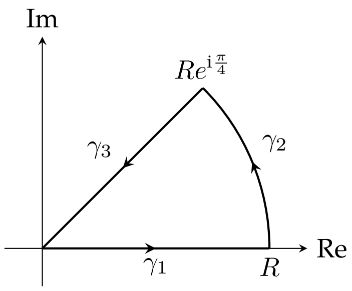
Stein 2.6.1 Prove that

$$\int_0^\infty \sin(x^2)\,\mathrm{d}x = \int_0^\infty \cos(x^2)\,\mathrm{d}x = \frac{\sqrt{2\pi}}{4}.$$

These are the **Fresnel integrals**. Here, $\displaystyle\int_0^\infty$ is interpreted as $\displaystyle\lim_{R\to\infty}\int_0^R$.

Proof Consider the integral of $f(z) = e^{\mathrm{i}z^2}$ along the contour $\gamma \coloneqq \gamma_1 \cup \gamma_2 \cup \gamma_3$ as shown below.



The integral along γ_2 is

$$\int_{\gamma_2} f(z)\,\mathrm{d}z = \mathrm{i}R\int_0^{\frac{\pi}{4}} \exp\left(\mathrm{i}R^2e^{2\mathrm{i}\theta} + \mathrm{i}\theta\right)\mathrm{d}\theta = \mathrm{i}R\int_0^{\frac{\pi}{4}} e^{\mathrm{i}\left(R^2\cos 2\theta + \theta\right)}e^{-R^2\sin 2\theta}\,\mathrm{d}\theta.$$

By Jordan's inequality, we have $\sin x \geqslant \dfrac{2}{\pi}x$ for $x \in \left[0, \frac{\pi}{2}\right]$. Hence

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leqslant R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\,\mathrm{d}\theta \leqslant R\int_0^{\frac{\pi}{4}} e^{-\frac{4R^2\theta}{\pi}}\,\mathrm{d}\theta = \frac{\pi\left(1 - e^{-R^2}\right)}{4R} \xrightarrow{R\to+\infty} 0.$$

The integral along γ_3 is

$$\int_{\gamma_3} f(z)\,\mathrm{d}z = \int_R^0 \exp\left(\mathrm{i}r^2e^{\mathrm{i}\frac{\pi}{2}}\right)e^{\mathrm{i}\frac{\pi}{4}}\,\mathrm{d}r = -e^{\mathrm{i}\frac{\pi}{4}}\int_0^R e^{-r^2}\,\mathrm{d}r \xrightarrow{R\to+\infty} -\frac{\sqrt{\pi}e^{\mathrm{i}\frac{\pi}{4}}}{2}.$$

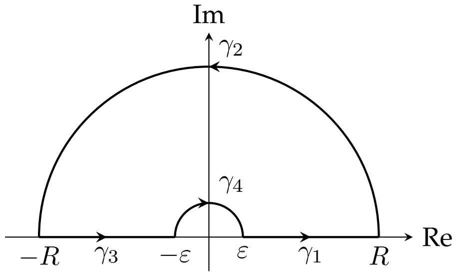
By Cauchy's theorem, we have

$$\int_0^\infty e^{\mathrm{i}x^2}\,\mathrm{d}x = \lim_{R\to+\infty}\int_{\gamma_1} f(z)\,\mathrm{d}z = -\lim_{R\to+\infty}\int_{\gamma_3} f(z)\,\mathrm{d}z = \frac{\sqrt{\pi}e^{\mathrm{i}\frac{\pi}{4}}}{2}.$$

Taking the real and imaginary parts, we get the desired result. □

Stein 2.6.2 Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$.

Proof Consider the integral of $f(z) = \dfrac{e^{\mathrm{i}z}}{z}$ along the contour $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as shown below.



The integral along γ_2 is

$$\int_{\gamma_2} f(z)\,\mathrm{d}z = \int_0^\pi \frac{\exp\left(\mathrm{i}Re^{\mathrm{i}\theta}\right)}{Re^{\mathrm{i}\theta}} \mathrm{i}Re^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}\int_0^\pi \exp(\mathrm{i}R\cos\theta - R\sin\theta)\,\mathrm{d}\theta.$$

By Jordan's inequality, we have $\sin x \geqslant \dfrac{2}{\pi}x$ for $x \in \left[0, \frac{\pi}{2}\right]$. Hence

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| &\leqslant \int_0^\pi e^{-R\sin\theta}\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,\mathrm{d}\theta \\ &\leqslant 2\int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}}\,\mathrm{d}\theta = \frac{\pi\left(1-e^{-R}\right)}{R} \xrightarrow{R\to+\infty} 0.\end{aligned}$$

Similarly, the integral along γ_4 is

$$\int_{\gamma_4} f(z)\,\mathrm{d}z = -\mathrm{i}\int_0^\pi \exp(\mathrm{i}\varepsilon\cos\theta - \varepsilon\sin\theta)\,\mathrm{d}\theta,$$

and it follows that

$$\left|\pi\mathrm{i} + \int_{\gamma_4} f(z)\,\mathrm{d}z\right| = \left|\int_0^\pi [1-\exp(\mathrm{i}\varepsilon\cos\theta - \varepsilon\sin\theta)]\,\mathrm{d}\theta\right| \leqslant \pi \max_{0\leqslant\theta\leqslant\pi} |1-\exp(\mathrm{i}\varepsilon\cos\theta - \varepsilon\sin\theta)|,$$

which tends to zero as $\varepsilon \to 0$. That is to say, the integral along γ_4 tends to $-\pi\mathrm{i}$ as $\varepsilon \to 0$. Therefore, we obtain by Cauchy's theorem

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = -\frac{1}{2}\,\mathrm{Im}\left\{\lim_{R\to+\infty,\,\varepsilon\to 0} \int_{\gamma_2\cup\gamma_4} f(z)\,\mathrm{d}z\right\} = \frac{1}{2}\,\mathrm{Im}(\pi\mathrm{i}) = \frac{\pi}{2}.$$

□

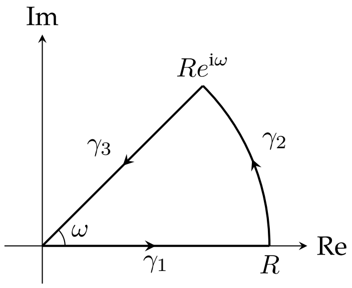
Stein 2.6.3 Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos bx\,\mathrm{d}x \quad \text{and} \quad \int_0^\infty e^{-ax}\sin bx\,\mathrm{d}x, \quad a > 0$$

by integrating e^{-Az}, $A = \sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = \frac{a}{A}$.

Solution Consider the integral of $f(z) = e^{-Az}$ along the contour $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3$ as shown below.

Here $\omega \in \left(0, \frac{\pi}{2}\right)$ and we assume first that $b > 0$, so that $\sin\omega = \frac{b}{A}$.



The integral along γ_2 is

$$\int_{\gamma_2} f(z)\,\mathrm{d}z = \mathrm{i}R\int_0^\omega \exp\left(-ARe^{\mathrm{i}\theta} + \mathrm{i}\theta\right)\mathrm{d}\theta = \mathrm{i}R\int_0^\omega \exp[\mathrm{i}(\theta - AR\sin\theta) - AR\cos\theta]\,\mathrm{d}\theta.$$

Since $\cos x \geqslant 1 - \dfrac{2}{\pi}x$ for $x \in \left[0, \frac{\pi}{2}\right]$, we have

$$\left|\int_{\gamma_2} f(z)\,\mathrm{d}z\right| \leqslant R\int_0^\omega e^{-AR\cos\theta}\,\mathrm{d}\theta \leqslant Re^{-AR}\int_0^\omega e^{\frac{2AR}{\pi}\theta}\,\mathrm{d}\theta = \frac{\pi}{2A}\left(e^{AR\left(\frac{2\omega}{\pi}-1\right)} - e^{-AR}\right),$$

which tends to zero as $R \to +\infty$ for $\omega < \frac{\pi}{2}$. The integral along γ_3 is

$$\begin{aligned}\int_{\gamma_3} f(z)\,\mathrm{d}z &= \int_R^0 \exp\left(-Are^{\mathrm{i}\omega} + \mathrm{i}\omega\right)\mathrm{d}r = -e^{\mathrm{i}\omega}\int_0^R \exp(-Ar\cos\omega - \mathrm{i}Ar\sin\omega)\,\mathrm{d}r \\ &= -\left(\frac{a}{A} + \mathrm{i}\frac{b}{A}\right)\int_0^R e^{-(a+b\mathrm{i})r}\,\mathrm{d}r.\end{aligned}$$

By Cauchy's theorem, we have

$$\lim_{R\to+\infty}\int_{\gamma_3} f(z)\,\mathrm{d}z = -\int_0^\infty e^{-Ax}\,\mathrm{d}x = -\frac{1}{A}.$$

Therefore, we obtain

$$\int_0^\infty e^{-(a+b\mathrm{i})r}\,\mathrm{d}r = \frac{1}{A}\left(\frac{a}{A} + \mathrm{i}\frac{b}{A}\right)^{-1} = \frac{1}{a+b\mathrm{i}}.$$

Taking the real and imaginary parts then gives

$$\int_0^\infty e^{-ax}\cos bx\,\mathrm{d}x = \frac{a}{a^2+b^2} \quad \text{and} \quad \int_0^\infty e^{-ax}\sin bx\,\mathrm{d}x = \frac{b}{a^2+b^2}.$$

These hold for $b \leqslant 0$ as well, as one can see from the integrands. □

Stein 2.6.4 Prove that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi \mathrm{i} x\xi}\,\mathrm{d}x$.

Proof In Example 1 of Section 2.3, we have shown that the above identity holds for $\xi \in \mathbb{R}$. Now, for $\xi = a + b\mathrm{i}$ with $a, b \in \mathbb{R}$, we have

$$\int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi\mathrm{i}x\xi}\,\mathrm{d}x = \int_{-\infty}^\infty e^{-\pi x^2 - 2\pi bx + 2\pi\mathrm{i}ax}\,\mathrm{d}x$$

$$
\begin{aligned}
&= e^{\pi b^2 - 2\pi \mathrm{i} ab} \int_{-\infty}^{\infty} e^{-\pi(x+b)^2 + 2\pi \mathrm{i} a(x+b)} \,\mathrm{d}x \\
&= e^{\pi b^2 - 2\pi \mathrm{i} ab - \pi a^2} = e^{-\pi \xi^2}.
\end{aligned}
$$

□

Stein 2.6.5 Suppose f is continuously *complex* differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that

$$\int_T f(z) \,\mathrm{d}z = 0.$$

This provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

Proof Let $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ where u and v are real-valued functions. Then

$$
\begin{aligned}
\int_T f(z) \,\mathrm{d}z = \int_T (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y) &= \int_T (u \,\mathrm{d}x - v \,\mathrm{d}y) + \mathrm{i} \int_T (u \,\mathrm{d}y + v \,\mathrm{d}x) \\
&= \int_{\text{Interior of } T} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \mathrm{d}y \wedge \mathrm{d}x + \mathrm{i} \int_{\text{Interior of } T} \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \mathrm{d}y \wedge \mathrm{d}x \\
&= 0
\end{aligned}
$$

by the Cauchy–Riemann equations. □

Stein 2.6.6 Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then

$$\int_T f(z) \,\mathrm{d}z = 0.$$

Proof Let γ_ε be a circle centered at w with radius ε contained in T. Since f is holomorphic in the region enclosed by T and γ_ε, we have

$$\int_T f(z) \,\mathrm{d}z = \int_{\gamma_\varepsilon} f(z) \,\mathrm{d}z.$$

By the boundedness of f near w, we have

$$\left| \int_{\gamma_\varepsilon} f(z) \,\mathrm{d}z \right| \leqslant 2\pi\varepsilon \max_{|z-w|=\varepsilon} |f(z)| \xrightarrow{\varepsilon \to 0} 0.$$

Therefore, we obtain the desired result. □

Stein 2.6.7 Suppose $f\colon \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2|f'(0)| \leqslant d. \tag{2.6.7–1}$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Proof The Cauchy integral formula gives

$$f'(0) = \frac{1}{2\pi \mathrm{i}} \int_{C_r} \frac{f(\zeta)}{\zeta^2} \,\mathrm{d}\zeta, \tag{2.6.7–2}$$

where C_r is the circle centered at the origin with radius r. Replace ζ by $-\zeta$ in (2.6.7–2) to get

$$f'(0) = -\frac{1}{2\pi\mathrm{i}}\int_{C_r}\frac{f(-\zeta)}{\zeta^2}\,\mathrm{d}\zeta. \tag{2.6.7–3}$$

Adding (2.6.7–2) and (2.6.7–3) gives

$$\begin{aligned}|2f'(0)| &= \left|\frac{1}{2\pi\mathrm{i}}\int_{C_r}\frac{f(\zeta)-f(-\zeta)}{\zeta^2}\,\mathrm{d}\zeta\right| \leqslant \frac{1}{2\pi}\int_{C_r}\left|\frac{f(\zeta)-f(-\zeta)}{\zeta^2}\right|\mathrm{d}\zeta \\ &\leqslant \frac{d}{2\pi}\int_{C_r}\frac{\mathrm{d}\zeta}{r^2} = \frac{d}{r}.\end{aligned}$$

Letting $r \to 1$ gives (2.6.7–1). It is clear that the equality holds when f is linear.

To show that the equality holds only when f is linear, we let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and denote

$$N(r) \coloneqq \frac{1}{\pi r^2}\int\limits_{\mathbb{B}(0,r)} |f'(z)|^2\,\mathrm{d}x\,\mathrm{d}y$$

for $r \in [0,1]$. If $f'(0) = 0$, then $d = 0$ and f is constant. Otherwise, we have

$$\lim_{r\to 0^+} N(r) = |f'(0)|^2 > 0.$$

This shows that f is locally injective near the origin, and by the area formula we have

$$\begin{aligned}\frac{\operatorname{Area}(f(\mathbb{B}(0,r)))}{\pi r^2} &= N(r) = \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}\left|\sum_{n=1}^{\infty} n a_n \rho^{n-1} e^{\mathrm{i}(n-1)\theta}\right|^2 \rho\,\mathrm{d}\theta\,\mathrm{d}\rho \\ &= \frac{1}{\pi r^2}\sum_{n,m=1}^{\infty} nm a_n \overline{a_m}\int_0^r\int_0^{2\pi} \rho^{n+m-2} e^{\mathrm{i}(n-m)\theta}\rho\,\mathrm{d}\theta\,\mathrm{d}\rho \\ &= \frac{1}{\pi r^2}\sum_{n=1}^{\infty} n^2|a_n|^2\int_0^r\int_0^{2\pi}\rho^{2n-1}\,\mathrm{d}\theta\,\mathrm{d}\rho \\ &= \sum_{n=1}^{\infty} n|a_n|^2 r^{2n-2}\end{aligned}$$

and

$$N'(r) = \sum_{n=2}^{\infty} n(2n-2)|a_n|^2 r^{2n-3}$$

for all r small enough. If f is not linear, i.e., there exists $n \geqslant 2$ such that $a_n \neq 0$, then $N'(r) > 0$ and $N(r)$ is strictly increasing in r for r small enough. Hence

$$N(0) = |f'(0)|^2 < N(r) = \frac{\operatorname{Area}(f(\mathbb{B}(0,r)))}{\pi r^2} \leqslant \frac{\pi[d(r)/2]^2}{\pi r^2} = \left(\frac{d(r)}{2r}\right)^2, \tag{2.6.7–4}$$

where the "$\leqslant$" sign is due to the isodiametric inequality. Meanwhile, by the maximum modulus principle, we have

$$\frac{d(r)}{r} = \sup_{\theta\in[0,2\pi]}\sup_{|z|=r}\left|\frac{f(e^{\mathrm{i}\theta}z)-f(z)}{z}\right|.$$

For any fixed θ, the function $\dfrac{f\left(e^{i\theta}z\right)-f(z)}{z}$ is holomorphic in $\mathbb{D}$. By the maximum modulus principle, the supremum of its modulus over $|z|=r$ is a nondecreasing function of r. Taking the supremum over θ, we conclude that $\dfrac{d(r)}{r}$ is a nondecreasing function of r. So if the equality holds in (2.6.7–1), then for small r we have

$$\frac{d(r)}{r}\leqslant\frac{d(1)}{1}=d=2|f'(0)|,$$

which contradicts (2.6.7–4). Therefore f must be linear. □