Stein 7.3.5 Consider the following function

$$\tilde{\zeta}(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

(1) Prove that the series defining $\tilde{\zeta}(s)$ converges for $\operatorname{Re}(s) > 0$ and defines a holomorphic function in that half-plane.

(2) Show that for $s > 1$ one has $\tilde{\zeta}(s) = \left(1 - 2^{1-s}\right)\zeta(s)$.

(3) Conclude, since $\tilde{\zeta}$ is given as an alternating series, that ζ has no zeros on the segment $0 < s < 1$. Extend this last assertion to $s = 0$ by using the functional equation.

Proof (1) Since the partial sums $\displaystyle\sum_{n=1}^{N}(-1)^n$ are bounded, Exercise 7.3.1 applies.

(2) When $s > 1$, since $\zeta(s)$ and $\tilde{\zeta}(s)$ are absolutely convergent (as infinite series), we have

$$\zeta(s) - \tilde{\zeta}(s) = \sum_{n=1}^{\infty}\left[\frac{1}{n^s} - \frac{(-1)^{n+1}}{n^s}\right] = \sum_{n=1}^{\infty} \frac{2}{(2n)^s} = 2^{1-s}\zeta(s).$$

(3) Note that at $s = 1$, the simple pole of $\zeta(s)$ cancels with the zero of $1 - 2^{1-s}$, so both sides of the identity in (2) are holomorphic functions on $\operatorname{Re}(s) > 0$ that agree on $\operatorname{Re}(s) > 1$. Thus this identity holds on the whole half-plane $\operatorname{Re}(s) > 0$. Focusing on $0 < s < 1$, we have

$$\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s} > 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence, the alternating series $\tilde{\zeta}(s)$ is strictly positive when $0 < s < 1$, and $\zeta(s) \neq 0$ on the segment $0 < s < 1$ by the identity in (2). Finally, the functional equation $\xi(s) = \xi(1-s)$, or equivalently,

$$\pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\tfrac{1-s}{2}\right)\zeta(1-s),$$

in Theorem 2.3 of Chapter 6, shows that $\zeta(0) \neq 0$ since the simple pole of $\zeta(1-s)$ at $s = 0$ cancels with the simple zero of $1/\Gamma\left(\frac{s}{2}\right)$. This concludes that $\zeta(s) \neq 0$ on the segment $0 < s < 1$. $\square$

Remark We have shown that $\zeta(0) = -\frac{1}{2}$ in Homework 11.

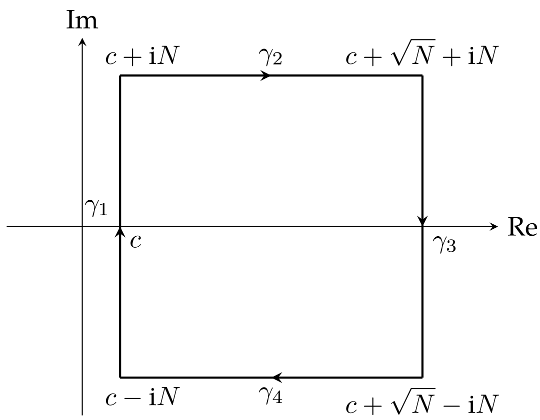
Stein 7.3.6 Show that for every $c > 0$

$$\lim_{N\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}N}^{c+\mathrm{i}N} a^s \frac{\mathrm{d}s}{s} = \begin{cases} 1, & \text{if } a > 1, \\ \frac{1}{2}, & \text{if } a = 1, \\ 0, & \text{if } 0 \leqslant a < 1. \end{cases}$$

This integral is taken over the vertical segment from $c - \mathrm{i}N$ to $c + \mathrm{i}N$.

Proof Let $f(s) = \dfrac{a^s}{s}$ be the integrand.

(1) For $0 \leqslant a < 1$, choose the rectangular contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as illustrated below.



Since $f(s)$ is holomorphic on $\mathbb{C} \setminus \{0\}$, we have $\displaystyle\int_\gamma f(s)\,\mathrm{d}s = 0$. For the integral along γ_2, one has

$$\left| \int_{\gamma_2} f(s)\,\mathrm{d}s \right| \leqslant \sqrt{N} \cdot \frac{a^c}{N} \xrightarrow{N\to\infty} 0.$$

By the same argument we have

$$\left| \int_{\gamma_4} f(s)\,\mathrm{d}s \right| \leqslant \sqrt{N} \cdot \frac{a^c}{N} \xrightarrow{N\to\infty} 0.$$

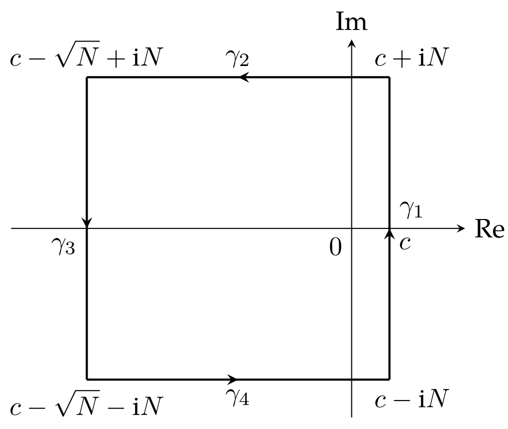
For the integral along γ_3, we have

$$\left| \int_{\gamma_3} f(s)\,\mathrm{d}s \right| \leqslant 2N \cdot \frac{a^{c+\sqrt{N}}}{c+\sqrt{N}} \xrightarrow[0\leqslant a<1]{N\to\infty} 0.$$

Therefore, letting $N \to \infty$ gives

$$\lim_{N\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}N}^{c+\mathrm{i}N} a^s \frac{\mathrm{d}s}{s} = 0 \quad \text{when } 0 \leqslant a < 1.$$

(2) For $a > 1$, choose the rectangular contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as illustrated below.



The residue of $f(s)$ at $s=0$ is 1, hence by the residue formula we have

$$\int_\gamma f(s)\,\mathrm{d}s = 2\pi\mathrm{i}\,\mathrm{Res}(f,0) = 2\pi\mathrm{i}.$$

For the integral along γ_2, one has

$$\left|\int_{\gamma_2} f(s)\,\mathrm{d}s\right| \leqslant \sqrt{N}\cdot\frac{a^c}{N} \xrightarrow{N\to\infty} 0.$$

By the same argument we have

$$\left|\int_{\gamma_4} f(s)\,\mathrm{d}s\right| \leqslant \sqrt{N}\cdot\frac{a^c}{N} \xrightarrow{N\to\infty} 0.$$

For the integral along γ_3, we have

$$\left|\int_{\gamma_3} f(s)\,\mathrm{d}s\right| \leqslant 2N\cdot\frac{a^{c-\sqrt{N}}}{\sqrt{N}-c} \xrightarrow[a>1]{N\to\infty} 0.$$

Therefore, letting $N\to\infty$ gives

$$\lim_{N\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}N}^{c+\mathrm{i}N} a^s\frac{\mathrm{d}s}{s} = 1 \quad \text{when } a>1.$$

(3) For $a=1$, we compute directly

$$\lim_{N\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}N}^{c+\mathrm{i}N}\frac{\mathrm{d}s}{s} = \frac{1}{2\pi\mathrm{i}}\lim_{N\to\infty}\mathrm{i}[\arg(c+\mathrm{i}N)-\arg(c-\mathrm{i}N)] = \frac{\pi}{2\pi} = \frac{1}{2}.$$

Here we choose the principal branch of the logarithm in the slit plane $\mathbb{C}\setminus(-\infty,0]$. □

Remark For $a\in(0,1)\cup(1,\infty)$, we can also use Lemma 2.4 in Chapter 7 to write

$$\begin{aligned}
\frac{1-1/a}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} a^s\frac{\mathrm{d}s}{s} &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} a^s\frac{\mathrm{d}s}{s} - \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} a^{s-1}\frac{\mathrm{d}s}{s} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} a^s\frac{\mathrm{d}s}{s} - \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} a^s\frac{\mathrm{d}s}{s+1} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} a^s\frac{\mathrm{d}s}{s(s+1)} \\
&= \begin{cases} 0, & \text{if } 0<a<1, \\ 1-1/a, & \text{if } a>1, \end{cases}
\end{aligned}$$

whence the result follows.

Stein 7.4.2 One of the "explicit formulas" in the theory of primes is as follows: if ψ_1 is the integrated Tchebychev function considered in Section 2, then

$$\psi_1(x) = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - E(x)$$

where the sum is taken over all zeros ρ of the zeta function in the critical strip. The error term is given by

$$E(x) = c_1 x + c_0 + \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)},$$

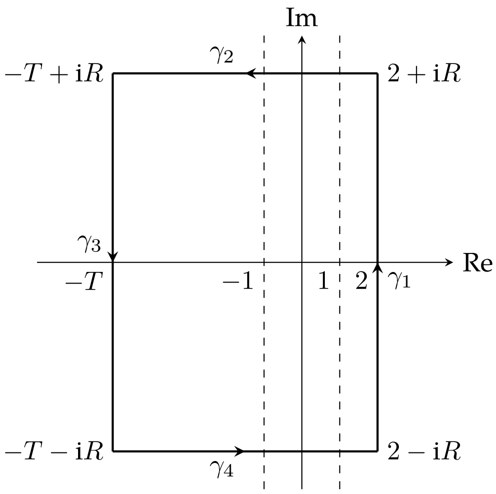
where

$$c_1 = \frac{\zeta'(0)}{\zeta(0)} \quad \text{and} \quad c_0 = -\frac{\zeta'(-1)}{\zeta(-1)}.$$

Proof By Proposition 2.3 in Chapter 7 we have

$$\psi_1(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \mathrm{d}s \quad \text{for all } c > 1.$$

Now fix $c = 2$ and consider the integral of $f(s) = \dfrac{x^{s+1}}{s(s+1)} \left(-\dfrac{\zeta'(s)}{\zeta(s)}\right)$ along the rectangular contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as illustrated below.



It is necessary to choose R with a little care, so that the horizontal sides of the rectangle shall avoid, as far as possible, the zeros of $\zeta(s)$ in the critical strip (see the discussion following ② on page 6). Similarly, here T is chosen to be a large odd integer, so that the left vertical side passes halfway between two of the trivial zeros of $\zeta(s)$.

We begin by calculating the residues of f at $1, 0, -1$, and all the zeros of ζ:

$$\begin{aligned}
\mathrm{Res}(f, 1) &= -\frac{x^2}{2} \operatorname{Ord}(\zeta, 1) = \frac{x^2}{2}, \\
\mathrm{Res}(f, 0) &= \lim_{s \to 0} \frac{x^{s+1}}{s+1} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) = -c_1 x \quad \text{where } c_1 = \frac{\zeta'(0)}{\zeta(0)}, \\
\mathrm{Res}(f, -1) &= \lim_{s \to -1} \frac{x^{s+1}}{s} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) = -c_0 \quad \text{where } c_0 = -\frac{\zeta'(-1)}{\zeta(-1)}, \\
\mathrm{Res}(f, -2k) &= -\frac{x^{-2k+1}}{-2k(-2k+1)} \operatorname{Ord}(\zeta, -2k) = -\frac{x^{1-2k}}{2k(2k-1)} \quad \text{for } k = 1, 2, 3, \cdots, \\
\mathrm{Res}(f, \rho) &= -\frac{x^{\rho+1}}{\rho(\rho+1)} \operatorname{Ord}(\zeta, \rho) \quad \text{for any nontrivial zero } \rho \text{ of } \zeta.
\end{aligned}$$

Here we used "Ord" to denote the order of the zero at the given point. Note that in the formula we are going to prove the nontrivial zeros of ζ are to be counted with multiplicities, i.e., each ρ appears in the summation as many times as its order, since we actually don't know whether they are simple or not.

In Exercise 7.3.8 we have shown that $(s-1)\zeta(s)$ is an entire function of growth order 1, thus by Theorem 2.1 in Chapter 5 we have $\displaystyle\sum_\rho \frac{1}{|\rho|^{1+\varepsilon}} < \infty$ for every $\varepsilon > 0$. Hence

$$\sum_\rho \left|\frac{x^{\rho+1}}{\rho(\rho+1)}\right| \leqslant \sum_\rho \frac{x^2}{|\rho|^2} < \infty.$$

Also, it is obvious that $E(x) = O(x)$ as $x \to \infty$. So we are allowed to apply the residue formula and let R and T tend to infinity to write

$$\psi_1(x) + \frac{1}{2\pi \mathrm{i}} \lim_{R,T\to\infty} \int\limits_{\gamma_2\cup\gamma_3\cup\gamma_4} f(s)\,\mathrm{d}s = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - E(x).$$

It remains to show that the integral of $f(s)$ along $\gamma_2\cup\gamma_3\cup\gamma_4$ vanishes as R and T tend to infinity. To achieve this, we need an estimate for $|\zeta'/\zeta|$. We will show that

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = \begin{cases} O(\log|2s|), & \text{if } \operatorname{Re}(s) \leqslant -1 \text{ and all disks of radius } \frac{1}{2} \text{ around the trivial zeros are excluded,} \\ O\Big(\log^2 R\Big), & \text{if } -1 < \operatorname{Re}(s) \leqslant 2 \text{ and } \operatorname{Im}(s) = R. \end{cases}$$

With this established, it is clear that the integral of $f(s)$ along $\gamma_2\cup\gamma_3\cup\gamma_4$ vanishes as R and T tend to infinity (note that $|\zeta'/\zeta|$ is symmetric about the real axis), thereby completing the proof.

Now let us prove these estimates.

Estimate I: $\operatorname{Re}(s) \leqslant -1$ **with open disks excluded** First recall two functional relations satisfied by $\Gamma(s)$:

- ◇ $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$.
- ◇ $\Gamma(s)\Gamma\big(s+\frac{1}{2}\big) = \sqrt{\pi}2^{1-2s}\Gamma(2s)$, which has been proved in Exercise 6.3.3.

Combined, one has

$$\begin{aligned}\Gamma\big(\tfrac{1-s}{2}\big) = \Gamma\big(1-\tfrac{1+s}{2}\big) &= \frac{\pi}{\sin\big(\pi\frac{1+s}{2}\big)\Gamma\big(\frac{1+s}{2}\big)} = \frac{\pi}{\cos\frac{\pi s}{2}\Gamma\big(\frac{1+s}{2}\big)} = \frac{\pi}{\cos\frac{\pi s}{2}}\cdot\frac{\Gamma\big(\frac{s}{2}\big)}{2^{1-s}\sqrt{\pi}\Gamma(s)} \\ &= \frac{\sqrt{\pi}\Gamma\big(\frac{s}{2}\big)}{2^{1-s}\cos\frac{\pi s}{2}\Gamma(s)},\end{aligned}$$

thus giving

$$\frac{\Gamma\big(\frac{s}{2}\big)}{\Gamma\big(\frac{1-s}{2}\big)} = \pi^{-\frac{1}{2}}2^{1-s}\cos\frac{\pi s}{2}\Gamma(s).$$

If this is used in the functional equation of $\zeta(s)$, we get

$$\zeta(1-s) = \frac{\pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big)\zeta(s)}{\pi^{-\frac{1-s}{2}}\Gamma\big(\frac{1-s}{2}\big)} = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s).$$

Taking the logarithmic derivative of both sides gives

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} = -\frac{\pi}{2}\tan\frac{\pi s}{2} + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)} - \log 2\pi.$$

Since we are interested in the left-hand side under $\mathrm{Re}(1-s) \leqslant -1$, the right-hand side can be considered only for $\mathrm{Re}(s) \geqslant 2$. The first term is bounded if s is not close to any odd integer, or more specifically, $|s-(2m+1)| \geqslant \frac{1}{2}$ for all $m \in \mathbb{N}$. Note that this is equivalent to

$$|(1-s)-(-2m)| \geqslant \frac{1}{2},$$

which is precisely satisfied by our assumption that all disks of radius $\frac{1}{2}$ around the trivial zeros are excluded. The third term is bounded since

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = \left|\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right| \leqslant \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} = -\frac{\zeta'(2)}{\zeta(2)} \quad \text{when } \mathrm{Re}(s) \geqslant 2. \tag{7.4.2–1}$$

Finally, the digamma function $\Gamma'(s)/\Gamma(s)$ is $O(\log|s|)$, and hence $O(\log 2|1-s|)$. Replacing $1-s$ by s in the above, we obtain **Estimate I**. The asymptotic estimate for the digamma function can be deduced from Exercise 6.3.13, where we have shown that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \frac{\mathrm{d}}{\mathrm{d}s} \log\Gamma(s) = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+s}\right).$$

Apply the Euler–Maclaurin summation formula to $(x+s)^{-1}$,

$$\sum_{n=0}^{N} \frac{1}{n+s} = \log(N+s) - \log s + \frac{1}{2s} + \frac{1}{2(s+N)} + O\big(|s|^{-2}\big).$$

Then

$$\sum_{n=1}^{N} \frac{1}{n+s} = \log(N+s) - \log s - \frac{1}{2s} + \frac{1}{2(s+N)} + O\big(|s|^{-2}\big).$$

Hence

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O\big(|s|^{-2}\big). \tag{7.4.2–2}$$

$\boxed{\textbf{Estimate II: } -1 < \mathrm{Re}(s) \leqslant 2 \textbf{ and } \mathrm{Im}(s) = R}$ We refer to two results which we shall prove later:

① For large R (not coinciding with the ordinate of a zero) and $-1 \leqslant \mathrm{Re}(s) \leqslant 2$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\mathrm{Im}(\rho)-R|<1} \frac{1}{s-\rho} + O(\log R). \tag{7.4.2–3}$$

② For any large R, the number of zeros ρ of ζ with $|\mathrm{Im}(\rho) - R| < 1$ is $O(\log R)$.

As a consequence of ②, among the ordinates of these zeros there must be a gap of length at least $C(\log R)^{-1}$ for some constant $C > 0$ independent of R. Hence by varying R by a bounded amount we can ensure that

$$|\mathrm{Im}(\rho) - R| \geqslant \frac{C'}{\log R}$$

for all zeros ρ of ζ. Now we apply ① with the present choice of R to get

$$|s-\rho| \geqslant |\mathrm{Im}(\rho) - R| \geqslant \frac{C'}{\log R},$$

and the number of summation terms is also $O(\log R)$. So on the *new* horizontal lines of integration we obtain **Estimate II**:

$$\frac{\zeta'(s)}{\zeta(s)} = O\Big(\log^2 R\Big) \quad \text{for } -1 \leqslant \mathrm{Re}(s) \leqslant 2.$$

Now we prove the two results ① and ② mentioned above. Define

$$\tilde{\xi}(s) = \frac{1}{2}s(s-1)\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\big(\tfrac{s}{2}\big)\zeta(s) = (s-1)\pi^{-\frac{s}{2}}\Gamma\big(\tfrac{s}{2}+1\big)\zeta(s), \tag{7.4.2–4}$$

then by the deduction in Exercise 7.3.8 we see $\tilde{\xi}(s)$ is an entire function of order 1. Hadamard's factorization theorem shows that

$$\tilde{\xi}(s) = e^{A+Bs}\prod_{\rho}\Big(1-\frac{s}{\rho}\Big)e^{\frac{s}{\rho}},$$

where the product is taken over all nontrivial zeros of ζ. Logarithmic differentiation of this gives

$$\frac{\tilde{\xi}'(s)}{\tilde{\xi}(s)} = B + \sum_{\rho}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big).$$

Since by our definition

$$\frac{\tilde{\xi}'(s)}{\tilde{\xi}(s)} = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'\big(\frac{s}{2}+1\big)}{\Gamma\big(\frac{s}{2}+1\big)} + \frac{\zeta'(s)}{\zeta(s)},$$

we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - B - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'\big(\frac{s}{2}+1\big)}{\Gamma\big(\frac{s}{2}+1\big)} - \sum_{\rho}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big). \tag{7.4.2–5}$$

By the asymptotic behavior (7.4.2–2) of the digamma function we see that the Γ term above is less than $A\log t$ if $t \geqslant 2$ and $1 \leqslant \sigma \leqslant 2$ for $s = \sigma + \mathrm{i}t$. Hence, in this region,

$$-\mathrm{Re}\Big(\frac{\zeta'(s)}{\zeta(s)}\Big) < A\log t - \sum_{\rho}\mathrm{Re}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big).$$

In this inequality we take $s = 2 + \mathrm{i}R$, and since $|\zeta'/\zeta|$ is bounded for such s as shown in (7.4.2–1), we obtain

$$\sum_{\rho}\mathrm{Re}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) < A\log R.$$

Note that $\mathrm{Re}\Big(\frac{1}{\rho}\Big) > 0$ for each ρ, and

$$\mathrm{Re}\Big(\frac{1}{s-\rho}\Big) = \mathrm{Re}\Big(\frac{1}{2+\mathrm{i}R-\rho}\Big) = \frac{2-\mathrm{Re}(\rho)}{[2-\mathrm{Re}(\rho)]^2+[R-\mathrm{Im}(\rho)]^2} \geqslant \frac{1}{4+[R-\mathrm{Im}(\rho)]^2},$$

we get

$$\sum_{\rho}\frac{1}{1+[R-\mathrm{Im}(\rho)]^2} = O(\log R).$$

As a consequence, we see that

$$\frac{1}{2}\#\{\rho : |\mathrm{Im}(\rho) - R| < 1\} \leqslant \sum_{|\mathrm{Im}(\rho)-R|<1} \frac{1}{1+[R-\mathrm{Im}(\rho)]^2} \leqslant \sum_{\rho} \frac{1}{1+[R-\mathrm{Im}(\rho)]^2} = O(\log R),$$

which implies ②. Also note as a byproduct that

$$\frac{1}{2} \sum_{|\mathrm{Im}(\rho)-R|\geqslant 1} \frac{1}{|\mathrm{Im}(\rho) - R|^2} \leqslant \sum_{|\mathrm{Im}(\rho)-R|\geqslant 1} \frac{1}{1+|R-\mathrm{Im}(\rho)|^2} \leqslant \sum_{\rho} \frac{1}{1+[R-\mathrm{Im}(\rho)]^2} = O(\log R),$$

hence we find

$$\sum_{|\mathrm{Im}(\rho)-R|\geqslant 1} \frac{1}{|\mathrm{Im}(\rho) - R|^2} = O(\log R). \tag{7.4.2–6}$$

By formula (7.4.2–5), applied at $s = \sigma + \mathrm{i}R$ (here $-1 < \sigma \leqslant 2$) and $2 + \mathrm{i}R$ and subtracted,

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \frac{\zeta'(2+\mathrm{i}R)}{\zeta(2+\mathrm{i}R)} - \frac{1}{s-1} + \frac{1}{1+\mathrm{i}R} + \frac{1}{2}\frac{\Gamma'\left(2+\frac{\mathrm{i}R}{2}\right)}{\Gamma\left(2+\frac{\mathrm{i}R}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} + \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+\mathrm{i}R-\rho}\right) \\ &= O(\log R) + \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+\mathrm{i}R-\rho}\right),\end{aligned}$$

where we have used (7.4.2–1) and (7.4.2–2) to estimate the ζ and Γ terms. Now we focus on the sum. For the terms with $|\mathrm{Im}(\rho) - R| \geqslant 1$, we have

$$\left|\frac{1}{s-\rho} - \frac{1}{2+\mathrm{i}R-\rho}\right| = \frac{2-\sigma}{|s-\rho||2+\mathrm{i}R-\rho|} \leqslant \frac{3}{|\mathrm{Im}(\rho) - R|^2},$$

and their contribution to the sum is $O(\log R)$ by (7.4.2–6). As for the terms with $|\mathrm{Im}(\rho) - R| < 1$, we have $|2 + \mathrm{i}R - \rho| \geqslant |(2 + \mathrm{i}R) - (1 + \mathrm{i}R)| = 1$, and the number of terms is $O(\log R)$ by ② above. Therefore we have proved ①. □