Stein 8.5.15 Here are two properties enjoyed by automorphisms of the upper half-plane.

(1) Suppose Φ is an automorphism of $\mathbb{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.

(2) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of three distinct points on the real axis with

$$x_1 < x_2 < x_3 \quad \text{and} \quad y_1 < y_2 < y_3.$$

Prove that there exists (a unique) automorphism Φ of $\mathbb{H}$ so that $\Phi(x_j) = y_j$, $j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$.

Proof (1) Since $\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{PSL}_2(\mathbb{R})$, we can represent Φ as $z \mapsto \dfrac{az+b}{cz+d}$. By the assumption, the equation $ax + b = x(cx + d)$ has three distinct real roots, which implies that $a = d$ and $b = c = 0$. Hence, Φ is the identity.

(2) Note that the map

$$T_{x_1,x_2,x_3} \colon \mathbb{H} \to \mathbb{H}, \quad z \mapsto \frac{(x_2 - x_3)(z - x_1)}{(x_2 - x_1)(z - x_3)}$$

is an automorphism of $\mathbb{H}$, since if we let

$$a = x_2 - x_3, \quad b = x_1(x_3 - x_2), \quad c = x_2 - x_1, \quad d = x_3(x_1 - x_2),$$

then

$$\begin{aligned} \mathrm{Im}(T_{x_1,x_2,x_3}(z)) &= \frac{(ad - bc)\,\mathrm{Im}(z)}{|cz + d|^2} \\ &= \frac{(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)\,\mathrm{Im}(z)}{|cz + d|^2} \\ &> 0 \quad \text{whenever } z \in \mathbb{H}. \end{aligned}$$

Moreover, T_{x_1,x_2,x_3} maps x_1, x_2, and x_3 to 0, 1, and ∞, respectively. Therefore, the composition

$$\Phi := T^{-1}_{y_1,y_2,y_3} \circ T_{x_1,x_2,x_3} \colon \mathbb{H} \to \mathbb{H}$$

is an automorphism of $\mathbb{H}$ that maps x_j to y_j for $j = 1, 2, 3$.

If Φ_1 and Φ_2 are two such automorphisms, then the composition $\Phi_2 \circ \Phi_1^{-1}$ is an automorphism of $\mathbb{H}$ that fixes x_1, x_2, and x_3. By part (1), we have $\Phi_2 \circ \Phi_1^{-1} = \mathrm{Id}_{\mathbb{H}}$, so that $\Phi_2 = \Phi_1$. ☐

Stein 8.5.16 Let

$$f(z) = \frac{\mathrm{i} - z}{\mathrm{i} + z} \quad \text{and} \quad f^{-1}(w) = \mathrm{i}\frac{1 - w}{1 + w}.$$

(1) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d such that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az + b}{cz + d} = f^{-1}\big(e^{\mathrm{i}\theta} f(z)\big).$$

(2) Given $\alpha \in \mathbb{D}$, find real numbers a, b, c, d so that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1}(\psi_\alpha(f(z))),$$

with ψ_α defined in Section 2.1.

(3) Prove that if g is an automorphism of the unit disc, then there exist real numbers a, b, c, d such that $ad - bc = 1$ and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1} \circ g \circ f(z).$$

Proof (1) Since

$$\begin{aligned}
f^{-1}\big(e^{\mathrm{i}\theta} f(z)\big) &= \mathrm{i}\frac{1 - e^{\mathrm{i}\theta}\frac{\mathrm{i}-z}{\mathrm{i}+z}}{1 + e^{\mathrm{i}\theta}\frac{\mathrm{i}-z}{\mathrm{i}+z}} = \mathrm{i}\frac{\mathrm{i} + z - e^{\mathrm{i}\theta}(\mathrm{i} - z)}{\mathrm{i} + z + e^{\mathrm{i}\theta}(\mathrm{i} - z)} \\
&= \mathrm{i}\frac{\big(1 + e^{\mathrm{i}\theta}\big)z + \mathrm{i}\big(1 - e^{\mathrm{i}\theta}\big)}{\big(1 - e^{\mathrm{i}\theta}\big)z + \mathrm{i}\big(1 + e^{\mathrm{i}\theta}\big)} \\
&= \mathrm{i}\frac{\Big(e^{\mathrm{i}\frac{\theta}{2}} + e^{-\mathrm{i}\frac{\theta}{2}}\Big)z + \mathrm{i}\Big(e^{-\mathrm{i}\frac{\theta}{2}} - e^{\mathrm{i}\frac{\theta}{2}}\Big)}{\Big(e^{-\mathrm{i}\frac{\theta}{2}} - e^{\mathrm{i}\frac{\theta}{2}}\Big)z + \mathrm{i}\Big(e^{\mathrm{i}\frac{\theta}{2}} + e^{-\mathrm{i}\frac{\theta}{2}}\Big)} \\
&= \frac{\big(\cos\frac{\theta}{2}\big)z + \sin\frac{\theta}{2}}{\big(-\sin\frac{\theta}{2}\big)z + \cos\frac{\theta}{2}},
\end{aligned}$$

we can take

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

(2) Since

$$\begin{pmatrix} -\mathrm{i} & \mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & \alpha \\ -\overline{\alpha} & 1 \end{pmatrix}\begin{pmatrix} -1 & \mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} = \begin{pmatrix} \mathrm{i}(\overline{\alpha} - \alpha) & \alpha + \overline{\alpha} - 2 \\ \alpha + \overline{\alpha} + 2 & \mathrm{i}(\alpha - \overline{\alpha}) \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{Im}(\alpha) & 2\,\mathrm{Re}(\alpha) - 2 \\ 2\,\mathrm{Re}(\alpha) + 2 & -2\,\mathrm{Im}(\alpha) \end{pmatrix},$$

whose determinant is $4\big(1 - |\alpha|^2\big)$, by the isomorphism between the Möbius group and $\mathrm{PGL}_2(\mathbb{C})$, we can take

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{1 - |\alpha|^2}}\begin{pmatrix} \mathrm{Im}(\alpha) & \mathrm{Re}(\alpha) - 1 \\ \mathrm{Re}(\alpha) + 1 & -\mathrm{Im}(\alpha) \end{pmatrix}.$$

(3) By Theorem 2.2, $g(z) = e^{\mathrm{i}\theta}\psi_\alpha(z)$ for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. Then $f^{-1} \circ g \circ f$ is the composition of the maps in (1) and (2), which means that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{1 - |\alpha|^2}}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \mathrm{Im}(\alpha) & \mathrm{Re}(\alpha) - 1 \\ \mathrm{Re}(\alpha) + 1 & -\mathrm{Im}(\alpha) \end{pmatrix}.$$

□

Stein 8.5.17 If $\psi_\alpha(z)=\dfrac{\alpha-z}{1-\overline{\alpha}z}$ for $|\alpha|<1$, prove that

$$\frac{1}{\pi}\iint_{\mathbb{D}}|\psi_\alpha'|^2\,\mathrm{d}x\,\mathrm{d}y=1 \quad\text{and}\quad \frac{1}{\pi}\iint_{\mathbb{D}}|\psi_\alpha'|\,\mathrm{d}x\,\mathrm{d}y=\frac{1-|\alpha|^2}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2},$$

where in the case $\alpha=0$ the expression on the right is understood as the limit as $|\alpha|\to 0$.

Proof Since $\psi_\alpha\in\mathrm{Aut}(\mathbb{D})$, by Proposition 2.3 in Chapter 1, we have

$$\pi=\mathrm{Area}(\psi_\alpha(\mathbb{D}))=\iint_{\psi_\alpha(\mathbb{D})}\mathrm{d}x\,\mathrm{d}y=\iint_{\mathbb{D}}|\psi_\alpha'|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Note that by the symmetry of the disc,

$$\iint_{\mathbb{D}}|\psi_\alpha'|\,\mathrm{d}x\,\mathrm{d}y=\iint_{\mathbb{D}}\frac{1-|\alpha|^2}{|1-\overline{\alpha}z|^2}\,\mathrm{d}x\,\mathrm{d}y=\iint_{\mathbb{D}}\frac{1-|\alpha|^2}{|1-|\alpha|z|^2}\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$\begin{aligned}\iint_{\mathbb{D}}\frac{\mathrm{d}x\,\mathrm{d}y}{|1-|\alpha|z|^2}&=\iint_{\mathbb{D}}\frac{\mathrm{d}x\,\mathrm{d}y}{(1-|\alpha|x)^2+(|\alpha|y)^2}\\&=\int_0^1\left(\int_0^{2\pi}\frac{\rho\,\mathrm{d}\theta}{(1-|\alpha|\rho\cos\theta)^2+(|\alpha|\rho\sin\theta)^2}\right)\mathrm{d}\rho\\&=\int_0^1\left(\int_0^{2\pi}\frac{\mathrm{d}\theta}{1-2|\alpha|\rho\cos\theta+(|\alpha|\rho)^2}\right)\rho\,\mathrm{d}\rho\\&=\int_0^1\left(\int_0^{2\pi}P_{|\alpha|\rho}(\theta)\,\mathrm{d}\theta\right)\frac{\rho}{1-(|\alpha|\rho)^2}\,\mathrm{d}\rho,\end{aligned}$$

where $P_{|\alpha|\rho}(\theta)$ is the Poisson kernel for the unit disc given by

$$P_{|\alpha|\rho}(\theta)=\frac{1-(|\alpha|\rho)^2}{1-2|\alpha|\rho\cos\theta+(|\alpha|\rho)^2}.$$

By the Poisson integral representation formula in Exercise 2.6.12,

$$\begin{aligned}\iint_{\mathbb{D}}|\psi_\alpha'|\,\mathrm{d}x\,\mathrm{d}y&=\left(1-|\alpha|^2\right)\int_0^1\left(\int_0^{2\pi}P_{|\alpha|\rho}(\theta)\,\mathrm{d}\theta\right)\frac{\rho}{1-(|\alpha|\rho)^2}\,\mathrm{d}\rho\\&=2\pi\left(1-|\alpha|^2\right)\int_0^1\frac{\rho}{1-|\alpha|^2\rho^2}\,\mathrm{d}\rho\\&=2\pi\left(1-|\alpha|^2\right)\cdot\frac{1}{2|\alpha|^2}\log\frac{1}{1-|\alpha|^2}\\&=\frac{\pi\left(1-|\alpha|^2\right)}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2}.\end{aligned}$$

□

Stein 8.5.19 Prove that the complex plane slit along the union of the rays $\bigcup_{k=1}^{n}\{A_k+\mathrm{i}y : y\leqslant 0\}$ is simply connected.

Proof We can assume $A_1,\cdots,A_n\in\mathbb{R}$. Given a closed curve in this domain, we can shift it upward by some amount so that it is contained in $\mathbb{H}$. Since $\mathbb{H}$ is simply connected, the shifted curve can be continuously deformed to a point in $\mathbb{H}$. This shows that $\mathbb{C} \setminus \bigcup_{k=1}^{n} \{A_k + \mathrm{i}y : y \leqslant 0\}$ is simply connected.□

Stein 8.6.3 The Schwarz–Pick lemma (see Exercise 8.5.13) is the infinitesimal version of an important observation in complex analysis and geometry.

For complex numbers $w \in \mathbb{C}$ and $z \in \mathbb{D}$ we define the **hyperbolic length** of w at z by

$$\|w\|_z = \frac{|w|}{1 - |z|^2},$$

where $|w|$ and $|z|$ denote the usual absolute values. This length is sometimes referred to as the **Poincaré metric**, and as a Riemann metric it is written as

$$\mathrm{d}s^2 = \frac{|\,\mathrm{d}z|^2}{(1 - |z|^2)^2}.$$

This idea is to think of w as a vector lying in the tangent space at z. Observe that for a fixed w, its hyperbolic length grows to infinity as z approaches the boundary of the disc. We pass from the infinitesimal hyperbolic length of tangent vectors to the global hyperbolic distance between two points by integration.

(1) Given two complex numbers z_1 and z_2 in the disc, we define the **hyperbolic distance** between them by

$$d(z_1, z_2) = \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \,\mathrm{d}t,$$

where the infimum is taken over all smooth curves $\gamma\colon [0,1] \to \mathbb{D}$ joining z_1 and z_2. Use the Schwarz–Pick lemma to prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$d(f(z_1), f(z_2)) \leqslant d(z_1, z_2) \quad \text{for any } z_1, z_2 \in \mathbb{D}.$$

In other words, holomorphic functions are distance-decreasing in the hyperbolic metric.

(2) Prove that automorphisms of the unit disc preserve the hyperbolic distance, namely

$$d(\varphi(z_1), \varphi(z_2)) = d(z_1, z_2), \quad \text{for any } z_1, z_2 \in \mathbb{D}$$

and any automorphism φ. Conversely, if $\varphi\colon \mathbb{D} \to \mathbb{D}$ preserves the hyperbolic distance, then either φ or $\overline{\varphi}$ is an automorphism of $\mathbb{D}$.

(3) Given two points $z_1, z_2 \in \mathbb{D}$, show that there exists an automorphism φ such that $\varphi(z_1) = 0$ and $\varphi(z_2) = s$ for some s on the segment $[0, 1)$ on the real line.

(4) Prove that the hyperbolic distance between 0 and $s \in (0, 1]$ is

$$d(0, s) = \frac{1}{2} \log \frac{1+s}{1-s}.$$

(5) Find a formula for the hyperbolic distance between any two points in the unit disc.

Proof (1) By the Schwarz–Pick lemma (see Exercise 8.5.13),

$$\frac{|f'(\gamma(t))|}{1 - |f(\gamma(t))^2|} \leqslant \frac{1}{1 - |\gamma(t)|^2}.$$

Therefore,

$$\begin{aligned} d(f(z_1), f(z_2)) &= \inf_{\substack{\Gamma \\ f(z_1)\to f(z_2)}} \int_0^1 \frac{|\Gamma'(t)|}{1-|\Gamma(t)|^2}\,\mathrm{d}t \leqslant \inf_{\substack{f\circ\gamma \\ z_1\to z_2}} \int_0^1 \frac{|(f\circ\gamma)'(t)|}{1-|f(\gamma(t))|^2}\,\mathrm{d}t \\ &\leqslant \inf_{\substack{\gamma \\ z_1\to z_2}} \int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\,\mathrm{d}t = d(z_1, z_2). \end{aligned} \tag{8.6.3–1}$$

(2) ① If $\varphi \in \operatorname{Aut}(\mathbb{D})$, then the first "$\leqslant$" in (8.6.3–1) is an equality. And by Exercise 8.5.13 (1), the second "$\leqslant$" in (8.6.3–1) is also an equality. Hence φ preserves the hyperbolic distance.

② Conversely, suppose $\varphi\colon \mathbb{D} \to \mathbb{D}$ preserves the hyperbolic distance. We first show that *each hyperbolic circle in $\mathbb{D}$ is also a Euclidean circle*. Let $C_r(z_0)$ be a hyperbolic circle centered at $z_0 \in \mathbb{D}$ with radius $r > 0$. By the above paragraph, the Blaschke factor ψ_{z_0} maps $C_r(z_0)$ to a hyperbolic circle centered at 0 with radius r, which is also a Euclidean circle. Since $\psi_{z_0}^{-1}$ is a Möbius transformation, it maps Euclidean circles to Euclidean circles. Therefore, $C_r(z_0)$ is a Euclidean circle.

By composing φ with the Blaschke factor $\psi_{\varphi(0)}$, we can assume $\varphi(0) = 0$. And by composing φ with a rotation, we can assume $\varphi\big(\frac{1}{2}\big) = \frac{1}{2}$. Consider the hyperbolic circles in $\mathbb{D}$ with centers 0 and $\frac{1}{2}$ passing through $\frac{\mathrm{i}}{2}$. By the previous discussion, these two (Euclidean) circles intersect exactly at $\pm\frac{\mathrm{i}}{2}$. Hence, $\varphi\big(\frac{\mathrm{i}}{2}\big)$ must be one of these points. By composing φ with the conjugation $z \mapsto \bar{z}$, we can assume $\varphi\big(\frac{\mathrm{i}}{2}\big) = \frac{\mathrm{i}}{2}$. The same reasoning shows that $\varphi(z) = \bar{z}$ for every $z \in \mathbb{D}$. And since $d\big(\varphi(z), \frac{\mathrm{i}}{2}\big) = d\big(z, \frac{\mathrm{i}}{2}\big)$, it must be $\varphi(z) = z$.

Unwinding the above argument, we see that either φ or $\overline{\varphi}$ is an automorphism of $\mathbb{D}$.

(3) One can compose the Blaschke factor ψ_{z_1} with an appropriate rotation to obtain the desired automorphism φ.

(4) Let $\gamma(t) = x(t) + \mathrm{i}y(t)\colon [0, a] \to \mathbb{D}$ be a smooth curve with $\gamma(0) = 0$ and $\gamma(a) = s \in (0, 1)$. Then

$$\int_0^a \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\,\mathrm{d}t \geqslant \int_0^a \frac{|x'(t)|}{1-|x(t)|^2}\,\mathrm{d}t = \int_0^a \frac{|\mathrm{d}x(t)|}{1-|x(t)|^2} \geqslant \int_0^s \frac{\mathrm{d}u}{1-u^2} = \frac{1}{2}\log\frac{1+s}{1-s}.$$

The equality holds if we take $\gamma(t) = \dfrac{st}{a}$. Thus, $d(0, s) = \dfrac{1}{2}\log\dfrac{1+s}{1-s}$.

(5) By (2), (3), and (4), we have

$$d(z_1, z_2) = d(0, |\psi_{z_1}(z_2)|) = \frac{1}{2}\log\frac{1+\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right|}{1-\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right|} = \frac{1}{2}\log\frac{|1-\overline{z_1}z_2|+|z_1-z_2|}{|1-\overline{z_1}z_2|-|z_1-z_2|}.$$

□