Stein 6.3.1 Prove that
$$\Gamma(s)=\lim_{n\to\infty}\frac{n^s n!}{s(s+1)\cdots(s+n)}$$
whenever $s\neq 0,-1,-2,\cdots$.

Proof By Theorem 1.7 of Chapter 6, we have
$$\frac{1}{\Gamma(s)}=e^{\gamma s}s\prod_{n=1}^{\infty}\Big(1+\frac{s}{n}\Big)e^{-\frac{s}{n}}\quad\text{or}\quad \Gamma(s)=e^{-\gamma s}s^{-1}\prod_{n=1}^{\infty}\frac{n}{n+s}e^{\frac{s}{n}}.$$
Using the definition of Euler's constant γ one can write
$$\begin{aligned}\Gamma(s)&=\lim_{N\to\infty}\exp\left\{-s\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)\right\}s^{-1}\prod_{n=1}^{N}\frac{n}{n+s}e^{\frac{s}{n}}\\&=\lim_{N\to\infty}e^{s\log N}\frac{N!}{s(s+1)\cdots(s+N)}=\lim_{N\to\infty}\frac{N^sN!}{s(s+1)\cdots(s+N)}.\end{aligned}$$
□

Stein 6.3.3 Show that Wallis's product formula can be written as
$$\sqrt{\frac{\pi}{2}}=\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}(2n+1)^{\frac12}.$$
As a result, prove the following identity:
$$\Gamma(s)\Gamma\big(s+\tfrac12\big)=\sqrt{\pi}2^{1-2s}\Gamma(2s).$$

Proof By Wallis's product formula we have
$$\frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n)^2-1}=\lim_{n\to\infty}\frac{2^{2n}(n!)^2(2n+1)}{[(2n+1)!!]^2}=\lim_{n\to\infty}\frac{2^{4n}(n!)^4(2n+1)}{[(2n+1)!]^2},$$
which implies the desired result. Now use the formula proved in Exercise 6.3.1 to get
$$\begin{aligned}\Gamma(s)\Gamma\big(s+\tfrac12\big)&=\lim_{n\to\infty}\frac{n^sn!}{s(s+1)\cdots(s+n)}\cdot\frac{n^{s+\frac12}n!}{\big(s+\frac12\big)\big(s+\frac32\big)\cdots\big(s+\frac12+n\big)}\\&=\lim_{n\to\infty}\frac{n^{2s+\frac12}(n!)^2 2^{2n+2}}{(2s)(2s+2)\cdots(2s+2n)(2s+1)(2s+3)\cdots(2s+2n+1)}\\&=\lim_{n\to\infty}\left(\frac{2^{2n}(n!)^2\sqrt{2n+1}}{(2n+1)!}\right)\left(\frac{(2n)^{2s}(2n)!}{(2s)(2s+1)\cdots(2s+2n)}\right)\frac{\sqrt{n}(2n+1)2^{2-2s}}{\sqrt{2n+1}(2s+2n+1)}\\&=\sqrt{\frac{\pi}{2}}\cdot\Gamma(2s)\cdot\frac{1}{\sqrt2}\cdot 2^{2-2s}\\&=\sqrt{\pi}2^{1-2s}\Gamma(2s).\end{aligned}$$
□

Remark The identity $\Gamma(s)\Gamma\big(s+\frac12\big)=\sqrt{\pi}2^{1-2s}\Gamma(2s)$ can be derived in another way using Exercise 6.3.13. Let $f(s)=\dfrac{\Gamma(s)\Gamma\big(s+\frac12\big)}{\Gamma(2s)}$. Then
$$\frac{\mathrm{d}^2\log f(s)}{\mathrm{d}s^2}=\sum_{n=0}^{\infty}\left[\frac{1}{(s+n)^2}+\frac{1}{\big(s+n+\frac12\big)^2}-\frac{4}{(2s+n)^2}\right]=\sum_{n=0}^{\infty}\left[\frac{1}{\big(s+\frac{n}{2}\big)^2}-\frac{1}{\big(s+\frac{n}{2}\big)^2}\right]=0,$$

Hence $\log f(s) = As + B$ for some constant A, B, and $f(s) = e^{As+B}$. Substituting $s = 1$ and $s = \frac{1}{2}$ one gets $A = -2\log 2$ and $B = \log 2 + \log\sqrt{\pi}$, so $f(s) = \sqrt{\pi}2^{1-2s}$.

Stein 6.3.5 Use the fact that $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin \pi s}$ to prove that

$$\left|\Gamma\left(\tfrac{1}{2} + \mathrm{i}t\right)\right| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}}, \quad \text{whenever } t \in \mathbb{R}.$$

Proof By the definition of $\Gamma(s)$ we have $\Gamma(\bar{s}) = \overline{\Gamma(s)}$, so substituting $s = \frac{1}{2} + \mathrm{i}t$ we get

$$\frac{\pi}{\sin \pi\left(\frac{1}{2} + \mathrm{i}t\right)} = \Gamma\left(\tfrac{1}{2} + \mathrm{i}t\right)\Gamma\left(\tfrac{1}{2} - \mathrm{i}t\right) = \left|\Gamma\left(\tfrac{1}{2} + \mathrm{i}t\right)\right|^2.$$

Then the desired result follows from

$$\frac{\pi}{\sin \pi\left(\frac{1}{2} + \mathrm{i}t\right)} = \frac{\pi}{\cos(\mathrm{i}\pi t)} = \frac{2\pi}{e^{\pi t} + e^{-\pi t}}.$$

□

Stein 6.3.7 The Beta function is defined for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$ by

$$\mathrm{B}(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} \,\mathrm{d}t.$$

(1) Prove that $\mathrm{B}(\alpha, \beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(2) Show that $\mathrm{B}(\alpha, \beta) = \displaystyle\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} \,\mathrm{d}u$.

Proof (1) A change of variables gives

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} \,\mathrm{d}t \,\mathrm{d}s \\ &\xlongequal[t=u(1-r)]{s=ur} \int_0^\infty \int_0^1 (ur)^{\beta-1} [u(1-r)]^{\alpha-1} e^{-u} u \,\mathrm{d}r \,\mathrm{d}u \\ &= \int_0^\infty u^{\alpha+\beta-1} e^{-u} \,\mathrm{d}u \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} \,\mathrm{d}r \\ &= \Gamma(\alpha+\beta)\mathrm{B}(\alpha, \beta).\end{aligned}$$

(2) Substituting $t = \dfrac{1}{1+u}$ in the integral we get

$$\mathrm{B}(\alpha, \beta) = \int_0^\infty \left(\frac{u}{1+u}\right)^{\alpha-1} \left(\frac{1}{1+u}\right)^{\beta-1} \frac{1}{(1+u)^2} \,\mathrm{d}u = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} \,\mathrm{d}u.$$

□

Stein 6.3.12 This exercise gives two simple observations about $1/\Gamma$.

(1) Show that $\dfrac{1}{|\Gamma(s)|}$ is not $O\left(e^{c|s|}\right)$ for any $c > 0$.

(2) Show that there is no entire function $F(s)$ with $F(s) = O\left(e^{c|s|}\right)$ that has simple zeros at $s =$

$0, -1, -2, \cdots, -n, \cdots$, and that vanishes nowhere else.

Proof (1) Using $s\Gamma(s) = \Gamma(s+1)$, for $k \in \mathbb{N}$, we have

$$\Gamma\left(-k-\tfrac{1}{2}\right) = \frac{\Gamma\left(-k+\frac{1}{2}\right)}{-k-\frac{1}{2}} = \cdots = \frac{\sqrt{\pi}}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-k-\frac{1}{2}\right)}.$$

Hence

$$\left|\frac{1}{\Gamma\left(-k-\frac{1}{2}\right)}\right| = \frac{\frac{3}{2}\cdot\frac{5}{2}\cdots\left(k+\frac{1}{2}\right)}{2\sqrt{\pi}} \geqslant \frac{k!}{2\sqrt{\pi}}.$$

If $\dfrac{1}{|\Gamma(s)|}$ is $O\left(e^{c|s|}\right)$ for some $c > 0$, then there exists $C > 0$ such that

$$k! \leqslant Ce^{c\left(k+\frac{1}{2}\right)} \quad \text{for all } k \in \mathbb{N},$$

which is impossible since $\lim\limits_{k\to\infty} k!e^{-c\left(k+\frac{1}{2}\right)} = +\infty$.

(2) Suppose that $F(s)$ is such a function with growth order $\leqslant 1$, then by Hadamard's factorization theorem we have

$$F(s) = e^{As+B} s \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}.$$

Comparing this with the Weierstrass product for $\Gamma(s)$ in Theorem 1.7 of Chapter 6 we get

$$\frac{1}{\Gamma(s)} = F(s)e^{(\gamma-A)s-B},$$

but this contradicts (1) by our assumptions on $F(s)$. □