \iff z = (2n+1 \mp a)i \quad \text{for } n \in \mathbb{Z}. \end{aligned}$$

When $0 < a < 1$, we have

$$(\cosh \pi z + \cos \pi a)'\big|_{z=(2n+1\mp a)i} = \pi \sinh[\pi(2n+1\mp a)i] \neq 0.$$

Thus, the poles of $f(z)$ are simple. Now, let us consider the integral of $f(z)$ over the contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as shown below.



Note that $(1 \pm a)$i are the only two poles of $f(z)$ in the region enclosed by the contour γ. The corresponding residues are given by

$$\begin{aligned}\text{Res}(f,(1-a)\mathrm{i}) &= \lim_{z\to(1-a)\mathrm{i}}[z-(1-a)\mathrm{i}]f(z) = \frac{e^{-2\pi\mathrm{i}(1-a)\mathrm{i}\xi}\sin\pi a}{\pi\sinh[\pi(1-a)\mathrm{i}]}\\ &= \frac{e^{2\pi(1-a)\xi}\sin\pi a}{\pi\mathrm{i}\sin(\pi a)} = \frac{e^{2\pi(1-a)\xi}}{\pi\mathrm{i}}\end{aligned}$$

and

$$\begin{aligned}\text{Res}(f,(1+a)\mathrm{i}) &= \lim_{z\to(1+a)\mathrm{i}}[z-(1+a)\mathrm{i}]f(z) = \frac{e^{-2\pi\mathrm{i}(1+a)\mathrm{i}\xi}\sin\pi a}{\pi\sinh[\pi(1+a)\mathrm{i}]}\\ &= -\frac{e^{2\pi(1+a)\xi}\sin\pi a}{\pi\mathrm{i}\sin(\pi a)} = -\frac{e^{2\pi(1+a)\xi}}{\pi\mathrm{i}}.\end{aligned}$$

We dispense with the integrals of f on the vertical sides by showing that they go to zero as R tends to infinity. Indeed, if $z = R + \mathrm{i}y$ with $0 \leqslant y \leqslant 2$, then

$$\left|e^{-2\pi\mathrm{i}z\xi}\right| = e^{2\pi y\xi} \leqslant e^{4\pi|\xi|},$$

and

$$|\cosh\pi z| = \left|\frac{e^{\pi z}+e^{-\pi z}}{2}\right| \geqslant \frac{1}{2}\left||e^{\pi z}| - \left|e^{-\pi z}\right|\right| \geqslant \frac{1}{2}\left(e^{\pi R} - e^{-\pi R}\right) \xrightarrow{R\to\infty} \infty,$$

which show that the integral over γ_2 goes to 0 as $R\to\infty$. A similar argument applies to γ_4. Finally, we see that if I denotes the integral we wish to calculate, then the integral of f over γ_3 evaluates to

$$\int_R^{-R} e^{-2\pi\mathrm{i}(x+2\mathrm{i})\xi}\frac{\sin\pi a}{\cosh\pi(x+2\mathrm{i})+\cos\pi a}\,\mathrm{d}x = -e^{4\pi\xi}I.$$

In the limit as R tends to infinity, the residue formula gives

$$I - e^{4\pi\xi}I = 2\left[e^{2\pi(1-a)\xi} - e^{2\pi(1+a)\xi}\right] = -4e^{2\pi\xi}\sinh 2\pi a\xi,$$

and since $1 - e^{4\pi\xi} = -e^{2\pi\xi}\left(e^{2\pi\xi} - e^{-2\pi\xi}\right) = -2e^{2\pi\xi}\sinh 2\pi\xi$, we find that

$$I = \frac{-4e^{2\pi\xi}\sinh 2\pi a\xi}{-2e^{2\pi\xi}\sinh 2\pi\xi} = \frac{2\sinh 2\pi a\xi}{\sinh 2\pi\xi}. \qquad \square$$