Stein 1.4.2 Let $\langle\cdot,\cdot\rangle$ denote the usual inner product in $\mathbb{R}^2$. In other words, if $Z = (x_1, y_1)$ and $W = (x_2, y_2)$, then

$$\langle Z, W\rangle = x_1x_2 + y_1y_2.$$

Similarly, we may define a Hermitian inner product $(\cdot,\cdot)$ in $\mathbb{C}$ by

$$(z, w) = z\overline{w}.$$

The term Hermitian is used to describe the fact that $(\cdot,\cdot)$ is not symmetric, but rather satisfies the relation

$$(z, w) = \overline{(w, z)} \quad \text{for all } z, w \in \mathbb{C}.$$

Show that

$$\langle z, w\rangle = \frac{1}{2}[(z, w) + (w, z)] = \mathrm{Re}(z, w),$$

where we use the usual identification $z = x + \mathrm{i}y \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.

Proof Suppose $z = x_1 + \mathrm{i}y_1$ and $w = x_2 + \mathrm{i}y_2$. Then

$$(z, w) = z\overline{w} = (x_1 + \mathrm{i}y_1)(x_2 - \mathrm{i}y_2) = (x_1x_2 + y_1y_2) + \mathrm{i}(x_2y_1 - x_1y_2)$$

and

$$\mathrm{Re}(z, w) = x_1x_2 + y_1y_2 = \langle z, w\rangle.$$ □

Stein 1.4.4 Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\succ$ between complex numbers so that:

(1) For any two complex numbers z, w, one and only one of the following is true: $z \succ w$, $w \succ z$ or $z = w$.

(2) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.

(3) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \succ 0$, then $z_1 \succ z_2$ implies $z_1z_3 \succ z_2z_3$.

Proof Consider the three complex numbers $0, \pm\mathrm{i}$. If $\mathrm{i} \succ 0 \succ -\mathrm{i}$, then by (3) one has $\mathrm{i}\cdot\mathrm{i} \succ \mathrm{i}\cdot(-\mathrm{i})$, that is, $-1 \succ 1$. Then applying (3) again gives $-1\cdot\mathrm{i} \succ 1\cdot\mathrm{i}$. Hence, one has $-\mathrm{i} \succ \mathrm{i}$ while $\mathrm{i} \succ -\mathrm{i}$, contradicting (1). The other cases can be similarly shown to be impossible. □

Stein 1.4.5 A set Ω is said to be **pathwise connected** if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an *open* set Ω is pathwise connected if and only if Ω is connected.

(1) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z\colon [0, 1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t^* = \sup_{0\leqslant t\leqslant 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \leqslant s < t\}.$$

Arrive at a contradiction by considering the point $z(t^*)$.

(2) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega_1$ as desired.

The proof actually shows that the regularity and type of curves we used to define pathwise connectedness can be relaxed without changing the equivalence between the two definitions when Ω is open. For instance, we may take all curves to be continuous, or simply polygonal lines (a polygonal line is a piecewise-smooth curve which consists of finitely many straight line segments).

Proof (1) Since Ω_1 is open, for any t_0 with $z(t_0) \in \Omega_1$, there exists $t_1 > t_0$ such that $z(t_1) \in \Omega_1$. This shows $z(t^*) \notin \Omega_1$. Similarly, $z(t^*) \notin \Omega_2$. Hence, $z(t^*) \notin \Omega_1 \cup \Omega_2 = \Omega$, a contradiction.

(2) For any $p \in \Omega_1$, choose an open ball $\mathbb{B}(p, r) \subset \Omega$. Since any point in $\mathbb{B}(p, r)$ can be joined to p by a line segment in $\mathbb{B}(p, r)$, it follows that $\mathbb{B}(p, r) \subset \Omega_1$. Hence, Ω_1 is open. Similarly, Ω_2 is open. Obviously Ω_1 and Ω_2 are disjoint and their union is Ω. Since we can choose a ball around w that is contained in Ω (which is pathwise connected), it follows that Ω_1 is non-empty. Therefore, by the connectedness of Ω, we conclude that $\Omega = \Omega_1$, i.e., Ω is pathwise connected. □

Stein 1.4.6 Let Ω be an open set in $\mathbb{C}$ and $z \in \Omega$. The **connected component** (or simply the **component**) of Ω containing z is the set $\mathcal{C}_z$ of all points w in Ω that can be joined to z by a curve entirely contained in Ω.

(1) Check first that $\mathcal{C}_z$ is open and connected. Then, show that $w \in \mathcal{C}_z$ defines an equivalence relation, that is: (i) $z \in \mathcal{C}_z$, (ii) $w \in \mathcal{C}_z$ implies $z \in \mathcal{C}_w$, and (iii) if $w \in \mathcal{C}_z$ and $z \in \mathcal{C}_\zeta$, then $w \in \mathcal{C}_\zeta$.

Thus Ω is the union of all its connected components, and two components are either disjoint or coincide.

(2) Show that Ω can have only countably many distinct connected components.

(3) Prove that if Ω is the complement of a compact set, then Ω has only one unbounded component.

Proof (1) The same proof as in Exercise 1.4.5 (2) shows that $\mathcal{C}_z$ is open. Then by Exercise 1.4.5 (1), the pathwise connectedness of $\mathcal{C}_z$ implies that $\mathcal{C}_z$ is connected. The three properties are obvious from the definition of pathwise connectedness.

(2) If Ω has uncountably many connected components, then by (1) we obtain uncountably many disjoint open balls in Ω. This contradicts the countability of rational points in $\mathbb{C}$.

(3) Choose a closed disc containing the compact set Ω^c, so that the complement of this disc is open and connected, and hence is contained in a component of Ω. This component is obviously the unique unbounded component of Ω. □

Stein 1.4.7 The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(1) Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

(2) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F\colon z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

① F maps the unit disc to itself (that is, $F\colon \mathbb{D} \to \mathbb{D}$), and is holomorphic.

② F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

③ $|F(z)| = 1$ if $|z| = 1$.

④ $F\colon \mathbb{D} \to \mathbb{D}$ is bijective.

Proof (1) We have

$$\begin{gathered} |z| < 1 \text{ and } |w| < 1 \implies \left(1-|z|^2\right)\left(1-|w|^2\right) > 0 \iff |w|^2 + |z|^2 < 1 + |w|^2|z|^2 \\ \iff |w|^2 + |z|^2 - w\bar{z} - \overline{w}\bar{z} < 1 - w\bar{z} - \overline{w}z + |w|^2|z|^2 \iff (w-z)(\overline{w}-\bar{z}) < (1-\overline{w}z)(1-w\bar{z}) \\ \iff |w-z|^2 < |1-\overline{w}z|^2 \overset{\bar{z}w\neq 1}{\iff} \left|\frac{w-z}{1-\overline{w}z}\right| < 1 \end{gathered}$$

and

$$\begin{gathered} |z| = 1 \text{ or } |w| = 1 \iff \left(1-|z|^2\right)\left(1-|w|^2\right) = 0 \iff |w|^2 + |z|^2 = 1 + |w|^2|z|^2 \\ \iff |w|^2 + |z|^2 - w\bar{z} - \overline{w}\bar{z} = 1 - w\bar{z} - \overline{w}z + |w|^2|z|^2 \iff (w-z)(\overline{w}-\bar{z}) = (1-\overline{w}z)(1-w\bar{z}) \\ \iff |w-z|^2 = |1-\overline{w}z|^2 \overset{\bar{z}w\neq 1}{\iff} \left|\frac{w-z}{1-\overline{w}z}\right| = 1. \end{gathered}$$

(2) ① The implications in the proof of (1) show that F maps $\mathbb{D}$ to $\mathbb{D}$, and F is holomorphic since it is a rational function whose denominator is never zero in $\mathbb{D}$.

② Direct computation gives $F(0) = \dfrac{w-0}{1-0} = w$ and $F(w) = \dfrac{w-w}{1-|w|^2} = 0$.

③ This is shown in the proof of (1).

④ It is direct to check that $F^2 = \mathrm{Id}_{\mathbb{D}}$, so F is bijective. □

Stein 1.4.9 Show that in polar coordinates, the Cauchy–Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + \mathrm{i}\theta \quad \text{where } z = re^{\mathrm{i}\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof If we let $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases}$ then $\begin{pmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix}$ and hence

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\dfrac{1}{r}\sin\theta \\ \sin\theta & \dfrac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta\dfrac{\partial u}{\partial r} - \dfrac{1}{r}\sin\theta\dfrac{\partial u}{\partial \theta} \\ \sin\theta\dfrac{\partial u}{\partial r} + \dfrac{1}{r}\cos\theta\dfrac{\partial u}{\partial \theta} \end{pmatrix}.$$

Similarly, we have

$$\begin{pmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta\dfrac{\partial v}{\partial r} - \dfrac{1}{r}\sin\theta\dfrac{\partial v}{\partial \theta} \\ \sin\theta\dfrac{\partial v}{\partial r} + \dfrac{1}{r}\cos\theta\dfrac{\partial v}{\partial \theta} \end{pmatrix}.$$

The Cauchy–Riemann in Cartesian coordinates is then equivalent to

$$\begin{cases} \cos\theta\dfrac{\partial u}{\partial r} - \dfrac{1}{r}\sin\theta\dfrac{\partial u}{\partial \theta} = \sin\theta\dfrac{\partial v}{\partial r} + \dfrac{1}{r}\cos\theta\dfrac{\partial v}{\partial \theta}, \\ \sin\theta\dfrac{\partial u}{\partial r} + \dfrac{1}{r}\cos\theta\dfrac{\partial u}{\partial \theta} = -\cos\theta\dfrac{\partial v}{\partial r} + \dfrac{1}{r}\sin\theta\dfrac{\partial v}{\partial \theta}. \end{cases}$$

Rewrite these equations as

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial r} - \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the desired result follows.

For the logarithm function, consider the functions $u = \log r$ and $v = \theta$. Then

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = 0 = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Therefore, the logarithm function is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$. □

Stein 1.4.15 Abel's theorem. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{r\to 1,\, r<1} \sum_{n=1}^{\infty} r^n a_n = \sum_{n=1}^{\infty} a_n.$$

In other words, if a series converges, then it is Abel summable with the same limit.

Proof Let $A_k = \sum_{n=1}^{k} a_n$ and $A = \lim_{k\to\infty} A_k$. Then by Exercise 1.4.14 we have

$$\sum_{n=1}^{N} r^n a_n = r^N A_N - \sum_{n=1}^{N-1} \left(r^{n+1} - r^n\right) A_n = r^N A_N + (1-r) \sum_{n=1}^{N-1} r^n A_n.$$

After subtracting a constant from a_1, we may assume that $A = 0$. Letting $N \to \infty$ gives

$$\sum_{n=1}^{\infty} r^n a_n = (1-r) \sum_{n=1}^{\infty} r^n A_n.$$

Given $\varepsilon > 0$, pick M large enough so that $|A_n| < \varepsilon$ for all $n \geqslant M$ and note that

$$\left| (1-r) \sum_{n=M}^{\infty} r^n A_n \right| \leqslant \varepsilon |1-r| \sum_{n=M}^{\infty} |r|^n = \varepsilon |1-r| \frac{r^M}{1-|r|}.$$

Therefore, we have

$$\limsup_{r\to 1,\, r<1} \left| \sum_{n=1}^{\infty} r^n a_n \right| \leqslant \limsup_{r\to 1,\, r<1} |1-r| \left| \sum_{n=1}^{M-1} r^n A_n \right| + \varepsilon \limsup_{r\to 1,\, r<1} \frac{|1-r| r^M}{1-|r|} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

Stein 1.4.16 Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when:

(1) $a_n = (\log n)^2$.

(2) $a_n = n!$.

(3) $a_n = \dfrac{n^2}{4^n + 3n}$.

(4) $a_n = \dfrac{(n!)^3}{(3n)!}$.

(5) Find the radius of convergence of the **hypergeometric series**

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \cdots$.

(6) Find the radius of convergence of the Bessel function of order r:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n},$$

where r is a positive integer.

Solution Denote the radius of convergence by R.

(1) Since $\limsup_{n\to\infty}(\log n)^{\frac{2}{n}} = \exp\left(\limsup_{n\to\infty}\frac{2\log n}{n}\right) = 1$, we have $R = 1$.

(2) By Stirling's formula we get $\limsup_{n\to\infty}(n!)^{\frac{1}{n}} = \limsup_{n\to\infty}(2n\pi)^{\frac{1}{2n}}\frac{n}{e} = +\infty$. Hence $R = 0$.

(3) Since $\limsup_{n\to\infty}\left(\frac{n^2}{4^n+3n}\right)^{\frac{1}{n}} = \limsup_{n\to\infty}\frac{n^{\frac{2}{n}}}{4} = \frac{1}{4}$, we have $R = 4$.

(4) By Stirling's formula we get

$$\limsup_{n\to\infty}\left[\frac{(n!)^3}{(3n)!}\right]^{\frac{1}{n}} = \limsup_{n\to\infty}\left[\frac{(2n\pi)^{\frac{3}{2}}\left(\frac{n}{e}\right)^{3n}}{\sqrt{6n\pi}\left(\frac{3n}{e}\right)^{3n}}\right]^{\frac{1}{n}} = \limsup_{n\to\infty}\left[\frac{(2n\pi)^{\frac{3}{2}}}{\sqrt{6n\pi}3^{3n}}\right]^{\frac{1}{n}} = \frac{1}{27}$$

Hence $R = 27$.

(5) The series terminates if either α or β is a nonpositive integer, in which case the function reduces to a polynomial and $R = +\infty$. Otherwise, we use the following useful limit for asymptotic approximations as $x \to \infty$:

$$\Gamma(x+\alpha) \sim \Gamma(x)x^{\alpha}, \quad \alpha \in \mathbb{C},$$

which yields

$$\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} \sim \frac{\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{\Gamma(\beta+n)}{\Gamma(\beta)}}{n!\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}} \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}n^{\alpha+\beta-\gamma}.$$

Since

$$\limsup_{n\to\infty}\left(\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}n^{\alpha+\beta-\gamma}\right)^{\frac{1}{n}} = 1,$$

we have $R = 1$.

(6) By Stirling's formula we get

$$\limsup_{n\to\infty}\left|\frac{(-1)^n}{n!(n+r)!2^{2n+r}}\right|^{\frac{1}{2n+r}} = \limsup_{n\to\infty}\left(\frac{1}{2\pi n\left(\frac{2n}{e}\right)^{2n+r}}\right)^{\frac{1}{2n+r}} = 0.$$

Hence $R = +\infty$. □