Stein 8.5.1 A holomorphic mapping $f\colon U \to V$ is a **local bijection** on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z, so that $f\colon D \to f(D)$ is a bijection.

Prove that a holomorphic map $f\colon U \to V$ is a local bijection on U if and only if $f'(z) \neq 0$ for all $z \in U$.

Proof The "only if" part is guaranteed by Proposition 1.1. Suppose $f'(z_0) \neq 0$ and let $w_0 = f(z_0)$. Then we can find $r > 0$ such that $f(z) \neq w_0$ whenever $0 < |z - z_0| \leqslant r$. Define

$$m \coloneqq \min_{|z-z_0|=r} |f(z) - w_0| > 0.$$

Then for $z_0 \in \partial\mathbb{B}(z_0, r)$ and $w \in \mathbb{B}(w_0, m)$, we have

$$|[f(z) - w] - [f(z) - w_0]| = |w - w_0| < m \leqslant |f(z) - w_0|.$$

Now, Rouché's theorem implies that $f(z) - w$ and $f(z) - w_0$ have the same number of zeros in $\mathbb{B}(z_0, r)$, which is exactly one. This shows that $f\colon \mathbb{B}(z_0, r) \to \mathbb{B}(w_0, m)$ is a bijection. $\square$

Remark In fact, the inverse function theorem applies when $f'(z_0) \neq 0$.

Stein 8.5.5 Prove that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from the half-disc $\{z = x + \mathrm{i}y : |z| < 1,\ y > 0\}$ to the upper half-plane.

Proof Since

$$f(x + \mathrm{i}y) = -\frac{1}{2}\left(x + \mathrm{i}y + \frac{1}{x + \mathrm{i}y}\right) = -\frac{x}{2}\left(1 + \frac{1}{x^2 + y^2}\right) - \frac{\mathrm{i}y}{2}\left(1 - \frac{1}{x^2 + y^2}\right),$$

it is clear that f maps the upper half-disc into the upper half-plane. Note that the equation $f(z) = w$ reduces to the quadratic equation $z^2 + 2wz + 1 = 0$, which has two distinct roots z_1 and z_2 in $\mathbb{C} \setminus \mathbb{R}$ whenever $w \in \mathbb{H}$. By Vieta's formulas, $z_1 + z_2 = -2w$ and $z_1 z_1 = 1$. Hence, exactly one of the roots z_1 or z_2 lies in the upper half-disc. This shows that

$$f\colon \{z = x + \mathrm{i}y : |z| < 1,\ y > 0\} \to \mathbb{H}$$

is a bijective holomorphic function. $\square$

Stein 8.5.9 Prove that the function u defined by

$$u(x, y) = \mathrm{Re}\left(\frac{\mathrm{i} + z}{\mathrm{i} - z}\right) \quad \text{and} \quad u(0, 1) = 0$$

is harmonic in the unit disc and vanishes on its boundary. Note that u is not bounded in $\mathbb{D}$.

Proof Since $\dfrac{\mathrm{i} + z}{\mathrm{i} - z}$ is holomorphic in $\mathbb{D}$, it follows by Exercise 1.4.11 that u is harmonic in $\mathbb{D}$. The boundary values can be determined using Thales's theorem in geometry, which states that the angle subtended by a diameter is always a right angle, and thus has a cosine value of zero. $\square$

Stein 8.5.10 Let $F\colon \mathbb{H} \to \mathbb{C}$ be a holomorphic function that satisfies

$$|F(z)| \leqslant 1 \quad \text{and} \quad F(\mathrm{i}) = 0.$$

Prove that

$$|F(z)| \leqslant \left|\frac{z-\mathrm{i}}{z+\mathrm{i}}\right| \quad \text{for all } z \in \mathbb{H}.$$

Proof Consider the map $G\colon \mathbb{D} \to \mathbb{H},\ w \mapsto \mathrm{i}\dfrac{1-w}{1+w}$, as given in Theorem 1.2. The composition $F \circ G\colon \mathbb{D} \to \mathbb{C}$ satisfies

$$|F \circ G(w)| \leqslant 1 \quad \text{and} \quad F \circ G(0) = 0.$$

By the Schwarz lemma, we have

$$|F \circ G(w)| = \left|F\left(\mathrm{i}\frac{1-w}{1+w}\right)\right| \leqslant |w|.$$

Substituting $w = \dfrac{\mathrm{i}-z}{\mathrm{i}+z}$ gives the desired inequality. □

Stein 8.5.11 Show that if $f\colon \mathbb{B}(0, R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \leqslant M$ for some $M > 0$, then

$$\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \leqslant \frac{|z|}{MR}.$$

Proof Without loss of generality, we can assume that f is not constant. Then the maximum modulus principle implies that $|f(z)| < M$ for all $z \in \mathbb{B}(0, R)$. Now, consider the map $g\colon \mathbb{D} \to \mathbb{D},\ z \mapsto \dfrac{f(Rz)}{M}$. It suffices to show that

$$\left|\frac{g(0) - g(z)}{1 - \overline{g(0)}g(z)}\right| \leqslant |z|,$$

which is implemented by the Schwarz lemma, since the left-hand side is the composition of g with the Blaschke factor $\psi_{g(0)}$. □

Stein 8.5.12 A complex number $w \in \mathbb{D}$ is a **fixed point** for the map $f\colon \mathbb{D} \to \mathbb{D}$ if $f(w) = w$.

(1) Prove that if $f\colon \mathbb{D} \to \mathbb{D}$ is analytic and has two distinct fixed points, then f is the identity, that is, $f(z) = z$ for all $z \in \mathbb{D}$.

(2) Must every holomorphic function $f\colon \mathbb{D} \to \mathbb{D}$ have a fixed point?

Proof (1) Suppose z_1 and z_2 are two distinct fixed points of f in $\mathbb{D}$. Consider the Blaschke factor $\psi_{z_1}(z) = \dfrac{z_1 - z}{1 - \overline{z_1}z}$. Then the composition $g := \psi_{z_1} \circ f \circ \psi_{z_1}^{-1}\colon \mathbb{D} \to \mathbb{D}$ has two distinct fixed points 0 and $\psi_{z_1}(z_2)$ in $\mathbb{D}$. By the Schwarz lemma, $g = \mathrm{Id}_{\mathbb{D}}$, which implies $f = \mathrm{Id}_{\mathbb{D}}$.

(2) The composition f as illustrated in the following diagram is an automorphism of $\mathbb{D}$ which has no

fixed points:

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "z\mapsto \mathrm{i}\frac{1-z}{1+z}"] \arrow[d, dashed, "f"'] & \mathbb{H} \arrow[d, "z\mapsto z+1"] \\
\mathbb{D} & \mathbb{H} \arrow[l, "z\mapsto \frac{\mathrm{i}-z}{\mathrm{i}+z}"]
\end{tikzcd}$$

□

Stein 8.5.13 The **pseudo-hyperbolic distance** between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|.$$

(1) Prove that if $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\rho(f(z), f(w)) \leqslant \rho(z,w) \quad \text{for all } z, w \in \mathbb{D}.$$

Moreover, prove that if f is an automorphism of $\mathbb{D}$ then f preserves the pseudo-hyperbolic distance

$$\rho(f(z), f(w)) = \rho(z,w) \quad \text{for all } z, w \in \mathbb{D}.$$

(2) Prove that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leqslant \frac{1}{1-|z|^2} \quad \text{for all } z \in \mathbb{D}.$$

This result is called the Schwarz–Pick lemma. See Problem 8.6.3 for an important application of this lemma.

Proof (1) Consider the Blaschke factor $\psi_\alpha(z) = \dfrac{z-\alpha}{1-\overline{\alpha}z}$ for $\alpha \in \mathbb{B}(0,1)$. Then it suffices to prove that

$$\left|\psi_{f(w)} \circ f(z)\right| \leqslant |\psi_w(z)|,$$

which is equivalent to

$$\left|\psi_{f(w)} \circ f \circ \psi_w^{-1}(z)\right| \leqslant |z|.$$

This is a direct consequence of the Schwarz lemma. If $f \in \mathrm{Aut}(\mathbb{D})$, then we also have

$$\rho\big(f^{-1}(z), f^{-1}(w)\big) \leqslant \rho(z,w),$$

so that the equality holds for all $z, w \in \mathbb{D}$.

(2) By (1) we have

$$\left|\frac{f(z)-f(w)}{z-w}\right| \leqslant \left|\frac{1-\overline{f(w)}f(z)}{1-\overline{w}z}\right|,$$

and by letting $w \to z$ we obtain the desired inequality. □

Stein 8.5.14 Prove that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ take the form

$$e^{\mathrm{i}\theta}\frac{z-\beta}{z-\overline{\beta}}, \quad \theta \in \mathbb{R} \text{ and } \beta \in \mathbb{H}.$$

Proof By Theorem 1.2, the map $g(z) = \mathrm{i}\dfrac{1-z}{1+z}$ is a conformal map from $\mathbb{D}$ to $\mathbb{H}$. Now, given any conformal mapping $f\colon \mathbb{H} \to \mathbb{D}$, by Theorem 2.2, the composition $f \circ g \in \mathrm{Aut}(\mathbb{D})$ takes the form

$$f(g(z)) = e^{\mathrm{i}\gamma} \frac{\alpha - z}{1 - \overline{\alpha} z},$$

where $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. This shows that

$$f(z) = \frac{1+\alpha}{1-\overline{\alpha}} e^{\mathrm{i}\gamma} \frac{z - \mathrm{i}\frac{1-\alpha}{1+\alpha}}{z - \overline{\mathrm{i}\frac{1-\alpha}{1+\alpha}}} = e^{\mathrm{i}\theta} \frac{z - \beta}{z - \overline{\beta}}$$

for some $\theta \in \mathbb{R}$ and $\beta = \mathrm{i}\dfrac{1-\alpha}{1+\alpha} \in \mathbb{H}$, since $\left|\dfrac{1+\alpha}{1-\overline{\alpha}}\right| = 1$. $\square$