Stein 2.6.9 Let Ω be a bounded connected open subset of $\mathbb{C}$, and $\varphi\colon \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \quad \text{and} \quad \varphi'(z_0) = 1$$

then φ is linear.

Proof By considering the function $\varphi(z + z_0) - z_0$ on $\Omega - z_0$, we can assume without loss of generality that $z_0 = 0$. Suppose to the contrary that φ is not linear. Then there exists $n \geqslant 2$ with $a_n \neq 0$, such that

$$\varphi(z) = z + a_n z^n + O\big(z^{n+1}\big)$$

near 0. It follows by induction that if we set $\varphi_k = \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then

$$\varphi_k(z) = z + k a_n z^n + O\big(z^{n+1}\big)$$

near 0. Since $\varphi_k(\Omega) \subset \Omega$ and Ω is bounded, there exists $M > 0$ such that $|\varphi_k| \leqslant M$ on Ω for all k. Choose an open ball $\mathbb{B}(0, r)$ such that its closure is contained in Ω. By Cauchy's inequalities, one has

$$n!k|a_n| = \left|\varphi_k^{(n)}(z)\right| \leqslant \frac{n!M}{r^n},$$

that is,

$$|a_n| \leqslant \frac{M}{kr^n}$$

for any $k \geqslant 1$. Then by letting $k \to \infty$, we get $a_n = 0$, a contradiction. $\square$

Stein 2.6.11 Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(1) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f\big(Re^{\mathrm{i}\varphi}\big) \operatorname{Re}\left(\frac{Re^{\mathrm{i}\varphi} + z}{Re^{\mathrm{i}\varphi} - z}\right) \mathrm{d}\varphi.$$

(2) Show that

$$\operatorname{Re}\left(\frac{Re^{\mathrm{i}\gamma} + r}{Re^{\mathrm{i}\gamma} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

Proof (1) Let $\zeta = Re^{\mathrm{i}\varphi}$. Then $\mathrm{d}\zeta = \mathrm{i}\zeta\,\mathrm{d}\varphi$ and it suffices to show that

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\zeta|=R} f(\zeta) \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right) \frac{\mathrm{d}\zeta}{\zeta}. \tag{2.6.11–1}$$

When $|z| < R$, the function $\dfrac{f(\zeta)}{\zeta - \frac{R^2}{\bar{z}}}$ is a holomorphic function of ζ on D_R. Hence,

$$\int\limits_{|\zeta|=R} \frac{f(\zeta)}{\zeta - \frac{R^2}{\bar{z}}}\,\mathrm{d}\zeta = 0.$$

Therefore, by the Cauchy integral formula, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \int\limits_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \int\limits_{|\zeta|=R} f(\zeta)\left(\frac{1}{\zeta - z} + \frac{1}{\frac{\zeta\bar{\zeta}}{\bar{z}} - \zeta}\right) \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \int\limits_{|\zeta|=R} f(\zeta)\left(\frac{1}{\zeta - z} + \frac{\bar{z}}{\zeta(\bar{\zeta} - \bar{z})}\right) \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \int\limits_{|\zeta|=R} f(\zeta)\frac{\zeta\bar{\zeta} - z\bar{z}}{\zeta(\zeta - z)(\bar{\zeta} - \bar{z})}\, \mathrm{d}\zeta. \end{aligned}$$

The identity (2.6.11–1) is then proved by noting that

$$\begin{aligned} \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right) &= \frac{1}{2}\left(\frac{\zeta + z}{\zeta - z} + \frac{\bar{\zeta} + \bar{z}}{\bar{\zeta} - \bar{z}}\right) \\ &= \frac{1}{2}\frac{\zeta\bar{\zeta} - \zeta\bar{z} + z\bar{\zeta} - z\bar{z} + \zeta\bar{\zeta} - z\bar{\zeta} + \zeta\bar{z} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})} \\ &= \frac{\zeta\bar{\zeta} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})}. \end{aligned} \tag{2.6.11–2}$$

(2) Setting $\zeta = Re^{\mathrm{i}\gamma}$ and $z = r$ in (2.6.11–2) gives

$$\operatorname{Re}\left(\frac{Re^{\mathrm{i}\gamma} + r}{Re^{\mathrm{i}\gamma} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

□

Stein 2.6.13 Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

Proof From the hypothesis, we see that

$$\mathbb{C} = \bigcup_{n=0}^{\infty} \left\{ z \in \mathbb{C} : f^{(n)}(z) = 0 \right\}.$$

Since $\mathbb{C}$ is uncountable, there exists $n \geqslant 0$ such that the set $Z_n := \left\{ z \in \mathbb{C} : f^{(n)}(z) = 0 \right\}$ is uncountable. This implies that Z_n has an accumulation point and so $f^{(n)}$ is identically zero by the identity theorem. Therefore, f is a polynomial. □

Stein 2.6.15 Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \quad \text{whenever } |z| = 1,$$

then f is constant.

Proof Consider the function

$$g(z) := \begin{cases} f(z), & \text{if } |z| < 1, \\ \dfrac{1}{\overline{f(1/\bar{z})}}, & \text{if } |z| \geqslant 1. \end{cases}$$

It is direct to check that g is continuous on $\mathbb{C}$. Since $f(z)$ is holomorphic and non-vanishing in $\mathbb{D}$, the function $\dfrac{1}{f(1/z)}$ is holomorphic whenever $|z| > 1$. If $|z_0| > 1$, then the power series expansion of $\dfrac{1}{f(1/z)}$ near $\overline{z_0}$ gives

$$\frac{1}{f(1/z)} = \sum_{n=0}^{\infty} a_n (z - \overline{z_0})^n.$$

Then

$$\frac{1}{\overline{f(1/\bar{z})}} = \sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n.$$

Therefore, $g(z)$ is holomorphic when $|z| > 1$. Now, one can proceed as in the proof of Theorem 5.5 and apply Morera's theorem to show that g is holomorphic in $\mathbb{C}$. Thus, g is entire and bounded, so by Liouville's theorem, g (and hence f) is constant. □

Stein 2.7.3 Morera's theorem states that if f is continuous in $\mathbb{C}$, and $\displaystyle\int_T f(z)\,\mathrm{d}z = 0$ for all triangles T, then f is holomorphic in $\mathbb{C}$. Naturally, we may ask if the conclusion still holds if we replace triangles by other sets.

(1) Suppose that f is continuous on $\mathbb{C}$, and

$$\int_C f(z)\,\mathrm{d}z = 0$$

for every circle C. Prove that f is holomorphic.

(2) More generally, let Γ be any toy contour, and $\mathcal{F}$ the collection of all translates and dilates of Γ. Show that if f is continuous on $\mathbb{C}$, and

$$\int_\gamma f(z)\,\mathrm{d}z = 0 \quad \text{for all } \gamma \in \mathcal{F}$$

then f is holomorphic. In particular, Morera's theorem holds under the weaker assumption that $\displaystyle\int_T f(z)\,\mathrm{d}z = 0$ for all equilateral triangles.

Proof Consider the molifier $\varphi_\varepsilon(z) := \varepsilon^{-2}\varphi\big(\frac{z}{\varepsilon}\big)$, where

$$\varphi\colon \mathbb{C} \simeq \mathbb{R}^2 \to \mathbb{R}, \quad z \mapsto \begin{cases} c\exp\left(-\dfrac{1}{(|z|^2 - 1/4)^2}\right), & |z| < \frac{1}{2}, \\ 0, & |z| \geqslant \frac{1}{2}. \end{cases}$$

Here $c > 0$ is a suitable normalizing constant. Then the convolution

$$f_\varepsilon(z) := f * \varphi_\varepsilon(z) = \int_{\mathbb{R}^2} f(z - w)\varphi_\varepsilon(w)\,\mathrm{d}w \in \mathcal{C}^\infty(\mathbb{R}^2),$$

and f_ε converges uniformly to f on any compact subset of $\mathbb{C}$ as $\varepsilon \to 0$. Thus, by Theorem 5.2, it suffices to show that f_ε is holomorphic in each case. So we may assume that f is smooth.

(1) Since f is twice real differentiable, we can write

$$f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + O\big(|z - z_0|^2\big)$$

for all $z \in \mathbb{B}(z_0, \varepsilon)$, with some $\varepsilon > 0$. By the assumption, we have

$$\begin{aligned} 0 &= \int\limits_{|z-z_0|=\varepsilon} f(z)\,\mathrm{d}z \\ &= \int\limits_{|z-z_0|=\varepsilon} \big[f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + O\big(|z - z_0|^2\big)\big]\,\mathrm{d}z \\ &= \int\limits_{|z-z_0|=\varepsilon} \big[b(\overline{z - z_0}) + O\big(|z - z_0|^2\big)\big]\,\mathrm{d}z \\ &= \int_0^{2\pi} b\varepsilon e^{-\mathrm{i}\theta}\mathrm{i}\varepsilon e^{\mathrm{i}\theta}\,\mathrm{d}\theta + O\big(\varepsilon^3\big) \\ &= 2\pi\mathrm{i}b\varepsilon^2 + O\big(\varepsilon^3\big), \end{aligned}$$

which implies $b = 0$. Therefore, $\dfrac{\partial f}{\partial \bar{z}} = 0$ and f is holomorphic.

(2) Since $f = u + \mathrm{i}v$ is $\mathcal{C}^1$, for any open ball $\mathbb{B}(z_0, r)$, we have

$$0 = \int\limits_{|z-z_0|=r} f(z)\,\mathrm{d}z = \int\limits_{|z-z_0|=r} (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i} \int\limits_{|z-z_0|=r} (u\,\mathrm{d}y + v\,\mathrm{d}x).$$

With this and the Green's theorem, we obtain

$$\begin{aligned} 0 = -\int\limits_{|z-z_0|=r} (u\,\mathrm{d}x - v\,\mathrm{d}y) &= \iint\limits_{\mathbb{B}(z_0,r)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y, \\ 0 = \int\limits_{|z-z_0|=r} (u\,\mathrm{d}y + v\,\mathrm{d}x) &= \iint\limits_{\mathbb{B}(z_0,r)} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Dividing the above equations by πr^2 and letting $r \to 0$ gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This shows that the real and imaginary parts of f satisfy the Cauchy–Riemann equations, so f is holomorphic. □

Stein 2.7.4 Prove the converse to Runge's theorem: if K is a compact set whose complement is not connected, then there exists a function f holomorphic in a neighborhood of K which cannot be approximated uniformly by polynomials on K.

Proof Pick a point z_0 in a bounded component of K^{c}, and let $f(z) = \frac{1}{z-z_0}$. Then there is some $M > 0$ such that $|z - z_0| < M$ for all $z \in K$. If f can be approximated uniformly by polynomials on K, then we can find a polynomial p such that

$$|f(z) - p(z)| < \frac{1}{M}$$

for all $z \in K$. This implies that

$$|(z - z_0)p(z) - 1| < \frac{|z - z_0|}{M} < 1$$

for all $z \in K$. Since $(z - z_0)p(z) - 1$ is entire, and the bounded component containing z_0 is enclosed by K by the Jordan curve theorem, we have

$$|(z - z_0)p(z) - 1| < 1 \tag{2.7.4–1}$$

for all z in the union of K and the bounded component containing z_0 by the maximum modulus principle. Now, taking $z = z_0$ in (2.7.4–1) leads to a contradiction. □