Stein 6.3.14 This exercise gives an asymptotic formula for $\log n!$. A more refined asymptotic formula for $\Gamma(s)$ as $s \to \infty$ (Stirling's formula) is given in Appendix A.

(1) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x+1} \log\Gamma(t)\,\mathrm{d}t = \log x, \quad \text{for } x > 0,$$

and as a result

$$\int_x^{x+1} \log\Gamma(t)\,\mathrm{d}t = x\log x - x + c.$$

(2) Show as a consequence that $\log\Gamma(n) \sim n\log n$ as $n \to \infty$. In fact, prove that $\log\Gamma(n) \sim n\log n + O(n)$ as $n \to \infty$.

Proof (1) For $x > 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x+1} \log\Gamma(t)\,\mathrm{d}t = \log\Gamma(x+1) - \log\Gamma(x) = \log\frac{\Gamma(x+1)}{\Gamma(x)} = \log x,$$

and by integrating both sides we get the second formula.

(2) Since $\log\Gamma(t)$ is monotonically increasing when $t \geqslant 1$, we have

$$\log\Gamma(n) \leqslant \int_n^{n+1} \log\Gamma(t)\,\mathrm{d}t \leqslant \log\Gamma(n+1) = \log n + \log\Gamma(n).$$

This implies that

$$(n-1)\log n - n + c \leqslant \log\Gamma(n) \leqslant n\log n - n + c,$$

which gives the desired result. □

Stein 6.3.15 Prove that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x.$$

Proof For $x > 0$ we have

$$\frac{1}{e^x - 1} = \sum_{n=1}^\infty e^{-nx}.$$

Substituting this into the integral and applying Fubini's theorem, we get

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\,\mathrm{d}x \overset{t=nx}{=\!=} \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t = \zeta(s)\Gamma(s).$$

□

Stein 6.3.16 Use Exercise 6.3.15 to give another proof that $\zeta(s)$ is continuable in the complex plane with only singularity as a simple pole at $s = 1$.

Proof Use Exercise 6.3.15 to write

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x + \frac{1}{\Gamma(s)}\int_1^\infty \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x.$$

The second integral defines an entire function because of exponential decay near infinity, while

$$\int_0^1 \frac{x^{s-1}}{e^x-1}\,\mathrm{d}x = \int_0^1 x^{s-2}\sum_{m=0}^{\infty}\frac{B_m}{m!}x^m\,\mathrm{d}x = \sum_{m=0}^{\infty}\frac{B_m}{m!}\int_0^1 x^{s+m-2}\,\mathrm{d}x = \sum_{m=0}^{\infty}\frac{B_m}{m!(s+m-1)},$$

where B_m denotes the m-th Bernoulli number defined by

$$\frac{x}{e^x-1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}x^m.$$

Since $\dfrac{z}{e^z-1}$ is holomorphic for $|z|<2\pi$, and the right-hand side above has the same radius of convergence as $\displaystyle\sum_{m=0}^{\infty}\frac{B_m}{m!(s+m-1)}z^m$ when $s\neq 1$, we conclude that $\displaystyle\sum_{m=0}^{\infty}\frac{B_m}{m!(s+m-1)}$ converges for all $s\in\mathbb{C}\setminus\{1\}$. And from $B_0=1$ we see that $s=1$ becomes a simple pole of $\zeta(s)$. □

Stein 6.4.2 Prove that for $\mathrm{Re}(s)>0$

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}\,\mathrm{d}x$$

where $\{x\}$ is the fractional part of x.

Proof We have

$$\begin{aligned}
\mathrm{RHS} &= \frac{s}{s-1} - s\sum_{n=1}^{\infty}\int_n^{n+1}\frac{x-n}{x^{s+1}}\,\mathrm{d}x \\
&= \frac{s}{s-1} - s\int_1^{\infty}\frac{\mathrm{d}x}{x^s} + s\sum_{n=1}^{\infty}\int_n^{n+1}\frac{n}{x^{s+1}}\,\mathrm{d}x \\
&= \frac{s}{s-1} - \frac{s}{s-1} + \sum_{n=1}^{\infty} n\left[\frac{1}{n^s} - \frac{1}{(n+1)^s}\right] \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{s-1}} - \sum_{n=1}^{\infty}\frac{n}{(n+1)^s} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{s-1}} - \sum_{n=2}^{\infty}\frac{n-1}{n^s} \\
&= 1 + \sum_{n=2}^{\infty}\frac{1}{n^s} = \mathrm{LHS}.
\end{aligned}$$

□

Stein 6.4.3 If $Q(x)=\{x\}-\frac{1}{2}$, then we can write the expression in Problem 6.4.2 as

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\int_1^{\infty}\frac{Q(x)}{x^{s+1}}\,\mathrm{d}x.$$

Let us construct $Q_k(x)$ recursively so that

$$\int_0^1 Q_k(x)\,\mathrm{d}x = 0,\quad \frac{\mathrm{d}Q_{k+1}}{\mathrm{d}x} = Q_k(x),\quad Q_0(x)=Q(x)\quad\text{and}\quad Q_k(x+1)=Q_k(x).$$

Then we can write

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\int_1^\infty \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}Q_k(x)\right)x^{-s-1}\,\mathrm{d}x,$$

and a k-fold integration by parts gives the analytic continuation for $\zeta(s)$ when $\mathrm{Re}(s) > -k$.

Proof The two identities are clear from what we have proved in Problem 6.4.2 and the recursive definition of $Q_k(x)$. Assume first $\mathrm{Re}(s) > 0$, integration by parts gives

$$\begin{aligned}
&\int_1^\infty \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}Q_k(x)\right)x^{-s-1}\,\mathrm{d}x \\
=&\sum_{n=1}^\infty \int_n^{n+1} \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}Q_k(x)\right)x^{-s-1}\,\mathrm{d}x \\
=&\sum_{n=1}^\infty \left\{\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}Q_k(x)\right)x^{-s-1}\Big|_n^{n+1} + (s+1)\int_n^{n+1}\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}Q_k(x)\right)x^{-s-2}\,\mathrm{d}x\right\} \\
=&\sum_{n=1}^\infty Q_1(x)x^{-s-1}\Big|_n^{n+1} + (s+1)\int_1^\infty \left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}Q_k(x)\right)x^{-s-2}\,\mathrm{d}x \\
=&-Q_1(0) + (s+1)\int_1^\infty \left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}Q_k(x)\right)x^{-s-2}\,\mathrm{d}x \\
=&-Q_1(0) + (s+1)\left\{-Q_2(0) + (s+2)\int_1^\infty \left(\frac{\mathrm{d}^{k-2}}{\mathrm{d}x^{k-2}}Q_k(x)\right)x^{-s-3}\,\mathrm{d}x\right\} \\
=&\cdots \\
=&-Q_1(0) - \sum_{m=2}^k Q_m(0)(s+1)\cdots(s+m-1) + (s+1)(s+2)\cdots(s+k)\int_1^\infty Q_k(x)x^{-s-k-1}\,\mathrm{d}x.
\end{aligned}$$

Substituting this into the formula of $\zeta(s)$ we get

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} + s\sum_{m=1}^k Q_m(0)s(s+1)\cdots(s+m-1) - s(s+1)\cdots(s+k)\int_1^\infty Q_k(x)x^{-s-k-1}\,\mathrm{d}x.$$

Since $Q_k(x)$ is bounded on $\mathbb{R}$ by its periodicity, the integral converges for $\mathrm{Re}(s) > -k$, which gives the analytic continuation for $\zeta(s)$ when $\mathrm{Re}(s) > -k$. ☐

Stein 7.3.1 Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence of real numbers such that the partial sums

$$A_n = a_1 + \cdots + a_n$$

are bounded. Prove that the Dirichlet series

$$\sum_{n=1}^\infty \frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in this half-plane.

Proof Summation by parts gives

$$\sum_{n=1}^N \frac{a_n}{n^s} = \sum_{n=1}^N \frac{A_n - A_{n-1}}{n^s} = \frac{A_N}{N^s} - \sum_{n=1}^{N-1} A_n\left[\frac{1}{(n+1)^s} - \frac{1}{n^s}\right].$$

Assume $|A_n| \leqslant M$ for all $n \in \mathbb{N}$. Then

$$\left|\frac{A_N}{N^s}\right| \leqslant \frac{M}{N^{\mathrm{Re}(s)}} \xrightarrow{N\to\infty} 0$$

uniformly on every compact subset of the half-plane $\mathrm{Re}(s) > 0$. Applying the mean value theorem to z^{-s}, one gets

$$\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| \leqslant \frac{|s|}{n^{\mathrm{Re}(s)+1}}.$$

Therefore, on every compact subset K of the half-plane $\mathrm{Re}(s) > 0$, we have

$$\sum_{n=1}^{\infty}\left|A_n\left[\frac{1}{(n+1)^s} - \frac{1}{n^s}\right]\right| \leqslant \sum_{n=1}^{\infty}\frac{M|s|}{n^{\mathrm{Re}(s)+1}} \leqslant MS\sum_{n=1}^{\infty}\frac{1}{n^{\delta+1}},$$

where

$$S = \max_{s\in K}|s| < +\infty \quad \text{and} \quad \delta = \min_{s\in K}\mathrm{Re}(s) > 0.$$

These two estimates gives the uniform convergence of the series on every compact subset of the half-plane $\mathrm{Re}(s) > 0$, which implies the holomorphicity of the function defined by this series. □

Stein 7.3.2 The following links the multiplication of Dirichlet series with the divisibility properties of their coefficients.

(1) Show that if $\{a_m\}$ and $\{b_k\}$ are two bounded sequences of complex numbers, then

$$\left(\sum_{m=1}^{\infty}\frac{a_m}{m^s}\right)\left(\sum_{k=1}^{\infty}\frac{b_k}{k^s}\right) = \sum_{n=1}^{\infty}\frac{c_n}{n^s} \quad \text{where } c_n = \sum_{mk=n} a_m b_k.$$

The above series converge absolutely when $\mathrm{Re}(s) > 1$.

(2) Prove as a consequence that one has

$$[\zeta(s)]^2 = \sum_{n=1}^{\infty}\frac{d(n)}{n^s} \quad \text{and} \quad \zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty}\frac{\sigma_a(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s-a) > 1$, respectively. Here $d(n)$ equals the number of divisors of n, and $\sigma_a(n)$ is the sum of the a-th powers of the divisors of n. In particular, one has $\sigma_0(n) = d(n)$.

Proof (1) The convolution identity is obtained by noticing that $m^{-s}k^{-s} = n^{-s}$ if and only if $mk = n$. Assume that $\{a_m\}$ and $\{b_k\}$ are bounded by M. Then $|c_n| \leqslant Md(n)$. A classical result of the arithmetic functions $d(n)$ states that

$$\limsup_{n\to\infty}\frac{\log d(n)\log\log n}{\log n} = \log 2.$$

Hence $d(n) \leqslant C\log n$ for some constant $C > 0$, and

$$\sum_{n=1}^{\infty}\left|\frac{c_n}{n^s}\right| \leqslant MC\sum_{n=1}^{\infty}\frac{\log n}{n^{\mathrm{Re}(s)}} < +\infty \quad \text{when } \mathrm{Re}(s) > 1.$$

(2) Taking $a_m = b_k \equiv 1$ gives the first identity. For the second identity, note that

$$\zeta(s-a) = \sum_{k=1}^{\infty} \frac{1}{k^{s-a}} = \sum_{k=1}^{\infty} \frac{k^a}{k^s}.$$

It then follows by (1) that

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}, \quad \text{where } c_n = \sum_{k|n} k^a = \sigma_a(n).$$

□

Remark In (1), one can also use the fact that the Cauchy product of two absolutely convergent series is absolutely convergent.

Stein 7.3.3 In line with Exercise 7.3.2, we consider the Dirichlet series for $1/\zeta$.

(1) Prove that for $\mathrm{Re}(s) > 1$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n = p_1, \cdots p_k, \text{ and the } p_j \text{ are distinct primes}, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $\mu(nm) = \mu(n)\mu(m)$ whenever n and m are relatively prime.

(2) Show that

$$\sum_{k|n} \mu(k) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof The proof of (1) is based on the formula in (2).

(1) By the Dirichlet convolution formula in Exercise 7.3.2 (1), we have

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \left(\sum_{m=1}^{\infty} \frac{1}{m^s} \right) \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \right) = \sum_{n=1}^{\infty} \frac{c_n}{n^s},$$

where

$$c_n = \sum_{k|n} \mu(k) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

Hence

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \quad \text{for } \mathrm{Re}(s) > 1.$$

(2) The case $n = 1$ is clear. Now assume $n > 1$ and write $n = p_1^{r_1} \cdots p_m^{r_m}$ where $p_1, \cdots, p_m$ are distinct

primes and $r_1, \cdots, r_m$ are positive integers. Since $\mu(n)$ is a multiplicative function, we have

$$\sum_{k|n}\mu(k) = \sum_{0\leqslant s_i\leqslant r_i}\mu(p_1^{s_1}\cdots p_m^{s_m}) = \mu(1) + \sum_{s_i=0,1}\mu(p_1^{s_1}\cdots p_m^{s_m})$$
$$= 1 + \sum_{k=1}^{m}\binom{m}{k}(-1)^k = (1-1)^m = 0. \qquad \square$$

Remark One can prove (1) directly by using the Euler product formula for $\zeta(s)$ to write

$$\frac{1}{\zeta(s)} = \prod_{p\text{ prime}}\left(1-\frac{1}{p^s}\right),$$

and the result is clear from the definition of the Möbius function $\mu(n)$. Then (2) follows by noting that

$$1 = \zeta(s)\cdot\frac{1}{\zeta(s)} = \left(\sum_{m=1}^{\infty}\frac{1}{m^s}\right)\left(\sum_{\ell=1}^{\infty}\frac{\mu(\ell)}{\ell^s}\right) = \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{k|n}\mu(k)$$

when $\text{Re}(s) > 1$, and that the result holds trivially for $n = 1$.

Extra 1 Calculate $\zeta(0)$.

Solution By Theorem 2.4 of Chapter 6, the zeta function $\zeta(s)$ has a simple pole at $s = 1$, so

$$\lim_{s\to 1} s\zeta(1-s) = -1.$$

Let us recall two functional equations satisfied by $\Gamma(s)$:

- $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$.
- $\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2s}\Gamma(2s)$, which has been proved in Exercise 6.3.3.

Combined, one has

$$\Gamma\left(\tfrac{1-s}{2}\right) = \Gamma\left(1-\tfrac{1+s}{2}\right) = \frac{\pi}{\sin\left(\pi\frac{1+s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)} = \frac{\pi}{\cos\frac{\pi s}{2}\Gamma\left(\frac{1+s}{2}\right)} = \frac{\pi}{\cos\frac{\pi s}{2}}\cdot\frac{\Gamma\left(\frac{s}{2}\right)}{2^{1-s}\sqrt{\pi}\Gamma(s)}$$
$$= \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)}{2^{1-s}\cos\frac{\pi s}{2}\Gamma(s)},$$

thus giving

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \pi^{-\frac{1}{2}}2^{1-s}\cos\frac{\pi s}{2}\Gamma(s).$$

If this is used in the functional equation of $\zeta(s)$, we get

$$\zeta(1-s) = \frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)} = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s).$$

Therefore, we have

$$-1 = \lim_{s\to 0} s\zeta(1-s) = \lim_{s\to 0} 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}s\Gamma(s)\zeta(s)$$

$$
\begin{aligned}
&= \lim_{s\to 0} 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s+1)\zeta(s) \\
&= 2\zeta(0),
\end{aligned}
$$

which gives $\zeta(0) = -\frac{1}{2}$. □

Remark We can also use Exercise 6.3.16 to write

$$
\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{B_m}{m!(s+m-1)} + \frac{1}{\Gamma(s)}\int_1^{\infty}\frac{x^{s-1}}{e^x-1}\,\mathrm{d}x \\
&= \frac{B_1}{s\Gamma(s)} + \frac{1}{\Gamma(s)}\sum_{m\geqslant 0,\, m\neq 1}\frac{B_m}{m!(s+m-1)} + \frac{1}{\Gamma(s)}\int_1^{\infty}\frac{x^{s-1}}{e^x-1}\,\mathrm{d}x \\
&= \frac{B_1}{\Gamma(s+1)} + \frac{1}{\Gamma(s)}\sum_{m\geqslant 0,\, m\neq 1}\frac{B_m}{m!(s+m-1)} + \frac{1}{\Gamma(s)}\int_1^{\infty}\frac{x^{s-1}}{e^x-1}\,\mathrm{d}x,
\end{aligned}
$$

whence $\zeta(0) = B_1 = -\frac{1}{2}$ by taking the limit $s \to 0$ and noting that $1/\Gamma(s)$ vanishes at $s = 0$.